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Bubblesort, stacksort and their duals

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ABSTRACT

Let B and S be, respectively, the base steps of *bubblesort* and *stacksort*, and call $\tilde{B}$ and $\tilde{S}$ their dual versions via the reverse-complement map. We find some unexpected commutation properties between the classical operators and their duals, and we also prove that the set of permutations sortable by a prescribed number of iterations of B and $\tilde{B}$ is a pattern class.

1. Introduction

Bubblesort is one of the most well-known elementary sorting algorithms. In a single step B of bubblesort, a permutation is scanned from left to right, and two consecutive elements are interchanged if the smaller follows the greater.

Another very simple sorting algorithm makes use of a stack. In a single step of this algorithm, the input permutation is scanned from left to right, and each element σ_i of the permutation is compared with the element τ on the top of the stack: if $\sigma_i < \tau$ (or if the stack is empty), then σ_i is pushed to the top of the stack; otherwise, τ moves to the rightmost position of the output permutation. Even for this algorithm, many single passes S are typically required before the sorting is complete.

As one might expect, if we mix together some steps of an algorithm with some steps of a completely different one, like B and S, the action of the resulting hybrid algorithm depends, in general, on the order we have used to perform the different steps. Despite this, denoting by $\tilde{B}$ and $\tilde{S}$ the dual versions of B and S via the reverse-complement map, we will prove that – quite surprisingly – they commute with the classical B and S. More precisely, the output of an algorithm consisting of some steps S and some steps $\tilde{B}$ depends only on the number of steps of each type, and not on their relative order. The same holds for B and $\tilde{B}$ together, and for $\tilde{S}$ and B, as stated in Theorem 4.8.

The analysis of sorting algorithms, and especially of stack-sorting ones, has often been related to the study of permutation patterns. Knuth [5] observed for the first time that the set of permutations sortable by one pass of stacksort is a pattern class. After this, many results have been found on different variations and generalizations of the original problem (see the 2003 survey by Bóna [4] for further details). Very recently, many interesting results about pattern classes related to bubblesort have been found by Albert et al. [2] and by Barnabei et al. [3]. At the end of this paper, as a generalization of some of the results contained in [2,3], we prove that the set of permutations sortable by a hybrid algorithm consisting of some passes of B and some of its dual $\tilde{B}$ is the class of permutations avoiding a set of inflations of 21.

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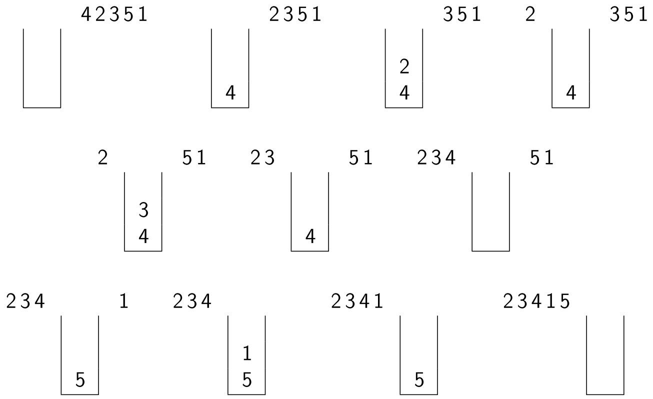

Fig. 1. The action of S on the input permutation $\sigma = 42\,351$.

2. Bubblesort and stacksort

In a single step B of bubblesort, a permutation σ is scanned from left to right, and two consecutive elements σ_i and σ_{i+1} are interchanged if $\sigma_i > \sigma_{i+1}$. For instance, if $\sigma = 26314875$ then $B(\sigma) = 23146758$. It is also possible to give a recursive definition for the operator B: if $\sigma = \alpha n \beta$, where n is the greatest element of σ, then

$$B(\alpha n \beta) = B(\alpha)\beta n.$$

In order to sort a permutation using a stack, one of the possible definitions of a single step S of the algorithm (that we will call *stacksort* for simplicity) is the following. First of all, the first element σ_1 is pushed into the stack. Then, scanning the input permutation σ from left to right, we compare σ_i with the element τ on the top of the stack: if $\sigma_i < \tau$ (or if the stack is empty), then σ_i is pushed to the top of the stack; otherwise, τ is popped out and put at the rightmost position of the output permutation $S(\sigma)$. At the end, all the elements still lying into the stack are popped out from top to bottom. See Fig. 1 for an example.

