

Fractional triangle decompositions in graphs with large minimum degree

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Abstract

A triangle decomposition of a graph is a partition of its edges into triangles. A fractional triangle decomposition of a graph is an assignment of a non-negative weight to each of its triangles such that the sum of the weights of the triangles containing any given edge is one. We prove that for all $\epsilon > 0$, every large enough graph on n vertices with minimum degree at least $(0.9 + \epsilon)n$ has a fractional triangle decomposition. This improves a result of Garaschuk that the same result holds for graphs with minimum degree at least $0.956n$. Together with a recent result of Barber, Kühn, Lo and Osthus, this implies that for all $\epsilon > 0$, every large enough triangle divisible graph on n vertices with minimum degree at least $(0.9 + \epsilon)n$ admits a triangle decomposition.

1 Introduction

Decomposition and packing problems are central and classical problems in combinatorics, in particular, in design theory. Kirkman's theorem [6] from the middle of 19th century gives a necessary and sufficient condition on the existence of a Steiner triple system with a certain number of elements. In the language of graph theory, Kirkman's result asserts that every complete graph with an odd number of vertices and a number of edges divisible by three can be decomposed into triangles. Barber, Kühn, Lo and Osthus [1] showed that the same conclusion is true for large graphs satisfying necessary divisibility conditions if their minimum degree is not too far from the number of their vertices. In this short paper, we study the fractional variant of the problem and we use it to improve the bound obtained by Barber et al.

Let us fix the terminology we are going to use. A *graph* is a pair of sets (V, E) such that elements of E are unordered pairs of elements of V . The elements of V are called *vertices* and the elements of E are called *edges*. We denote by uv (or vu) the edge with vertices u and v . We denote by $|G|$ the number of vertices of G . Two vertices contained in the same edge are said to be *adjacent* or to be *neighbours*. Two edges that share a vertex are said to be *adjacent*. The *degree* of a vertex v is equal to the number of neighbours of v . Let $\gcd(G)$ denote the greatest common divisor of the degrees of the vertices of G .

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijection b from V_1 to V_2 such that uv is an edge of G_1 if and only if $b(u)b(v)$ is an edge of G_2

for every two vertices u and v of G_1 . The *complete graph* K_k is the graph with k vertices all mutually adjacent. The graph K_3 is also called a *triangle*. A graph $G_1 = (V_1, E_1)$ is a *subgraph* of $G_2 = (V_2, E_2)$ if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. The subgraphs of G_2 isomorphic to G_1 will be referred to as copies of G_1 .

Let H be a graph. An *H -decomposition* of a graph G is a set of subgraphs of G isomorphic to H that are edge disjoint such that each edge of G is contained in one of them. A graph is *H -decomposable* if it admits an H -decomposition. A K_3 -decomposition is also called a *triangle decomposition* and a graph is *triangle decomposable* if it is K_3 -decomposable. A graph G is *H -divisible* if $\text{gcd}(G)$ is a multiple of $\text{gcd}(H)$, and the number of edges of G is a multiple of the number of edges of H . It is easy to see that every H -decomposable graph is H -divisible, but the converse is not true. Kirkman [6] proved that every K_3 -divisible complete graph is K_3 -decomposable. The fact that for all H , every H -divisible complete graph is H -decomposable remained an open problem for over one hundred years before it was solved by Wilson [8].

The first generalisation to graphs that are near complete is due to Gustavsson [4]. He proved that for every graph H , there exist $n_0(H)$ and $\epsilon(H)$ such that every H -divisible graph with $n \geq n_0(H)$ vertices and minimum degree at least $(1 - \epsilon(H))n$ is H -decomposable. This has been generalised to hypergraphs in a recent result of Keevash [5]. The best that is known to date for a general graph H is due to Barber et al. [1], who proved that for all $\epsilon > 0$, every sufficiently large H -divisible graph on n vertices with minimum degree at least $(1 - \frac{1}{16|H|^2(|H|-1)^2} + \epsilon)n$ is H -decomposable. For some particular classes of graphs, the exact asymptotic minimum degree threshold is known [1][9].

