COMBINATORICS OF MAPPINGS
(Graph Homomorphisms and Their Use)

Jaroslav Nešetřil*

Abstract

This text was prepared for the Spring School in Combinatorics 2000. It is a follow up of a paper [61] which was based on a course delivered by the author at NCTS, National Chiao Tung University, Taiwan in Febuary 1999. We survey results related to structural aspects of graph homomorphism. As we want to demonstrate this forms today a compact collection of results and methods which perhaps deserve its name : structural combinatorics. Due to the space limitations we concentrate on a sample of areas only: representation of algebraic structures by combinatorial ones (graphs), the poset of colour classes and corresponding algorithmic questions which lead to homomorphism dualities as a blend of algebraic and complexity approach.

In the last year the development in this area was a rapid one and we can complement this survey by 4 more results (which play a key role in our presentation): characterization of concrete categories (due to Freyd and Vinárek), new construction of a rigid graph on every set [62], new combinatorial proof of universality of the poset of all color classes [63] and a theorem proved in [73] which perhaps provide a proper setting for Müller's theorem on uniquelly colorable graphs. While the former two results are inserted in the text, the later two results are a bit moire involved and are presented in the form of two appendices.

These 4 places are suggested topics for an (advanced) reading at SS. The reader will find other palces which are suitable for an independent lecture (say about good characterizations or about representations).

1 Introduction

Graph theory receives its mathematical motivation from the two main areas of mathematics: algebra and geometry (topology) and it is fair to say that the graph notions

*Partially supported by GAUK Grant 158, EU Socrates IP Grant

stood at the birth of algebraic topology. Consequently various operations and comparisions for graphs stress either its algebraic part (e.g. various products) or geometrical part (e.g. contraction, subdivision). It is only natural that the key place in the modern graph theory is played by (fortunate) mixtures of both approaches as exhibited best by the various modifications of the notion of graph minor. However from the algebraic point of view perhaps the most natural graph notion is the following notion of a homomorphism:

Given two graphs G and G' a *homomorphism* f of G to G' is any mapping $f : V(G) \to V(G')$ which satisfies the following condition :

$$[x, y] \in E(G) \text{ implies } [f(x), f(y)] \in E(G')$$

.

This condition should be understood as follows: on both sides of the implication one considers the same type of edges (undirected or directed). The analogous definitions give the notions of the homomorphisms for hypergraphs (set systems) and relational systems (of a given type; that will be specified later).

The existence of a homomorphism from G to H is denoted by $G \to H$, in this case we also say that *G is homomorphic to H*, the non-existence of such a homomorphism is denoted by $G \not\to H$ and in such a case we say that *G fails to be homomorphic to H*. If G is homomorphic to H and also H is homomorphic to G then we say that G and H are a *homomorphism equivalent* (or simply *hom-equivalent*) and we denote this by $G \sim H$.

The homomorphism is an algebraic notion which in graph theory founds its way to problems related to products, reconstruction and chromatic polynomials, just to name a few.

A combinatorial approach is motivated usually by the chromatic number connection expressed by the following observation which holds for every undirected graph G:

$$G \to K_k \text{ if and only if } \chi(G) \leq k.$$

An algebraical approach leads to *groups*,*monoids*, *posets* and *catogories*.

In categories, which is the most general of these concepts, we speak about *objects* (for example graphs), *morphisms* (which are for example homomorphisms) and *composition* (which is just the composition of mappings).

Abstractly, we can think of this situation as an oriented multigraph with labeled arcs together with "composition of arrows".

Fig 1

However this (categorical) setting stresses some particular features and gives rise to a new perspective. A spectacular example we want to introduce now:

Given two graphs G, H we denote by $Hom(G,H)$ the set of all homomorphisms from G to H; formally $Hom(G,H) = \{f; f : G \to H\}$ (sometimes the notation $< G,H >$ is used).

We also denote by $h(G,H)$ the cardinality of the set $Hom(G,H)$; formally, we put $h(G,H) = |Hom(G,H)|$. We also denote by $\mathbf{h}(H)$ (sometimes the notation $< H >$ is used) the infinite vector whose coordinates are indexed by finite graphs; we consider non isomorphic graphs only.More formally,

$$\mathbf{h}(H) = (h(G,H); G \text{ finite graph},)$$

.

This formal approach (which typically involves large sets and which in this setting relates to invariants like Tutte-Grothendieck polynomial and problems in the statistical physics) has been a remarkable success in many respect. In addition to recent applications to problems related to statistical physics, see e.g. [80], [9], [6], we want to stress purely combinatorial problems. We want to motivate this by the following two results:

Theorem 1.1 (Lovász [51]) *For any two graphs G ,H holds :*

$$\mathbf{h}(G) = \mathbf{h}(H) \textit{ if and only if } G \cong H$$

.

It has been shown by Lovász [52] that Theorem 1 holds in most "combinatorial" categories covering particularly relation systems of arbitrary type.

Theorem 1.2 (Müller [56]) *If G is a graph with n vertices and $m > nlogn$ edges then G is edge reconstructive.*

Concerning Theorem 2 let us first recall the famous edge reconstruction conjecture (see e.g. survey by Nash - Williams [**?**]):

Conjecture 1 *For undirected graphs G, H with at least 4 edges the following two statements are equivalent*

(i) *G and H are isomorphic;*

(ii) *There exists a bijection $\iota : E(G) \to E(H)$ such that $G - e \cong H - \iota(e)$ for every $e \in E(G)$.*

This conjecture fails to be true for several (in fact 4) small graphs, so this is why we assume that G and H have as least 4 edges. Obviously (i) implies (ii) and thus the validity of the opposite implication is the core of the conjecture.

Now it is well known that the above condition (ii) is equivalent to the following condition (ii'):

There exists a bijection

$$\iota : \{A; A \subset E(G)\} \longrightarrow \{B; B \subset E(H)\}$$

such that $(V(G), A)$ is isomorphic to $(V(H), \iota(A))$ for every $A \subset E(G)$.

The edge reconstruction conjecture is related to Ulam's vertex reconstruction conjecture and Müller's result is one of the strongest results supporting its validity. As an illustration of the techniques we prove the following (less technical) results of Lovász [52] which motivated Müller's result.