In order to describe the action of S in progress we can also use the notation

$$\textit{output}\ \langle \textit{stack}]\ \textit{input},$$

where $\langle$ is the top of the stack and $]$ the bottom. For instance,

$$\begin{aligned} S(26314875) &= \langle 2]\,6314875 = 2\,\langle 6]\,314875 = 2\,\langle 36]\,14\,875 = 2\,\langle 136]\,4875 \\ &= 213\,\langle 46]\,875 = 21346\,\langle 8]\,75 = 21346\,\langle 78]\,5 = 21346\,\langle 578] = 21346578. \end{aligned}$$

According to our notation, if we write $x\,\langle y]$ (with empty input) we refer unambiguously to the last passage of S, just before the final emptying of the stack. Even for stacksort, it is possible to give a recursive definition of the operator S: if $\sigma = \alpha n \beta$, then

$$S(\alpha n \beta) = S(\alpha)S(\beta)n.$$

We can give equivalent definitions of B and S also referring to the local left-to-right maxima. We recall that an element of a permutation σ is a *left-to-right maximum* if it is greater than all the elements to its left. Hence, the set of all left-to-right maxima of σ is

$$\mathrm{Max}_{lr}(\sigma) = \{\sigma_i : \sigma_i > \sigma_j, \forall j < i\}.$$

We can write σ highlighting its left-to-right maxima $M_1, M_2, \ldots, M_k$, where $M_k = n$, in the following way:

$$\sigma = M_1\, u_1\, M_2\, u_2 \cdots M_k\, u_k = (M_\alpha\, u_\alpha)_\alpha,$$

where u_α are (possibly empty) words. Hence,

$$B((M_\alpha\, u_\alpha)_\alpha) = (u_\alpha\, M_\alpha)_\alpha \tag{1}$$

and

$$S((M_\alpha\, u_\alpha)_\alpha) = (S(u_\alpha)\, M_\alpha)_\alpha. \tag{2}$$

3. Dual sorting operators

A variant of the classic bubblesort operator is the *dual bubblesort operator* $\tilde{B}$, which reads the permutation from right to left and interchanges two consecutive elements σ_i and σ_{i+1} if $\sigma_i > \sigma_{i+1}$. Knuth ([6], Section 5.2.2) introduces $\tilde{B}$ in the definition of a slightly more efficient version of bubblesort, the so called *cocktail-shaker sort*: in this algorithm, steps B and $\tilde{B}$ are alternated, and this reduces the average number of comparisons. Very recently, the cocktail-shaker sort and other combinations of the operators B and $\tilde{B}$ have been studied by Barnabei et al. [3].

Observe that, if $\sigma = \alpha 1 \beta$, then

$$\tilde{B}(\alpha 1 \beta) = 1 \alpha \tilde{B}(\beta).$$

We remark that, denoting by ρ the usual reverse-complement operator

$$\rho(\sigma)_i = n + 1 - \sigma_{n+1-i},$$

we have

$$\tilde{B} = \rho \circ B \circ \rho. \tag{3}$$

Similarly, we can define the *dual stacksort operator* as

$$\tilde{S} = \rho \circ S \circ \rho. \tag{4}$$

In other words, the operator $\tilde{S}$ reads the input permutation from right to left and compares σ_i with the element τ on the top of the stack: if $\sigma_i > \tau$ (or if the stack is empty), then σ_i is pushed to the top of the stack; otherwise, τ is popped out and put at the leftmost position of the output permutation $\tilde{S}(\sigma)$. At the end, as for S, all the elements still lying into the stack are popped out from top to bottom. We can visualize the action of $\tilde{S}$ in progress using the notation

$$\textit{input}\ [\textit{stack}\rangle\ \textit{output}.$$

We can give equivalent definitions of $\tilde{B}$ and $\tilde{S}$ referring to the local right-to-left minima. An element of σ is a *right-to-left minimum* if it is smaller than all the elements to its right:

$$\mathrm{Min}_{rl}(\sigma) = \{\sigma_i : \sigma_i < \sigma_j,\ \forall j > i\}.$$

Observe that

$$\rho(\mathrm{Max}_{lr}(\sigma)) = \mathrm{Min}_{rl}(\rho(\sigma)) \quad \text{and} \quad \rho(\mathrm{Min}_{rl}(\sigma)) = \mathrm{Max}_{lr}(\rho(\sigma)). \tag{5}$$

Writing σ as

$$\sigma = v_h\, m_h \cdots v_2\, m_2\, v_1\, m_1 = (v_\beta\, m_\beta)_\beta,$$

it follows that

$$\tilde{B}((v_\beta\, m_\beta)_\beta) = (m_\beta\, v_\beta)_\beta \tag{6}$$

and

$$\tilde{S}((v_\beta\, m_\beta)_\beta) = (m_\beta\, \tilde{S}(v_\beta))_\beta. \tag{7}$$

4. Commutation results

From now on, all inequalities involving sequences are intended to hold for all the elements belonging to the sequences: for example, $\alpha < \beta$ means that every element of α is smaller than every element of β.