A *fractional H -decomposition* of a graph G is an assignment of non-negative weights to the copies of H in G such that for an edge e , the sum of the weights of the copies of H that contain e is equal to one. A graph is *fractionally H -decomposable* if it admits a fractional H -decomposition. A graph can be fractionally H -decomposable without being H -divisible. A fractional K_3 -decomposition is also called a *fractional triangle decomposition* and a graph is *fractionally triangle decomposable* if it is fractionally K_3 -decomposable. For all $r \geq 2$, Yuster [10] proved that every graph on n vertices with minimum degree at least $(1 - \frac{1}{9r^{10}})n$ is fractionally K_r -decomposable, and Dukes [2] proved that the same result holds for sufficiently large graphs on n vertices with minimum degree at least $(1 - \frac{1}{16r^2(r-1)^2})n$.

In this paper we will focus on triangle decompositions of graphs with large minimum degree. The following conjecture is due to Nash-Williams [7]:

Conjecture 1 (Nash-Williams [7]). *Let G be a K_3 -divisible graph with n vertices and minimum degree at least $\frac{3}{4}n$. If n is large enough, then G is K_3 -decomposable.*

The best result towards a proof of Conjecture 1 is due to Barber et al. [1].

Theorem 2 (Barber et al. [1]). *There exists an n_0 such that every K_3 -divisible graph G on $n \geq n_0$ vertices with minimum degree at least $0.956n$ is K_3 -decomposable.*

The proof of Theorem 2 relies on a result on fractional K_3 -decomposability, which we now state. The following appears as a conjecture in [3].

Conjecture 3 (Garaschuk [3]). *Let G be a graph with n vertices and minimum degree at least $\frac{3}{4}n$. If n is large enough, then G is fractionally K_3 -decomposable.*

The best known result towards proving Conjecture 1 was established by Garaschuk [3].

Theorem 4 (Garaschuk [3]). *Let G be a graph with n vertices and minimum degree at least $0.956n$. The graph G admits a fractional triangle decomposition.*

In this paper we use a different method to prove the following.

Theorem 5. *Let $\epsilon > 0$. There exists an n_0 such that every graph with $n \geq n_0$ vertices and minimum degree at least $(\frac{9}{10} + \epsilon)n$ admits a fractional triangle decomposition.*

In [1], a particular case of Theorem 11.1 and Lemma 12.3 imply the following.

Theorem 6 (Barber et al. [1]). *Suppose there exist n_0 and δ such that every graph on $n \geq n_0$ vertices with minimum degree at least δn is fractionally K_3 -decomposable. For all $\epsilon > 0$, there exist n_1 such that every K_3 -divisible graph on $n \geq n_1$ vertices with minimum degree at least $(\max(\delta, \frac{3}{4}) + \epsilon)n$ is K_3 -decomposable.*

Together with Theorem 6, our result improves Theorem 2.

Theorem 7. *Let $\epsilon > 0$. There exists an n_0 such that every K_3 -divisible graph on $n \geq n_0$ vertices with minimum degree at least $(\frac{9}{10} + \epsilon)n$ is K_3 -decomposable.*

2 Proof of Theorem 5

Let $\delta < \frac{1}{10}$ and fix a graph G with n vertices and minimum degree at least $(1 - \delta)n$. Suppose the graph G has at least one triangle with three vertices of degree at least $(1 - \delta)n + 2$. Let G' be the graph G where the edges of one such triangle are removed. Observe that G' has minimum degree at least $(1 - \delta)n$ and that if G' has a fractional triangle decomposition, then G has one too. Up to doing this operation several times, we can assume that G has no triangle with three vertices of degree at least $(1 - \delta)n + 2$. Let m be the number of edges of G .

Initially, we give the same weight w_Δ to every triangle such that the sum of the weights of the triangles is equal to $3m$. We will modify the weights of the triangles to obtain a fractional triangle decomposition. We will do so in a way that the total sum of the weights is preserved.

We define the weight of an edge e to be the sum of the weights of the triangles that contain e . Given H a copy of K_4 in G , and two non-adjacent edges e_1 and e_2 in H , let us call $(H, \{e_1, e_2\})$ a *rooted K_4* of G . We will use the following procedure to modify the weights of the edges of a rooted K_4 of G :

Let $(H, \{e_1, e_2\})$ be a rooted K_4 of G . By removing a weight w from the two triangles of H that contain e_1 and adding the same weight w to each of the other two triangles (i.e. those that contain e_2), we transfer a weight of $2w$ from e_1 to e_2 . The weights of all the other edges of the graph remain unchanged (see Figure 1).