Theorem 1.3.*([52]) Let G be a graph with n vertices and m edges, let $m > \binom{n}{2}/2$. Then G is edge-reconstructible.*

Proof. Let $G = (V, E)$, $H = (V, E')$ be graphs such that the condition (ii') holds:

Let $\iota : \{A, A \subset E\} \to \{B; B \subset E'\}$

be a bijection such that

$$(V, A) \cong (V, \iota(A)).$$

We shall need two more definition:

Given two graphs G_1, G_2 we denote by $i(G_1, G_2)$ the number of injective homomorphisms $f : G_1 \longrightarrow G_2$. $\overline{G}$ denotes the complement of graph G: $V(\overline{G}) = V(G)$,

$E(\overline{G}) = \binom{V(G)}{2} - E(G)$.

Using Inclusion-Exclusion Principle we can express the number of injective homomorphisms as follows:

$$(1) \qquad i(G, \overline{H}) = i((V, \emptyset), (V, \emptyset)) - \sum_{e_1 \in E} i((V, \{e_1\}), H) + \sum_{e_1 \neq e_2 \in E} i((V, \{e_1\, e_2\}), H) - \ldots + (-1)^{|E|} i(G, H).$$

Similarly

$$(2) \qquad i(H, \overline{H}) = i((V, \emptyset), (V, \emptyset)) - \sum_{e_1 \in E'} i((V, \{e_1\}), H) + \sum_{e_1 \neq e_2 \in E'} i((V, \{e_1\, e_2\}), H) - \ldots + (-1)^{|E'|} i(H, H).$$

Obviously $|E| = |E'|$ but also, according to our assumption (ii'), for every $1 \leq k \leq |E|$ the bijection ι associates to any set $\{e_1, \ldots, e_k\}$ a set $\iota(\{e_1, \ldots, e_k\}) = \{e'_1, \ldots, e'_k\}$ such that $(V, \{e_1, \ldots, e_k\}) \cong (V, \{e'_1, \ldots, e'_k\})$.

Thus the terms on the right side of expressions (2) and (3) are pairwise the same. Thus

$$(3) \qquad i(G, \overline{H}) - i(H, \overline{H}) = (-1)^{|E|} (i(G, H) - i(H, H)).$$

However the left side of (4) is zero as both G and H have too many edges. On the other side $inj(H, H)$ ¿ 0 and thus $inj(G, H) \neq 0$ too.

Let us sumarize : we proved that there exist injective homomorphism $G \longrightarrow H$.

Now we can repeat the same proof for pairs $inj(G, \overline{G})$ and $inj(H, \overline{G})$ and we obtain similarly $inj(H, G) \neq 0$.

But then, as our graphs are finite, we have $G \cong H$. (Alternatively, it suffices to prove $inj(G, H) \neq 0$ as we know that both G and H have the same number of edges. Thus any injective homomorphism, i.e. subgraph, is necessary isomorphism.) ♡

Perhaps this example of use of *combinatorics of maps* provides a good motivation for this paper where we want to introduce more examples. Due to the space limitations we have to concentrate on a few sample areas only. In Chapter 2 we deal with algebraic aspects of graph homomorphisms from the point of view of categories (of graphs and their homomorphisms) and posets (induced by the existence of homomorphism). Particularly we review the recent development related to the notion of density (and we give 3 proofs of the fundamental result for undirected graphs.

In Chapter 3 we survey complexity questions, both hard and polynomial instances of the basic decision problem (the existence of an H-colloring). We close with a characterization of finitary dualities which as an analogy for colorings of the Robertson-Seymour-Thomas program. We close the paper with yet another view relating the results to fundamental results (and insights) of P. Erdös.

The paper is organized as follows:

This paper is organized as follows:

Chapter 2: Ordering by Homomorphism (structure of color classes)

1. Categories and Representations
2. Concreteness
3. Universality
4. Independent families
5. Density and gaps

Chapter 3: Paradoxes of Complexity

1. Hard Cases
2. Polynomial cases and Homomorphism Dualities
3. Finitary Dualities
4. Gaps and Dualities
5. Final view

Appendix 1: On Extendability and Universality properties of $\mathcal{C}$.
Appendix 2: On Sparse Graphs with Given Homomorphisms

2 Ordering by Homomorphisms (Structure of Color Classes)

2.1 Categories and representations

Consider *all* finite graphs together with all homomorphisms between them. What we can say about a structure of such a situation?

This is certainly a complicated situation, on the first glance undescribally complicated. But there is a simple basic structure which underlies this situation and in fact this is a common structure to all similar situations. This underlying structure is called a *category*.

In order to define a category $\mathcal{K}$ we always specify *objects* (we denote them by capital letters A, B, C, ...) and *morphisms*.

Morphisms are labeled arrows denoted by $A \xrightarrow{f} B$ or $f : A \longrightarrow B$. What this means is that each morphism $f : A \longrightarrow B$ has specified two objects $d(f) = A$ and $r(f) = B$ (the *domain* and the *range* of f). Denote by $< A, B >$, or $Hom(A, B)$, the set of all morphisms f satisfying $d(f) = A$, $r(f) = B$ (more precisely we should write $< A, B >_{\mathcal{K}}$). As we work exclusively with finite objects (like finite graphs), we assume that for any pair of objects A, B the set $< A, B >$ is *finite* (and of course it may be empty).

Two more features describe our situation:

For every triple A, B, C of objects we have a mapping $\circ :< B, C > \times < A, B > \longrightarrow < A, C >$ which assigns to morphisms f, g their *composition* $f \circ g$ (this composition $f \circ g$ need not be a composition of maps as even f, g need not be mappings). We further assume that the operation $\circ$ (it is in fact a partial operation) is *associative*: $(f \circ g) \circ h = f \circ (g \circ h)$ whenever one of the sides of the equality is defined.

And finally we are given to very object A a morphism $\mathbf{1}_A$ which satisfies

$$\mathbf{1}_A \circ f = f \text{ and } g \circ \mathbf{1}_A = g$$

whenever the left hand side makes sense. $\mathbf{1}_A$ is called the *identity* on A.

This completes the description of our situation. If objects, morphisms, composition and unit objects are specified and the above minimal requirements ($< A, B >$ finite set, identity and associativity of $\circ$) then we say that we have an instance of a *category*. (We specified the notion of category in finite set theory, we make no attempts to generalize to infinity; so we have countably many objects and morphisms but between any two objects only finitely many morphisms. This a paper on finite combinatorics.)