In order to prove that some of the operators defined above commute, we first give some simple results.

Remark 4.1. If $\mathcal{M}$ is both a left-to-right maximum and a right-to-left minimum of a permutation $\sigma = \alpha \mathcal{M} \beta$ (α and β possibly empty), then

(i) $\alpha < \mathcal{M} < \beta$;
(ii) $\mathcal{M}$ lies in the $\mathcal{M}$-th position of σ;
(iii) $\mathcal{M}$ immediately precedes a left-to-right maximum M;
(iv) $\mathcal{M}$ immediately follows a right-to-left minimum m;
(v) the following relations hold:

$$B(\alpha \mathcal{M} \beta) = B(\alpha)\, \mathcal{M}\, B(\beta), \qquad \tilde{B}(\alpha \mathcal{M} \beta) = \tilde{B}(\alpha)\, \mathcal{M}\, \tilde{B}(\beta)$$
$$S(\alpha \mathcal{M} \beta) = S(\alpha)\, \mathcal{M}\, S(\beta), \qquad \tilde{S}(\alpha \mathcal{M} \beta) = \tilde{S}(\alpha)\, \mathcal{M}\, \tilde{S}(\beta).$$

Proof. Statements (i)–(iv) are straightforward. Concerning (v), observe that σ can be written as $\sigma = \alpha' m \mathcal{M} M \beta'$, and hence the application of B, S and their duals – see (1), (2), (6) and (7) – does not affect the relative order between α, $\mathcal{M}$ and β. $\square$

Lemma 4.2. *For every permutation σ the following relations hold:*

(i) $\mathrm{Max}_{lr}(\sigma) \subset \mathrm{Max}_{lr}(B(\sigma))$ *and* $\mathrm{Min}_{rl}(\sigma) \subset \mathrm{Min}_{rl}(B(\sigma))$;
(ii) $\mathrm{Max}_{lr}(\sigma) \subset \mathrm{Max}_{lr}(S(\sigma))$ *and* $\mathrm{Min}_{rl}(\sigma) \subset \mathrm{Min}_{rl}(S(\sigma))$;
(iii) $\mathrm{Max}_{lr}(\sigma) \subset \mathrm{Max}_{lr}(\tilde{B}(\sigma))$ *and* $\mathrm{Min}_{rl}(\sigma) \subset \mathrm{Min}_{rl}(\tilde{B}(\sigma))$;
(iv) $\mathrm{Max}_{lr}(\sigma) \subset \mathrm{Max}_{lr}(\tilde{S}(\sigma))$ *and* $\mathrm{Min}_{rl}(\sigma) \subset \mathrm{Min}_{rl}(\tilde{S}(\sigma))$.

Proof. (i) Let $\sigma = (M_\alpha\, u_\alpha)_\alpha$, whence $B(\sigma) = (u_\alpha\, M_\alpha)_\alpha$. Every $M_i \in \mathrm{Max}_{lr}(\sigma)$ is preceded, in $B(\sigma)$, by the same elements as in σ and by $u_i < M_i$: this yields $M_i \in \mathrm{Max}_{lr}(B(\sigma))$.

Let now $m \in \mathrm{Min}_{rl}(\sigma)$. If $m \in u_i$ for some i, in $B(\sigma)$ it is followed by the same elements as in σ and by $M_i > m$; hence, $m \in \mathrm{Min}_{rl}(\sigma)$ if and only if $m \in \mathrm{Min}_{rl}(B(\sigma))$. Otherwise, if $m = M_i$, then the elements following m in $B(\sigma)$ follow m also in σ, and thus $m \in \mathrm{Min}_{rl}(B(\sigma))$.

(ii) Let $\sigma = (M_\alpha\, u_\alpha)_\alpha$ and $S(\sigma) = (S(u_\alpha)\, M_\alpha)_\alpha$. Every $M_i \in \mathrm{Max}_{lr}(\sigma)$ is preceded, in $S(\sigma)$, by the same elements as in σ and by $S(u_i) < M_i$, and hence $M_i \in \mathrm{Max}_{lr}(S(\sigma))$. Now, let $m \in \mathrm{Min}_{rl}(\sigma)$. Immediately before we push m into the stack, the smaller elements inside are popped out and, immediately after, also m is popped out by the following element, which is greater. After this, the elements in the stack and those that are waiting to enter are all greater than m, and hence $m \in \mathrm{Min}_{rl}(S(\sigma))$.