To prevent the weight of any triangle from becoming negative, we have to restrict how much weight we can transfer using the procedure above. If for some w we use the procedure to transmit a weight of $2w$ from an edge to another one, then any triangle's weight is lowered by at most w for triangles that are in the K_4 , and does not change for other triangles. Moreover, since every triangle contains a vertex with degree at most $(\delta + 1)n$, any triangle is in at most $(1 - \delta)n$ copies of K_4 , and thus in at most $(1 - \delta)n$ oriented copies of K_4 (since for each K_4 there are three possible choices for the pair of edges). Since each triangle has an initial weight of w_Δ , if it sends weights of at most $\frac{2w_\Delta}{3(1-\delta)n}$ through each rooted K_4 that it is contained in, its final weight will be non-negative.

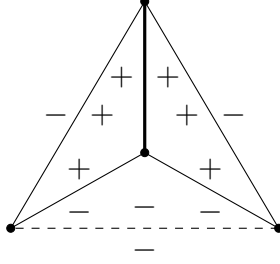


Figure 1: By removing some weight w from two triangles and adding w to the two other triangles, we remove $2w$ from the dashed edge and add $2w$ to the thick edge.

We express redistributing the weights as a flow problem in an auxiliary graph, which is denoted by \widehat{G} . The vertices of \widehat{G} are the edges of G . Two vertices in \widehat{G} are adjacent if they form a pair in a rooted K_4 and the edge between them is set to have the capacity $c = \frac{2w_\Delta}{3(1-\delta)n}$. Let E_c be the set of these edges. Moreover \widehat{G} has two additional vertices, which we will call the *supersource* and the *supersink*. Let T_e be the number of triangles of G that contains an edge e . If $T_e w_\Delta > 1$, then the vertex of \widehat{G} corresponding to e is joined to the supersource and the capacity of the corresponding edge of \widehat{G} is $T_e w_\Delta - 1$. Likewise, if $T_e w_\Delta < 1$, then the vertex of \widehat{G} corresponding to e is joined to the supersink and the capacity of the corresponding edge is $1 - T_e w_\Delta$. The vertices of G adjacent to the supersource are referred to as *sources* and those adjacent to the supersink as *sinks*. Let

$$M = \sum_{e \text{ source}} (T_e w_\Delta - 1) = \sum_{e \text{ sink}} (1 - T_e w_\Delta).$$

We will show that \widehat{G} has a flow of value M from the supersource to the supersink.

If \widehat{G} does not have a flow of value M , then it has a vertex cut (A_0, B_0) such that the supersource is contained in A_0 , the supersink in B_0 and the sum of the capacities of the edges from A_0 to B_0 is less than M . Let A be the edges of G corresponding to the vertices of A_0 and B the edges corresponding to the vertices of B_0 . Note that $|A| = |A_0| - 1$ and $|B| = |B_0| - 1$. Finally, let $k = |A|$, and observe that $|B| = m - k$.

Let T_A and T_B be the average T_e for e in A and in B respectively. Let $e = uv$ be an edge of G . Let W_e be the set of the vertices w such that uvw is a triangle. By the definition of T_e , $|W_e| = T_e$. Each vertex of W_e is non-adjacent to at most δn vertices of G , and thus is non-adjacent to at most δn vertices of W_e . So each vertex of W_e is adjacent to at least $T_e - \delta n$ vertices of W_e . Therefore e is in at least $\frac{T_e(T_e - \delta n)}{2}$ distinct copies of K_4 , and consequently e is in at least $\frac{T_e(T_e - \delta n)}{2}$ rooted K_4 .

Let e be a vertex of A . It is adjacent to at least $\frac{T_e(T_e - \delta n)}{2} - k$ vertices of B . Therefore the cut contains at least

$$\sum_{e \in A} \left(\frac{T_e(T_e - \delta n)}{2} - k \right)$$

edges of E_c . Similarly, it contains at least

$$\sum_{e \in B} \left(\frac{T_e(T_e - \delta n)}{2} - (m - k) \right)$$

edges of E_c . Moreover, for each source e that is in B and each sink e that is in A , the cut contains the edge between e and the supersource or the supersink. Recall that the

capacities of the edges of E_1 is $c = \frac{2w_\Delta}{3(1-\delta)n}$. Therefore the sum of the capacities of the edges of \widehat{G} is at least

$$\sum_{e \in A} \left(\frac{T_e(T_e - \delta n)}{2} - k \right) c + \sum_{e \text{ source} \in B} (T_e w_\Delta - 1) + \sum_{e \text{ sink} \in A} (1 - T_e w_\Delta).$$