Categories are abundand and so is the literature about them (we want to single out three books : MacLane classical modern [53], very elementary but rigorous [50] and [75] which is closest to our combinatorial setting).

Here are some examples:

SET = category of all finite sets and all mappings between them;

ORD = category of all finite linearly ordered sets and all monotone mappings between them (this is also called *simplicial category*);

GRA = category of all finite graphs and all their homomorphisms;

$(X, \leq)$ = the category induced by any partially ordered set : $x \longrightarrow y$ iff $x \leq y$; in this category $< x, y >$ consists from at most one morphism - such a category is called *thin*;

Any group $(X, \cdot, e)$ can be considered as a category with one object X only, morphisms $X \longrightarrow X$ are labeled by elements of group with composition defined as multiplication;

Any monoid $(X, \cdot, 1)$ can be treated similarly as a category with one object, (*monoid* is a semigroup with unite element).

As indicated by these examples, category theory is a *minimal calculus* common to most mathematical theories. It is a (rather schematic) world in which most mathematics (and mathematicians) live.

We mostly use categories and category theory to motivate, to formulate easily, general things which otherwise would be hard to describe. We usually do not use calculus of categories to prove a particular statement. But there are exceptions. Even in combinatorics there are exceptions and some of them we shall describe in this chapter. We need to compare categories. This is straightforward (by now; it

took some time before proper concepts were isolated):

Let $\mathcal{K}$, $\mathcal{L}$ be categories. A mapping F which maps

$$F : OBJECTS(\mathcal{K}) \longrightarrow OBJECTS(\mathcal{L})$$

$$F : MORPHISMS(\mathcal{K}) \longrightarrow MORPHISMS(\mathcal{L})$$

is called *functor* providing the following holds:

(i) $r(F(f)) = F(r(f))$

(ii) $d(F(f)) = F(d(f))$;

(iii) $F(f) \circ F(f) = F(f \circ g)$; (more exactly we should write $E(f) \circ_{\mathcal{L}} F(g) = F(f \circ_{\mathcal{K}} g)$)

(iv) $F(\mathbf{1}_A) = \mathbf{1}_{F(A)}$

for all morphisms f, g (providing that right hand side in (iii) is defined).
We write $F : \mathcal{K} \rightarrow \mathcal{L}$.

We say that a functor F is *faithful* providing it is 1-1 on every set $< A, B >$ of morphisms in $\mathcal{K}$. We say that a functor F is *embedding* providing F is 1-1 (both on $OBJECTS(\mathcal{K})$ and $MORPHISMS(\mathcal{K})$) and

$$\{F(f); f \in < A, B >\} = < F(A), F(B) >$$

Finally, we say that functor
$F : \mathcal{K} \longrightarrow \mathcal{L}$ is an *embedding of* $\mathcal{K}$ *into* $\mathcal{L}$ or that F is a *representation of* $\mathcal{K}$ *in* $\mathcal{L}$.

The notion of embedding or representation should capture the following results:

Theorem 2.1. *[[16]] Every group is isomorphic to the group of automorphisms of a graph.*

Theorem 2.2. *[[26]] Every monoid is isomorphic to the monoid endomorphisms of a graph.*

Theorem 2.3. *Every finite category can be represented by balanced directed graphs.*

All these theorems are true and in fact they are consequences of much more general approach which we shall now outline.

2.2 Concreteness - a combinatorial obstacle

The above results indicate that we are in a positive land. There are embeddings of categories in all possible directions. It seems that we can represent every category $\mathcal{K}$ by any other $\mathcal{L}$ if only $\mathcal{L}$ is sufficiently "non-trivial" (however there are exceptions, bounded degree, bounded genus, orientation, see [1], [2], [32]. The situation seems to be reminiscent to theory of NP-completeness where the initial joy(of being able to "narrow" $P - NP$ gap) turned to realistic scepticism (that virtualy "everything" seems to be NP-complete).

But for representation of categories there is one striking difference. There is a non-trivial necessary condition when a category is representable, say, by the category GRA of all finite graphs and all their homomorphisms. This is related to the following notion:

Definition 1 *A category K is said to be* concrete *if it is isomorphic to a subcategory of SET.*

Most categories ("from real life") are concrete as the morphism between their objects represent (special) mappings and one mapping usually does not correspond to two different morphisms between the same pairs of objects.

To be more precise, for example, GRA is a concrete category as every graph is determined by a set of vertices and edges and homomorphisms are just special mappings. When denoted properly this will be an inclusion.

But the situation is different and less easy if the morphisms in a category $\mathcal{K}$ are "abstract arrows". Then we have to find sets to represent objects and assign to morphisms mappings between corresponding sets. This is the case for example with any poset $(X, \leq)$: If we view $(X, \leq)$ as a category (this we did above; it was a thin category) then what we want is to replace each $x \in X$ by a set M_x and each pair $x \leq y$ by a mapping $f_{xy} : M_x \longrightarrow M_y$ so that composition holds : $f_{yz} \circ f_{xy} = f_{xz}$.

Having said that it is easy to gues such a representation. We can put

$$M_x = \{y; y \leq x\}$$

and $f_{xy}(z) = z$ for $z \leq x$ (or,equivalently, $z \in M_x$).

Similarly, to prove that a given group (or a given monoid), if considered as a (single object) category, is concrete, amounts to find a representation of a group (or monoid) by mappings. This is well known and for example left translations turn every group (or monoid) to *isomorphic* permutation group (or monoid of mappings) and that is what we wanted to prove. After realizing this perhaps the following question is justified:

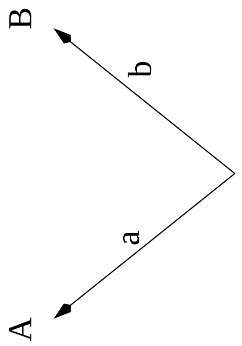
Problem 1 *Is every category concrete?*

This is a fundamental problem which took a full decade to solve. A nice combinatorics is involved and curiously enough this result misses most monographies which we quoted above. After all this perhaps belongs to more combinatorial context.