The proofs of (iii) and (iv) are straightforward by using (i), (ii) and relations (5). $\square$

Example 4.3. We check (i) and (ii) taking $\sigma = 315269784$. Observe that $\mathrm{Max}_{lr}(\sigma) = \{3, 5, 6, 9\}$ and $\mathrm{Min}_{rl}(\sigma) = \{1, 2, 4\}$. Since $B(\sigma) = 132567849$ and $S(\sigma) = 132567489$, we have that $\mathrm{Max}_{lr}(\sigma) \subset \mathrm{Max}_{lr}(B(\sigma)) = \{1, 3, 5, 6, 7, 8, 9\}$ and $\mathrm{Max}_{lr}(\sigma) \subset \mathrm{Max}_{lr}(S(\sigma)) = \{1, 3, 5, 6, 7, 8, 9\}$, $\mathrm{Min}_{rl}(\sigma) \subset \mathrm{Min}_{rl}(B(\sigma)) = \{1, 2, 4, 9\}$ and $\mathrm{Min}_{rl}(\sigma) \subset \mathrm{Min}_{rl}(S(\sigma)) = \{1, 2, 4, 8, 9\}$.

When bubblesort (or stacksort) acts on σ, some new right-to-left minima may arise. In the following lemma we prove that each of them lies immediately to the right of a (new or old) right-to-left minimum; of course, a similar (reversed) result holds for the dual operators $\tilde{B}$ and $\tilde{S}$.

Lemma 4.4. *For every permutation σ the following relations hold:*

(i) *in $B(\sigma)$, every $m \in \mathrm{Min}_{rl}(B(\sigma)) \setminus \mathrm{Min}_{rl}(\sigma)$ immediately follows an element $m' \in \mathrm{Min}_{rl}(B(\sigma))$;*
(ii) *in $S(\sigma)$, every $m \in \mathrm{Min}_{rl}(S(\sigma)) \setminus \mathrm{Min}_{rl}(\sigma)$ immediately follows an element $m' \in \mathrm{Min}_{rl}(S(\sigma))$;*
(iii) *in $\tilde{B}(\sigma)$, every $M \in \mathrm{Max}_{lr}(\tilde{B}(\sigma)) \setminus \mathrm{Max}_{lr}(\sigma)$ immediately precedes an element $M' \in \mathrm{Max}_{lr}(\tilde{B}(\sigma))$;*
(iv) *in $\tilde{S}(\sigma)$, every $M \in \mathrm{Max}_{lr}(\tilde{S}(\sigma)) \setminus \mathrm{Max}_{lr}(\sigma)$ immediately precedes an element $M' \in \mathrm{Max}_{lr}(\tilde{S}(\sigma))$.*

Proof. (i) Let $\sigma = (M_\alpha\, u_\alpha)_\alpha$ and $B(\sigma) = (u_\alpha\, M_\alpha)_\alpha$. The considerations made in the proof of case (i) of Lemma 4.2 imply that, if $m \in \mathrm{Min}_{rl}(B(\sigma)) \setminus \mathrm{Min}_{rl}(\sigma)$, then necessarily $m \in \mathrm{Max}_{lr}(\sigma)$ and thus, by Lemma 4.2, $m \in \mathrm{Max}_{lr}(B(\sigma))$. Hence, m is both a left-to-right maximum and a right-to-left minimum of $B(\sigma)$ and then (see Remark 4.1) it is preceded by a right-to-left minimum.

(ii) We focus on two consecutive right-to-left minima m_{i+1} and m_i of σ, where $m_{i+1} < m_i$. Thus, we can write $\sigma = u\, m_{i+1}\, v\, m_i\, w$, where u, v and w are (possibly empty) words, with $v, w > m_i$. After stacksort, every possible "new" right-to-left minimum between m_{i+1} and m_i, i.e. every possible $m \in \mathrm{Min}_{rl}(S(\sigma)) \setminus \mathrm{Min}_{rl}(\sigma)$, with $m_{i+1} < m < m_i$, must belong to u and must be popped out of the stack after m_{i+1} and before m_i. This holds if and only if, at the time m_{i+1} is pushed into the stack, there is a nonempty set $u' \subset u$, $m_{i+1} < u' < m_i$. In this case, m_{i+1} and all the elements of u' are popped out in increasing order when the first element of v is pushed (or when m_i is, if $v = \varnothing$), and they become all (consecutive) right-to-left minima of $S(\sigma)$.