At the same time, it is also at least

$$\sum_{e \in B} \left(\frac{T_e(T_e - \delta n)}{2} - (m - k) \right) c + \sum_{e \text{ source} \in B} (T_e w_\Delta - 1) + \sum_{e \text{ sink} \in A} (1 - T_e w_\Delta).$$

Since the sum of the capacities of the edges in the considered cut is less than M , we get that

$$\sum_{e \in A} \left(\frac{T_e(T_e - \delta n)}{2} - k \right) c + \sum_{e \text{ source} \in B} (T_e w_\Delta - 1) + \sum_{e \text{ sink} \in A} (1 - T_e w_\Delta) < M \quad (1)$$

and

$$\sum_{e \in B} \left(\frac{T_e(T_e - \delta n)}{2} - (m - k) \right) c + \sum_{e \text{ source} \in B} (T_e w_\Delta - 1) + \sum_{e \text{ sink} \in A} (1 - T_e w_\Delta) < M. \quad (2)$$

The inequalities (1) and (2) can be rewritten using that

$$M = \sum_{e \text{ source}} (T_e w_\Delta - 1)$$

and

$$M = \sum_{e \text{ sink}} (1 - T_e w_\Delta)$$

respectively as follows.

$$\sum_{e \in A} (T_e(T_e - \delta n) - 2k) c - 2 \sum_{e \in A} (T_e w_\Delta - 1) < 0 \quad (3)$$

$$\sum_{e \in B} (T_e(T_e - \delta n) - 2(m - k)) c - 2 \sum_{e \in B} (1 - T_e w_\Delta) < 0 \quad (4)$$

Since the summand is a convex function of T_e , we obtain the following.

$$(T_A(T_A - \delta n) - 2k) c - 2(T_A w_\Delta - 1) < 0 \quad (5)$$

$$(T_B(T_B - \delta n) - 2(m - k)) c - 2(1 - T_B w_\Delta) < 0 \quad (6)$$

The inequality (5) implies that

$$T_A(T_A - \delta n) + \frac{2}{c}(1 - T_A w_\Delta) < 2k. \quad (7)$$

The inequality (6) implies that

$$2k < 2m - T_B(T_B - \delta n) + \frac{2}{c}(1 - T_B w_\Delta). \quad (8)$$

We now combine the inequalities (7) and (8) and we substitute $c = \frac{2w_\Delta}{3(1-\delta)n}$ to get the following

$$T_A(T_A - \delta n) - (3n(1 - \delta)T_A) < 2m - T_B(T_B - \delta n) - (3n(1 - \delta)T_B) \quad (9)$$

Let e be an edge of G . Each end-vertex of e is non-adjacent to at most δn vertices of G . Hence, the edge e is contained in at least $n - 2\delta n$ triangles. Since e cannot be contained in more than n triangles, we get that $n - 2\delta n \leq T_e \leq n$. Consequently, we have $n - 2\delta n \leq T_A, T_B \leq n$.

A standard analytic argument shows that the left hand side of (9) is minimized when $T_A = n$ and the right hand side is maximized when $T_B = n - 2\delta n$. Consequently, it must hold that

$$n(n - \delta n) - 3(1 - \delta)n^2 < 2m - (n - 2\delta n)(n - 3\delta n) - (3n(1 - \delta)(n - 2\delta n)) \quad (10)$$

Note that we proved that in G , there is no triangle with three vertices of degree at least $(1 - \delta)n + 2$. Since G has minimum degree at least $(1 - \delta)n$, this implies that there are at most $2\delta n$ vertices of degree at least $(1 - \delta)n + 2$. Therefore we have $2m \leq (2\delta n)n + ((1 - 2\delta)n)((1 - \delta)n + 1)$

We get that $1 - 11\delta + 10\delta^2 < \frac{1-2\delta}{n}$. Since $\delta < \frac{1}{10}$, if n is large enough this leads to a contradiction. Since there exists a flow of value M in \widehat{G} , the weights of the triangles can be redistributed in a way that the triangles form a fractional decomposition of G . This finishes the proof of Theorem 5.

3 Conclusion

In this paper we proved that for all $\epsilon > 0$, there exists a constant n_0 such that every graph on $n \geq n_0$ vertices with minimum degree at least $(\frac{9}{10} + \epsilon)n$ is fractionally triangle decomposable. This implies that for all $\epsilon > 0$, there exists a constant n_0 such that every triangle divisible graph on $n \geq n_0$ vertices with minimum degree at least $(\frac{9}{10} + \epsilon)n$ is triangle decomposable.

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