The solution (and indeed the problem itself) started with John Isbell [40] when he discovered that the answer to the above problem is negative:

If a category $\mathcal{K}$ is concrete then it has to satisfy the following *Isbell's condition* which we are going to explain now:

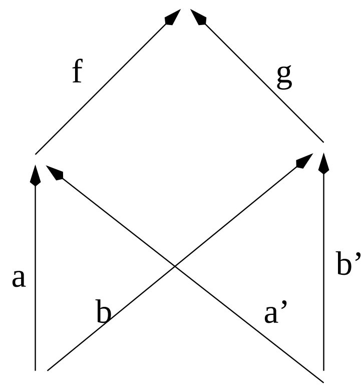
A *fork* in $\mathcal{K}$ is a pair (a, b) of morphisms of $\mathcal{K}$ which satisfy: $d(a) = d(b)$. More explicitely, for every two objects A, B of K denote by $L(A, B)$ the set of all forks (a, b) of morphisms of $\mathcal{K}$ which terminate in A, B, i.e. which satisfy: $r(a) = A$, $r(b) = B$ (and of course $d(a) = d(b)$; such forks we shall briefly call (A, B)-forks), see Fig.3.



On the set of all forks in $\mathcal{K}$ we define *Isbell equivalence* $\sim_{isb}$ as follows :

$(a,b) \sim_{isb} (a',b')$ iff for every pair (f,g) of morphisms $d(f) = A$, $d(g) = B$, $r(f) = r(g)$ holds: $fa = gb \Leftrightarrow fa' = gb'$

In the other words $(a,b) \sim_{isb} (a',b')$ if no pair of outgoing morphisms (f,g) distinguishes (a,b) from (a',b'); see Fig.4.



(Convince yourself that $\sim_{isb}$ is an equivalence.)

Now we have

Theorem 2.4. *(Isbell [40]) If $\mathcal{K}$ is a concrete a category then $\mathcal{K}$ satisfies the following* Isbell's Condition*:*

For any pair A, B of objects of $\mathcal{K}$ the equivalence $\sim_{isb}$ restricted to the set $L(A,B)$ (of all A,B-forks) has only finitely many classes.

Proof.

Let $\mathcal{K}$ be a concrete category. So we can identify $\mathcal{K}$ with a subset (more precisely a subcategory) of SET and thus we may assume without loss of generality that $\mathcal{K}$ is a category of (some) sets and of (some) mapping between them.

To any pair (a, b) of mappings with $d(a) = d(b) = C$, $r(a) = A$, $r(b) = B$ we associate the following relation $S(a, b)$ on $A \cup B$ (A, B are sets now):

$$S(a, b) = \{(a(u), b(u)); u \in C\}.$$

Assume now that $S(a, b) = S(a', b')$ and that $f \circ a = g \circ b$ Then obviously also $f \circ a' = g \circ b'$ (as if $f(a(u)) = g(b(u))$ then also $f(a'(u')) = g(b'(u'))$ for some u' satisfying $(a(u), b(u)) = (a'(u'), b'(u'))$). Thus we have that $S(a, b) = S(a', b')$ implies $(a, b) \sim_{isb} (a', b')$.

But then $S(a, b) \subseteq A \times B$ and thus the number of possible equivalence classes of $\sim$ on $V(A, B)$ is $\leq 2^{|A||B|}$ – a large yet finite number.

♡

Example:

Consider the following example of a very simple category $\mathcal{K}$: onjects are all integers and non-identical morphisms are all arrows (m, n) where $m < n$, moreover for every n we have two additional arrow $(-n, n)'$. Composition is defined by concatenation of arrows, i.e. $(n, p) \circ (m, n) = (m, p)$ with the exception of concatenation of arrows $(1, n)$ and $(-n, 1)$ when we define $(1, n) \circ (-n, 1) = (-n, n)'$. One can check that this is a category. However there are no equivalent forks which terminate in 0 and 1 and thus this category does not satisfy Isbell condition and thus it fails to be concrete.

Note that while every partial ordered set (considered as category) satisfies trivially Isbell condition (as $L(x, y)$ has only one equivalence class for every pair of objects x and y a "small" change of a linear order (as in this example) gives a non concrete category.

Thus there are additional structural combinatorial conditions for concrete categories. This we believe is a surprising fact. The complete solution of Concreteness Problem is provided by the following:

Theorem 2.5. *For a category K are the following two statements equivalent*

(1) *$\mathcal{K}$ is concrete*

(2) $\mathcal{K}$ *satisfies Isbell's Condition.*

This result was proved by J.Vinárek [78] (extending earlier result of P.Freyd [15]) . We proved (1) $\Rightarrow$ (2) only. (2) $\Rightarrow$ (1) is a harder but in a sense it may be viewed as an on-line version of the proof of the following special case which introduces an important construction. This is a good warm up for a general case.

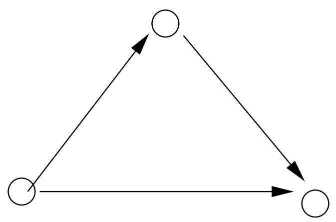
Theorem 2.6. *Every finite category $\mathcal{K}$ is concrete.*

Proof. We are given finitely many objects $A_1, \ldots, A_n$ together with (of course) finitely many morphism arrows $f : A_i \longrightarrow A_j, 1 \leq i, j \leq n$. Let us define functor which we will denote by symbol $<>$:

For object A_i we put $< A_i >= \bigcup_{k=1}^{n} < A_k, A_i >$

For morphism $f : A_i \longrightarrow A_j$ we put $< f > (\varphi) = f\varphi$ (for any morphism φ with $Rg(\varphi) = A_i$).

A little explanation which is needed is contained in the following diagram:



G

It holds $< \mathbf{1}_A >= \mathbf{1}_{<A>}$ and $< f \circ_{\mathcal{K}} g >=< f > \circ < g >$

For $i \neq j$ we have $< A_i > \neq < A_j >$ (as for example $1_{A_i} \notin < A_j >$ and by the same token for $f \neq g$ we have $< f > \neq < G >$. Thus $<>$ is an inclusion mapping of $\mathcal{K}$ in SET. ♡

The inclusion mapping $<>$ of $\mathcal{K}$ into SET is called *Cayley-Maclane Representation* from reason which will be evident in the next section.

It is perhaps surprising that the concreteness of a category is guaranteed by a single (structural) obstruction (i.e. Isbell's condition). The proof of Theorem 2.5 is a nice example of "combinatorics of maps" and as it short and selfcontained we include it here.