The proofs of (iii) and (iv) are straightforward by using (i), (ii) and relations (5). $\square$

Example 4.5. We check (i) and (ii) taking $\sigma = 236451987$. The set of right-to-left minima of σ is $\mathrm{Min}_{rl}(\sigma) = \{1, 7\}$. Moreover, $B(\sigma) = 234516879$ and $S(\sigma) = 234156789$, and hence $\mathrm{Min}_{rl}(B(\sigma)) = \{1, 6, 7, 9\}$ and $\mathrm{Min}_{rl}(S(\sigma)) = \{1, 5, 6, 7, 8, 9\}$. Now, it is easy to check that the "new" right-to-left minima, in both cases (6 and 9 in $B(\sigma)$ and 5, 6, 8, and 9 in $S(\sigma)$), immediately follow a right-to-left minimum of $B(\sigma)$ and $S(\sigma)$, respectively.

We now introduce the notion of *local sorting operator*, which will be useful in the proof of the next theorem. Let $\sigma = (M_\alpha\, u_\alpha)_\alpha = (v_\beta\, m_\beta)_\beta$, where M_i and m_j are the left-to-right maxima and right-to-left minima of σ, respectively, and define

$$B_{M_i}(\sigma) = M_1\, u_1 \cdots M_{i-1}\, u_{i-1}\, u_i\, M_i\, M_{i+1}\, u_{i+1} \cdots M_k\, u_k$$

and

$$\tilde{B}_{m_j}(\sigma) = v_h\, m_h \cdots v_{j+1}\, m_{j+1}\, m_j\, v_j\, v_{j-1}\, m_{j-1} \cdots v_1\, m_1.$$

Obviously,

$$B(\sigma) = B_{M_1} \circ B_{M_2} \circ \cdots \circ B_{M_k}(\sigma) \tag{8}$$

and

$$\tilde{B}(\sigma) = \tilde{B}_{m_1} \circ \tilde{B}_{m_2} \circ \cdots \circ \tilde{B}_{m_h}(\sigma). \tag{9}$$

Remark 4.6. The dependence of (8) and (9) on the local maxima and minima might cause a little trouble if we do not have any information about them. For example, if we want to rewrite $B \circ \tilde{B}(\sigma)$ with the above notations, we have to know both the right-to-left minima of σ and the left-to-right maxima of $\tilde{B}(\sigma)$. Despite this, every new left-to-right maximum M of $\tilde{B}(\sigma)$ is immediately to the left of another left-to-right maximum (see Lemma 4.4), and hence its corresponding local bubblesort B_M does not perform any interchange ($B_M = id$). More generally:

- $M \in \text{Max}_{lr}(\sigma)$ immediately precedes $M' \in \text{Max}_{lr}(\sigma) \Longrightarrow B_M = id$;
- $m \in \text{Min}_{rl}(\sigma)$ immediately follows $m' \in \text{Min}_{rl}(\sigma) \Longrightarrow \tilde{B}_m = id$.

Therefore, in order to rewrite $B \circ \tilde{B}$ or $\tilde{B} \circ B$ via the local sorting operators, we only need to know $\text{Max}_{lr}(\sigma)$ and $\text{Min}_{rl}(\sigma)$. This yields the following lemma.

Lemma 4.7. *Let $\sigma = (M_\alpha\, u_\alpha)_\alpha = (v_\beta\, m_\beta)_\beta$. Then*

$$\begin{aligned} B \circ \tilde{B} = \tilde{B} \circ B \quad &\Longleftrightarrow \quad B_{M_i} \circ \tilde{B}_{m_j} = \tilde{B}_{m_j} \circ B_{M_i} \qquad \forall\, i,\, j\,; \\ S \circ \tilde{B} = \tilde{B} \circ S \quad &\Longleftrightarrow \quad S \circ \tilde{B}_{m_j} = \tilde{B}_{m_j} \circ S \qquad \forall\, j. \end{aligned}$$

Theorem 4.8. *The following commutation properties hold:*

(i) $B \circ \tilde{B} = \tilde{B} \circ B$;
(ii) $S \circ \tilde{B} = \tilde{B} \circ S$ *(and hence* $\tilde{S} \circ B = B \circ \tilde{S}$*).*

Proof. (i) Let $\sigma = (M_\alpha\, u_\alpha)_\alpha = (v_\beta\, m_\beta)_\beta$, and choose M_i and m_j. By Lemma 4.7, it is sufficient to prove the commutativity of the local sorting operators B_{M_i} and $\tilde{B}_{m_j}$.