Proof. (of Theorem 2.5)

Let us make things easy: Let $\mathcal{K}$ be finitary category, the objects will be denoted by capital letters $A, B, X, Y, \dots$, its morphisms (which we now consider as "abstract arrows") will be denoted by lower case letters $a, b, f, g, \dots$. We can also enumerate objects as $A_1, A_2, \dots$ but to avoid unnecessary indexing we shall write simply $A < B$ if the index of A is smaller than the index of B. (We could also assume without loss of generality that objects are natural numbers.)

Recall the Isbell's condition:

Given two objects A, X we consider the set $L(A, X)$ of (A, X)-forks. (Observe the notation, A will be pivotal, X will play a role of an auxiliary variable.)

We assume that for every A, X the equivalence $\sim_{isb}$ has finitely many equivalence classes. This will be easier to handle if we select a set of representatives $R(A, X) \subset L(A, X)$ such that every (A, X)-fork (a, x) will be equivalent to exactly one "representative" fork from $R(A, X)$; this representative fork will be denoted by $< a, x >$. (Consequently $(a, x) \sim_{isb} (a', x')$ iff $< a, x >=< a', x' >$.)

We say that a fork $(a, x) \in L(A, B)$ is *minimal* if the morphism a factors through no $Y < X$. Explicitely: there is no $Y < X$ and arrows $c : C \to Y$ and $d : Y \to A$ such that $dc = a$.

Denote by $M(A, X)$ the set of all minimal forks in $L(A, X)$. We are on a good track as indicated by the following:

Claim 1 $M(A, X)$ is a finite set (for every pair A, X).

(To see this just observe that $M(A, X)$ is empty set whenever $A \leq X$ - a proof is provided by factorization $a = 1_A \circ a$)

The sets $M(A, X)$ will be building blocks of our inclusion of $\mathcal{K}$ in SET. However the sets $M(A, X)$ depend on the original ordering $<$ of objets and thus do not respect the equivalence $\sim_{isb}$ - the Isbell equivalence is "too large". (For example, it may happen that $< a, x >=< a', x' >, (a, x) \in M(A, X)$, while $(a', x') \notin M(A, X)$.) Thus we "hereditarily" enlarge the sets of representatives and hence in turn we refine the equivalence $\sim_{isb}$.

For a fork $(a, x) \in L(A, X)$ we denote by $[a, x]$ the following sequence:

$$(< a, x >, (\{< a, y >; y : d(a) \to Y\}; Y < X))$$

.

Let us give an alternative description: Given A and Y denote by a_Y the set of all representative forks $< a, y >$ where the arrow y ends in Y. Note that for every pair A, Y this is a finite set (by virtue of Isbel's condition). Using this we can simply write

$$[a, x] = (< a, x >, (a_Y; Y < X))$$

.

For all forks in the category $\mathcal{K}$ this defines the equivalence $\sim_{vin}$ by

$$(a,x) \sim_{vin} (a',x') iff [a,x] = [a',x']$$

.

Clearly $\sim_{vin}$ is a refinement of $\sim_{isb}$.

The key property of this construct is contained in the following

Claim The equivalence $\sim_{vin}$ is compatible with minimality of forks. Explicitely: If $(a,x) \sim_{vin} (a',x')$ and $(a,x) \in M(A,X)$ then also $(a',x') \in M(A,X)$.

Proof.(of Claim) To the contrary let us suppose that (a',x') is not a minimal fork in $L(A,X)$. Thus there exists $Y < X$ and arrows $y' : B' \to Y$ and $z : Y \to A$ such that $a' = zy'$ (here $B' = d(a')$). It is $< a', y' > \in a'_Y$ and as $a'_Y = a_Y$ there exists an arrow $y : B \to Y$ $(B = d(a))$ such that $< a, y > = < a', y' >$ (i.e. $(a,y) \sim_{isb} (a',y')$. However then $1_A a' = a' = zy'$ and thus also $1_A a = a = zy$. However this contradicts the minimality of the fork (a,x); see the following scheme. $\heartsuit$

Now we can define the inclusion of the category $\mathcal{K}$ into category SET of all sets:

To every object A of $\mathcal{K}$ we associate the set $F(A)$ which will consist from all sequences $[a,x]$ where (a,x) is a minimal (A,X)-fork together with a special "new" element O_A (different for every object A).

Formally, this can be written as

$$F(A) = \bigcup_{X \in Obj\mathcal{K}} \{[a,x]; (a,x) \in M(A,X)\} \cup \{O_A\}$$

.

Despite of this lengthy formula, for every A is $F(A)$ a finite set (as it is a finite union of finite sets).

For every morphism arrow $f : A \to B$ we define mapping $F(f) = f'$ as follows:

$f'([a,x] = [fa,x]$, if (fa,x) is a minimal fork (i.e. $\in M(B, r(x))$;

$f'([a,x]) = O_B$ otherwise;

$f(O_A) = O_B$.

We are not yet ready: we have to check that the mapping $f' = F(f)$ is indeed a mapping $F(A) \to F(B)$. For that one has to show that f' maps equivalnt forks to equivalent forks (and that $[a, x] = [a', x']$ implies $[fa, x] = [fa', x']$). However this is just to check definition and it is easy to see that $F(1_A) = 1_F(A)$. However we have to check also that if arrows f and g (abstractly) compose to h then the mappings $F(f)$ and $F(g)$ compose to $F(h)$. Here we need to consider several cases (paralel to the definition of mapping $F(f)$) and we leave it at that. ♡

2.3 Representations

Thus we know by Theorem 10 that all finite categories are concrete. Now we shall generalize this result (which gives existence of a faithfull functor) to the following results (which give embedding):

Theorem 2.7. *Any finite category is representable by graphs.*
Explicitely: For every finite category $\mathcal{K}$ there exists on embedding $F : \mathcal{K} \longrightarrow GRA$.

This theorem was proved in [27] and it extends representations of groups and monoids which were studied earlier.

The proof is a combination of two(by now) standard techniques: we first reduce the problem to relational systems (i.e. colored graphs) and then use a replacement trick to reduce colored graphs to graphs.

2.3.1 Relational systems instead of graphs

Definition 2 *An* m-relational system *S of order r is a pair $(X; (R_i; i = 1, \dots, m))$ where $R_i \subseteq X \times X$.*

(Alternatively, m-relational system is a directed graph with arc colored by m distinct colors.)

Given relational systems $S = (X, (R_i; i = 1, \dots, m))$ and $S' = (X', ((R'_i; i = 1, \dots, m))$ a *homomorphism* $f : S \longrightarrow S'$ is a mapping $f : X \longrightarrow X'$ which satisfies for every $i = 1, \dots, m$:

$$(x, y) \in R_i \Longrightarrow (f(x), f(y)) \in R'_i$$

We shall denote by $REL(r)$ the category of all finite m-relational systems and all homomorphisms between them.

Somehow it is easier to represent categories by relational systems. For example we have the following:

Theorem 2.8. *Every finite category can be represented for some m by $REL(m)$.*

Explicitly : For every finite category $\mathcal{K}$ there exists m and an embedding $F : \mathcal{K} \longrightarrow REL(r)$.

Proof. Let $\mathcal{K}$ has objects $\mathcal{A} = \{\mathcal{A}_\infty, \ldots, \mathcal{A}_\backslash\}$ and morphisms $\mathcal{M} = \{\{_\infty, \ldots, \{_\Updownarrow\}$. Let us define functor $F : \mathcal{K} \longrightarrow SET$ as follow

$$F(A) =< A >$$

$$F(f)(\varphi) = f \circ \varphi$$

(This is *Cayley-Maclane functor*, sometimes called *hom-functor*.)

On each set $< A >$ define relations $R_1^A, \ldots, R_m^A$ as follows
$(\varphi, \varphi') \in R_i$ iff $\varphi' = \varphi \circ f_i$

(This is the generalization of the right translation from groups to categories.) We shall prove that the above functor F is in fact an embedding $\mathcal{K} \longrightarrow REL(m)$.

For this it clearly suffices to prove:

1. For every $f_i : A \longrightarrow A'$ the mapping $F(f_i)$ is a homomorphism (in $REL(m)$)

$$(< A >, (R_i^A)) \longrightarrow (< A' >, (R_i^{A'}))$$

2. For every hommorphism

$$g : (< A >, (R_i^A)) \longrightarrow (< A' >, (R_i^{A'}))$$

there exists f_i such that $g = F(f_i)$.

However 1. is clear (as if $(\varphi, \varphi') \in R_j^A$ then $\varphi' = \varphi \circ f_j$ and $F(f_i)(\varphi) = f_i \circ \varphi$, $F(f_i)(\varphi') = f_i \circ \varphi' = f_i \circ \varphi \circ f_j$ and thus $(F(f_i)(\varphi), F(f_i)(\varphi') \in R_j^{A'})$.

For 2. we define f by $f = g(1_A) : A \longrightarrow A'$. It is a routine to check that $F(f) = g$. ♡

2.3.2 Indicator Construction (Amalgamation Technique)

The construction which we are going to introduce has many variants and many analogies in virtually any type of structures: algebraical, geometrical and combinatorial, see book [75] for many examples. But the nature of all these applications is similar, in many cases the same. So we can restrict ourselves to a simple illustrative example:

A graph I with two distinquished vertices a, b is called an *indicator.*

Given an oriented graph $G = (V, E)$ and an indicator (I, a, b) we define graph $G * (I, a, b) = (W, F)$ as follows:

$$W = (E \times V(I))/ \sim$$

where the equivalence $\sim$ is generated by the following pairs:

$$\begin{aligned}((x, y), a) &\sim (x, y'), a)\\ ((x, y), b) &\sim (x', y), b)\\ ((x, y), b) &\sim ((y, z), a)\end{aligned}$$

Thus the vertices of G are equivalence classes of the equivalence $\sim$. For a pair $(e, x) \in E \times V(I)$ its equivalence class will be denoted by $[e, x]$.

We put $\{[e, x], [e', x']\} \in F \Longleftrightarrow e = e'$ and $\{x, x'\} \in E(I)$.

This construction which is called an *arrow construction* is schematically indicated on Fig.6

From a homomorphism point of view the arrow construction has many convenient properties and in many instances one can guarantee that some properties of $G*(I, a, b)$

depend on the indicator (I, a, b) only. Particularly one can guarantee that for every oriented graph G in many cases holds that

$G \longrightarrow G'$ iff $G * (I, a, b) \longrightarrow G' * (I, a, b)$

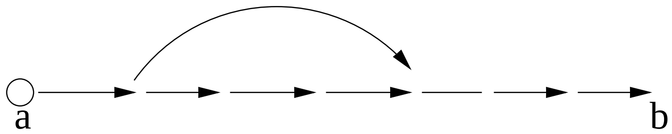
Even more so: for every homomorphism $g : G * (I, a, b) \longrightarrow G' * (I, a, b)$

there is a homomorphism $f : G \longrightarrow G'$ such that

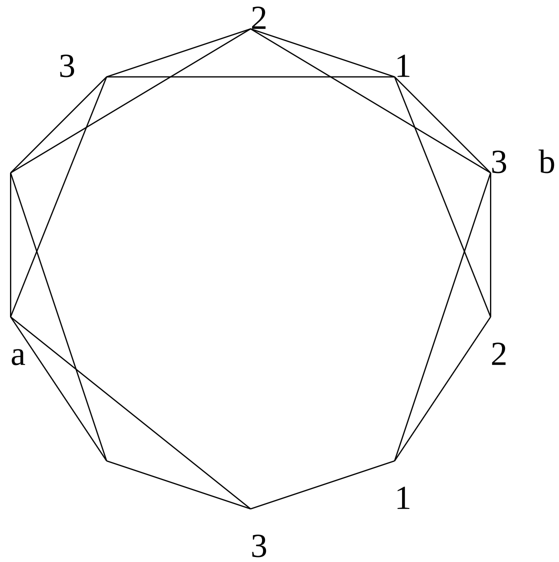
$$g([(u, v), x] = ([f(u), f(v)], x).$$

An indicator satisfying this property is called *rigid indicator*.

Examples of rigid indicators are easy to find. For example oriented graph I_1 on Fig.7



is rigid oriented indicator and also undirected graph I' on Fig.8 is an example of rigid indicator:



Let us prove this at least for the second graph $I = I'$ (see [23],[31]): The graph I has the following properties:

(i) $\chi(I) = 4$ and $\chi(I') < 4$ for every vertex deleted subgraph I' of I;

(ii) Every vertex x of I belonges to a triangle, moreover for every two vertices x and y or i there exists a path $x = x_0, x_1, x_2, \ldots, x_\epsilon = y$ such that $\{x_{i-1}, x_i, x_{i+1}\}$ forms a triangle in I.