If $M_i = m_j$, $M_i = m_{j+1}$ or $M_{i+1} = m_j$, then at least one of M_i and m_j is simultaneously a left-to-right maximum and a right-to-left minimum, and then (see Remarks 4.1 and 4.6) either B_{M_i} or $\tilde{B}_{m_j}$ is the identity map.

Now, suppose that $M_i \neq m_j$, $M_i \neq m_{j+1}$ and $M_{i+1} \neq m_j$. If $M_i\, u_i \cap v_j\, m_j = \varnothing$, or $M_i\, u_i \subset v_j$, or even $v_j\, m_j \subset u_i$, the commutative property follows immediately from the fact that the interchanges operated by B_{M_i} and $\tilde{B}_{m_j}$ do not cross each other. In the only two remaining cases, namely, $M_i\, w_1\, m_{j+1}\, w_2\, M_{i+1}\, w_3\, m_j$ and $m_{j+1}\, w_1\, M_i\, w_2\, m_j\, w_3\, M_{i+1}$, where the w_k are (possibly empty) words, the commutativity can be directly checked.

(ii) Choosing a right-to-left minimum m_j, by Lemma 4.7 it is sufficient to prove that S and $\tilde{B}_{m_j}$ commute. Let $\sigma = u\, v_j\, m_j\, w$, where $u = v_h\, m_h \cdots v_{j+1}\, m_{j+1}$ and $w = v_{j-1}\, m_{j-1} \cdots v_1\, m_1$. Obviously, $v_j > m_j$ and $w > m_j$.

Let $S(u) = u'\, \langle u''\, u'''\rbrack$, where $u'' < m_j$ and $u''' > m_j$, and let t', t'' and q be the sequences such that $\langle u'''\rbrack\, v_j = t'\, \langle t''\rbrack$ and $\langle t''\rbrack\, w = q$. Now, we have

$$\begin{aligned} S \circ \tilde{B}_{m_j}(\sigma) &= S \circ \tilde{B}_{m_j}(u\, v_j\, m_j\, w) = S(u\, m_j\, v_j\, w) = u'\, \langle u''\, u'''\rbrack\, m_j\, v_j\, w \\ &= u'u''\, \langle m_j\, u'''\rbrack\, v_j\, w = u'u''m_j\, t'\, \langle t''\rbrack\, w = u'u''m_j\, t'\, q \end{aligned}$$

and

$$\begin{aligned} \tilde{B}_{m_j} \circ S(\sigma) &= \tilde{B}_{m_j} \circ S(u\, v_j\, m_j\, w) = \tilde{B}_{m_j}(u'\, \langle u''\, u'''\rbrack\, v_j\, m_j\, w) \\ &= \tilde{B}_{m_j}(u'u''\, t'\, \langle t''\rbrack\, m_j\, w) = \tilde{B}_{m_j}(u'u''\, t'\, \langle m_j\, t''\rbrack\, w) \\ &= \tilde{B}_{m_j}(u'u''\, t'\, m_j\, q) = u'u''m_j\, t'\, q. \end{aligned}$$

Hence, $S \circ \tilde{B} = \tilde{B} \circ S$, and then, using (3) and (4), we get $\tilde{S} \circ B = B \circ \tilde{S}$. $\square$

Example 4.9. We show each passage of the proof of (ii) by taking $\sigma = 218\,356\,497$. Among the right-to-left minima of σ, we choose m_2. Then $\sigma = uv_2m_2w$, where $u = 2183$, $v_2 = 56$, $m_2 = 4$ and $w = 97$. Following the same notations used in the proof, we have that $S(u) = S(2183) = 12\, \langle 38\rbrack$, whence $u' = 12$, $u'' = 3$ and $u''' = 8$. Moreover, $\langle u'''\rbrack\, v_2 = \langle 8\rbrack\, 56 = 5\, \langle 68\rbrack$ (where $t' = 5$ and $t'' = 68$), and $\langle t''\rbrack\, w = \langle 68\rbrack\, 97 = 6879 = q$. Now, we have that

$$\begin{aligned} S \circ \tilde{B}_{m_2}(\sigma) &= S \circ \tilde{B}_4(218356497) = S(218345697) \\ &= 123\, \langle 48\rbrack\, 5697 = 12345\, \langle 68\rbrack\, 97 = 123456879 \end{aligned}$$

and

$$\begin{aligned}\tilde{B}_{m_2} \circ S(\sigma) &= \tilde{B}_4 \circ S(218356497) = \tilde{B}_4(12\,\langle 38]\,56497) = \tilde{B}_4(1235\,\langle 68]\,497)\\ &= \tilde{B}_4(1235\,\langle 468]\,97) = \tilde{B}_4(123546879) = 123456879.\end{aligned}$$