(iii) Identity is the only automorphism of I.(such graphs are called *asymmetric*).

Combining (i) and (iii) we get that the only homomorphism $I \longrightarrow I$ is the identity. Such graph is called a *rigid* graph. Moreover, this together with (ii) gives that for every oriented graph G the graph $G * (I, a, b)$ has the following property:

For every homomorphisms $f : I \to G * (I, a, b)$ there exists an edge $e \in E(G)$ such that $f(x) = [e, x]$ for every $x \in V(I)$.

Now this clearly implies that I is a rigid indicator.

Combining the above definition with the rigidity of the (symmetric) indicator I we just constructed an embedding F of the category $REL = REL(1)$ of all finite relations into category GRA of all finite undirected graphs:

Given an oriented graph G we put $F(G) = G * (I, a, b)$ and for a homomorphism $f : G \to G'$ we put $F(f)([u, v], x) = (f(u), f(v)], x)$.

Observe further that many combinatorial properties of graphs $G * (I, a, b)$ are determined *for any* graph G the graph $G * (I, a, b)$ satisfies:

(1) If I has a k-coloring so that the vertices a and b gets the same color, then $G * (I, a, b)$ is k-colorable too;

(2) If I has maximal clique size k and $a, b \notin E(I)$ then $G * (I, a, b)$ has maximal clique size is k

(3) If I has girth k and the distance of vertices a and b is $\geq 2k$ then $G * (I, a, b)$ has girth k.

Thus in fact we embedded the category REL into a category of 4-chromatic graphs with clique size 3. Or we could also say that we reduced relations and their homomorphisms to homomorphisms of undirected 4-chromatic graphs with clique size 3. This lead to study of rigid graphs and rigid indicators (more examples of rigid graphs are given e.g. in [8], [31], [66],[47]).

The situation is reminiscent to problems in Theoretical Computer Science we offer use such techniques to reduce one problem to another while preserving certain particular property.

Recall for example polynomial reductions which lead to NP-complete (and, say, isomorphism complete) problems. (In a sense the monograph [75] resembles [19] in that it provides a *catalogue* of structures and reductions between them.)

Let us return to our main theme: We do not have to use one indicator only. Suppose that $(I_1, a_1, b_1), (I_2, a_2, b_2), \ldots, (I_m, a_m, b_m)$ be indicators. Let $(X, (R_i; i = 1, \ldots, m))$ be an m-relational system. Let $(X, (R_i; i = 1, \ldots, r)) * ((I_i, a_i, b_i); i = 1, \ldots, m)$ (or shortly $(X, (R_i)) * (I_i, a_i, b_i))$ denotes the variant of arrow construction where we replace edge $e \in R_i$ by a copy of indicator (I_i, a_i, b_i). It is easy to modify the arrow construction (we use all the notation used above in the definition of arrow construction $G * (I, a, b)$):

$(X, (R_i)) * (I_i, a_i, b_i) = (W, F)$ where $W = \bigcup_{i=1}^{r} (R_i \times V(I_i)) / \sim$

where the equivalence $\sim$ is generated by the pairs
$((x, y), a_i) \sim ((x, y'), a_j)$ where $(x, y) \in R_i, (x, y') \in R_j$
$((x, y), b_i) \sim (x', y), b_j)$ where $(x, y) \in R_i, (x', y) \in R_j$
$(x, y), b_i) = ((y, z), a_j)$ where $(x, y) \in R_i, (y, z) \in R_j$.
(We could also briefly say that $((x, y), a_i) \sim ((x, y'), a_j)$ whenever these expressions belong to W).

$$F = \{\{[e, x], [e', x']\}; e = e' \in R_i, \{x, x'\} \in E(I_i)\}$$

We continue to proceede analogously to the above:

We say that the set of indicators $(I_1, a_1, b_1), \ldots, (I_m, a_m, b_m)$ is *rigid* if for every m-relational system $(X, (R_i))$ and for every i every homomorphism

$$f : I_i \longrightarrow (X, (R_i)) * (I_i, a_i, b_i)$$

has form $f(x) = [e, x]$ for some pair $e \in R_i$.

Examples of such indicators are easy to obatin. For example if we use the above oriented indicator I we can get set I_i by enlarging the length of the cycle. See schematic Fig.

It is a routine to prove that I_i have these properties. Note that an oriented rigid indicator (I, a, b) need not have a cycle and that the vertices a and b ma be on the same level. This implies that the arrow construction $G * (I, a, b)$ is a balanced graph for every graph G. An example of such a graph is given on the following Fig.

Combining the above construction together with Theorem 12 we finally obtain Theorems 2.1, 2.2, 2.3, and 2.7. More concretely we also proved:

Theorem 2.9 *Every finite category $\mathcal{K}$ is representable by oriented balanced graphs of given girth.*

2.4 Poset of Homomorphisms - Independent Families

We can consider the finite graphs with the relation $\leq$ induced by the existence of a homomorphism:

(4) $$G \leq H \text{ if and only if } G \to H$$

The relation $\leq$ is a quasiorder on the set of all graphs. However we can think of this set as a partially ordered set if we restrict our attention to minimal graphs which are called cores: A graph G is called a *core* [29] if every homomorphism $f : G \to G$ is an authomorphism. One can prove easily that for every graph G there exists up to an isomorphism unique subgraph G' such that G' is a core and $G \to G'$. The graph G' is called the *core of* G. Not suprisingly most graphs are cores (and it is NP-complete to decide whether a given graphs is a core or not).

We denote this poset by $\mathcal{C}$. $\mathcal{C}$ is called *Coloring Poset.*

$\mathcal{C}$ is a countable poset which is very rich. In fact we have the following:

Theorem 2.10 *$\mathcal{C}$ is universal countable poset. Explicitely, every countable poset is isomorphic to a subposet of $\mathcal{C}$.*

This is non-trivial result due to Z.Hedrlín, see [75]. No simple proof of this was until recently known. In the Appendix we shall present a new proof based on the extension properties of the poset $\mathcal{C}$.