Remark 4.10. We can rewrite the commutation properties stated in Theorem 4.8 by using ρ:

(i) $(B \circ \rho)^2 = (\rho \circ B)^2$;
(ii) $S \circ \rho \circ B \circ \rho = \rho \circ B \circ \rho \circ S$ (and hence $\rho \circ S \circ \rho \circ B = B \circ \rho \circ S \circ \rho$).

Proposition 4.11. *All the other possible pairs of operators, listed below, do not commute:*

(i) $B \circ S \neq S \circ B$ *(and hence* $\tilde{B} \circ \tilde{S} \neq \tilde{S} \circ \tilde{B}$*);*
(ii) $S \circ \tilde{S} \neq \tilde{S} \circ S$.

Proof. (i) Let $\sigma = 2431$: we have $B \circ S(\sigma) = 1234$, while $S \circ B(\sigma) = 2134$.
(ii) Taking $\sigma = 3421$, we have $S \circ \tilde{S}(\sigma) = 1234$, while $\tilde{S} \circ S(\sigma) = 1324$. $\square$

5. Sorting algorithms and pattern avoidance

A permutation σ is said to *avoid* a pattern τ if it does not contain any subsequence which is order isomorphic to τ. In this case, we write $\tau \not\leq \sigma$. For example, the permutation $\sigma = 5162374$ contains $\tau_1 = 3412$ (which is order isomorphic to 5623), but avoids $\tau_2 = 2431$.

A *pattern class with basis* Π is the set of permutations avoiding all the patterns $\pi \in \Pi$:

$$Av(\Pi) = \{\sigma : \pi \not\leq \sigma, \forall \pi \in \Pi\}.$$

The relationships between sorting algorithms and pattern-avoiding permutations have been studied by many authors: we briefly recall only some of the results involving bubblesort and stacksort. If A is any sorting operator, we denote by

$$Sort(A) = \{\sigma \mid A(\sigma) = 12\cdots n\}$$

the set of permutations sortable by A. Knuth [5] observed that

$$Sort(S) = Av(231).$$

In recent years, Albert et al. [2] found that

$$Sort(B) = Av(231, 321) \tag{10}$$

and

$$Sort(S \circ B) = Av(2341, 2431, 3241, 4231). \tag{11}$$

Moreover, as a generalization of (10), the authors proved that the permutations sortable by h passes of B are exactly those that avoid all the patterns of length $h+2$ whose final term is 1. In other words,

$$Sort(B^h) = Av(\Gamma_{h+2}), \tag{12}$$

where

$$\Gamma_{h+2} = \{\tau \in S_{h+2} \mid \tau_{h+2} = 1\}.$$

Very recently, Barnabei et al. [3] showed that

$$Sort(\tilde{B} \circ B) = Av(3412, 3421, 4312, 4321). \tag{13}$$

Now, by making use of the commutation properties proved in Theorem 4.8, we are able to prove that the permutations sortable by a fixed number of passes of B and $\tilde{B}$ can be expressed in terms of pattern-avoiding permutations. In order to describe the patterns involved, we need the following definition.

Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ and n permutations $\alpha_1, \alpha_2, \ldots, \alpha_n$, the *inflation* of σ by $\alpha_1, \alpha_2, \ldots, \alpha_n$ (denoted by $\sigma[\alpha_1, \alpha_2, \ldots, \alpha_n]$) is the permutation $\Sigma \in S_\ell$ (where $\ell = \sum_{i=1}^n |\alpha_i|$) which is obtained by replacing each element σ_i with a permutation τ_i such that:

- τ_i is order isomorphic to α_i;
- $\tau_i < \tau_j \iff \sigma_i < \sigma_j, \forall i, j = 1, \ldots, n$.

For example, $312\,[21, 213, 132] = 87213465$. We denote by

$$\sigma\, [\![\ell_1, \ldots, \ell_n]\!] = \{\sigma\,[\alpha_1, \ldots, \alpha_n] : |\alpha_i| = \ell_i,\ \forall\, i = 1, \ldots, n\}$$

the set of all possible inflations of σ by n permutations $\alpha_1, \alpha_2, \ldots, \alpha_n$ of fixed lengths $\ell_1, \ell_2, \ldots, \ell_n$. For instance, $231\,[\![2, 1, 2]\!] = \{34512, 34521, 43512, 43521\}$.

Observe that the result (12) can be described in terms of inflations as follows:

$$Sort(B^h) = Av(21\,[\![h + 1, 1]\!]). \tag{14}$$

This kind of pattern classes have also been studied in [1], where the authors give some very interesting results on classes of permutations avoiding particular inflations of 21. In the following, we will show that even the permutations that can be sorted by a fixed number of B and $\tilde{B}$ can be characterized using the same kind of pattern classes.

Lemma 5.1. *For every $h, k \geq 1$ the following equivalences hold:*

$$\begin{tikzcd}
 & \sigma \in Av(21\,[\![h+1, k+1]\!]) \arrow[dl, Leftrightarrow, "(1)"'] \arrow[dr, Leftrightarrow, "(2)"] & \\
B(\sigma) \in Av(21\,[\![h, k+1]\!]) \arrow[rr, Leftrightarrow, "(3)"'] & & \tilde{B}(\sigma) \in Av(21\,[\![h+1, k]\!])
\end{tikzcd}$$

Proof. We prove only equivalence (1), since (2) can be proved analogously and (3) follows from (1) and (2). For convenience, we will equivalently show that σ contains a pattern $\tau \in 21\,[\![h+1, k+1]\!]$ if and only if $B(\sigma)$ contains $\tau' \in 21\,[\![h, k+1]\!]$.

Let $\sigma = (M_\alpha\, u_\alpha)_\alpha$ contain $\tau \in 21\,[\![h+1, k+1]\!]$. Observe that the last $k+1$ elements of τ are not left-to-right maxima of σ, and thus they belong to some of the u_α. Hence, in $B(\sigma) = (u_\alpha\, M_\alpha)_\alpha$ their relative order is preserved, and the first of them (i.e. $\tau_{h+2} \in u_i$) follows the same elements than in σ, except for M_i. This implies that $B(\sigma)$ contains a subsequence $\tau' \in 21\,[\![h, k+1]\!]$. The proof of the converse is analogous. □

Theorem 5.2. *The set of permutations sortable by h steps of B and k steps of $\tilde{B}$ ($h, k \geq 0$), performed in any order, is the set*

$$Sort(B^h \circ \tilde{B}^k) = Av(21\,[\![h+1, k+1]\!]).$$

Proof. Applying Lemma 5.1 h times for B and k for $\tilde{B}$, we obtain that

$$\sigma \in Av(21\,[\![h+1, k+1]\!]) \iff B^h \circ \tilde{B}^k(\sigma) \in Av(21) = \{id\}. \quad \square$$

6. Conclusions and open problems

In this paper, we proved some commutation properties between the sorting operators B, S and their dual versions via the reverse-complement operator. As a generalization of (13) and (14), we also found a pattern class description of the permutations sortable by a fixed number of iterations of B and $\tilde{B}$.

Beyond this, many questions remain open: we list below just a few of them.

(1) We proved that B commutes with $\tilde{B}$, while S does not commute with $\tilde{S}$. Hence, it may be interesting to find other sorting operators which commute with their dual versions, and, if possible, to characterize them against those which do not.

(2) We did not describe the permutations sortable by h passes of B (or $\tilde{B}$) mixed with k passes of S (or $\tilde{S}$) in terms of pattern-avoiding permutations. The only result involving these operators, for $h = k = 1$ and S performed after B, is given by Albert et al. [2] (see (11)). Is it possible to give such a characterization for the other cases? And what about just S and $\tilde{S}$?

(3) For which other sets of permutations Π there exists an operator A_Π such that

$$Sort(A_\Pi) = Av(\Pi)?$$

(4) Let A be any sorting operator. Denote by

$$\ell_A = \min\{|\tau| : A(\tau) \neq id\}$$

the shortest length of A-unsortable permutations, and let

$$\Pi_A = \{\pi : A(\pi) \neq id \wedge |\pi| = \ell_A\}.$$

Then, for which operators A we have

$$Sort(A) = Av(\Pi_A)? \tag{15}$$

Observe that for S and for every operator of the form $B^h \circ \tilde{B}^k$ the equality (15) holds, while, for instance, for S^2 does not. In fact, $\Pi_{S^2} = \{2341, 3241\}$ while $Sort(S^2) \neq Av(2341, 3241)$. More precisely, the permutations sortable by two passes of stacksort are $Av(2341, 3\bar{5}241)$, where $3\bar{5}241$ denotes the patterns 3241 which are not part of a pattern 35241 (see [7] for a proof and further details).

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