Note that for finite posets this is much easier result which follows from the previous section. For finite posets one can also prove stronger results for example the following result proved in [71]

Theorem 2.11 *Every finite poset may be represented by homomorphisms between finite oriented paths.*

An extension of this result to countable posets is presently unknown and as proved in [71] this is equivalent to the on-line representability of finite posets.

Some particular cases of universality property of $\mathcal{C}$ proved to be more usefull than other and they were studied intensively. Two of such examples - independency and density - are subject of this and the following section.

We say that a set $G_1, G_2, \cdots, G_n, \cdots$ of graphs (finite or infinite) is *independent* if for no two graphs G_i, G_j, $i \neq j$ there is a homomorphism $G_i \to G_j$. Recall that a graph G is called *rigid* if the identity is the only homomorphism $G \to G$. We say that a set $G_1, G_2, \cdots, G_n, \cdots$ of graphs (finite or infinite) is *mutually rigid* if for no two graphs G_i, G_j there is a non-identical homomorphism $G_i \to G_j$.

We used the mutually rigid families in the above proof of Theorem 2.7. It s easy to construct an exponentially large set of mutually rigid graphs. This may be proved as follows:

Let $G = (V, E)$ be an undirected rigid graph with m edges. Let $G_1, \cdots, G_M$ be the all orientations of the graph G (i.e. $M = 2^m$). This set $G_1, \ldots, G_M$ is mutually rigid. This is easy to see as every homomorphism $f : G_i \to G_j$ is also a homomorphism $f : G \to G$ andus necessarily f is the identity and thus G_i is rigid and $i = j$.

Thus we have exponentially many mutually rigid oriented graphs and if we want to have undirected graphs with the same property we can consider the set $G_1 * (I, a, b), \cdots, G_M * (I, a, b))$ for an undirected rigid indicator $(I, a, b,)$.

However this is not the end of the story and we can ask what is the maximal size of a set of mutually rigid graphs on a given set X (of vertices). This clearly depends on the size of the set X only and thus denote by mr(k) the maximal size of the set of mutually rigid graphs on a set with k points. We have the following two basic results:

Theorem 2.12 (Mutually Rigid Families on Finite Sets)
$mr(k) = \binom{\binom{k}{2}}{\binom{k}{2}/2}(1 + o(1))$

This is a paper on finite combinatorics. Let us make a exception at this moment and let us state an infinite result related to our main theme.

Theorem 2.13 (Mutually Rigid Families on Infinite Sets) $mr(k) = 2^k$ *for every infinite k.*

The first result is due to Koubek and Rödl [47] and uses probabilistic tools. The second result will follow easily from the following important result [79]:

Theorem 2.14 *On every set there exists a rigid relation.*

Let us prove Theorem 2.14. What follows is a recent easy proof [62].

Proof. [of Theorem 2.14] It suffices to consider infinite sets only as the finite case is solved by directed paths.

Let X be an infinite set and assume that X is an ordinal $X = \{\beta; \beta \leq \alpha\}$. Let $X' = \{\beta'; \beta' \leq \alpha'\}$ be a disjoint copy of X. Further let $\{a, b, c, a', b', c'\}$ be six vertices disjoint with X and X'. For every ordinal $\beta \leq \alpha$ with countable cofinality let us choose a sequence $\{\beta_n\}$ such that $\sup \beta_n = \beta$.

We define the oriented graph (V, E) by the following set of arcs:

$(0, a), (a, b), (b, c), (c, 0), (0, b)$ and $(0', a'), (a', b'), (b', c'), (c', 0'), (a', c')$;

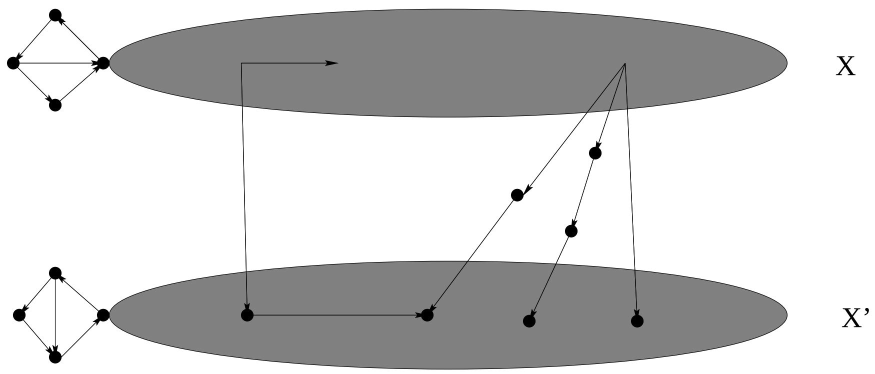
(β, γ) for all $\beta < \gamma \leq \alpha$ and (β', γ') for all $\beta' < \gamma' \leq \alpha'$;

(β, β') for all $\beta \leq \alpha$;

moreover for every ordinal $\beta \leq \alpha$ with countable cofinality let β be joined with β'_n by an oriented path of length 2n.

(All these paths are supposed to be vertex disjoint.)

Let V be the set of all vertices thus obtained. Clearly V is a countable union of sets of cardinality $\leq \alpha$ and thus V and X are in 1-1 correspondence.



Claim The oriented graph $G = (V, E)$ is rigid.

Proof.(of Claim) Let $f : V \to V$ be a homomorphism. As X and X' are transitive orientations of complete graphs, f restricted to both X and X' is an injection. The graph G is acyclic with the exception of vertices $\{a, b, c, 0\}$ and $\{a', b', c', 0'\}$. The graphs induced by these sets are isomorphic, but there is not homomorphism between them which would preserve vertices 0 and $0'$. However the mapping f restricted to

$\{a,b,c,0\}$ satisfies either $f(0)=0$ or $f(0)=0'$ (as vertices $0,0'$ are distinguished by large clique which includes them). This shows that both $f(0')=0$ and $f(0)=0'$ are impossible ant thus $f(0)=0, f(0')=0'$. Consequently f restricted to the set $\{a,a',b,b',c,c'\}$ is the identity. It follows that f maps X to X and X' to X'. As pairs (β,β') are the only arrows between X and X' we have that $(f(\beta))'=f((\beta)')$ for every $\beta \leq \alpha$. Let $\beta(0)$ be the smallest β for which $f(\beta)\neq\beta$. Then necessarily $f(\beta(0)) > \beta(0)$ (as if $f(\beta(0)) < \beta(0)$ then $f(f(\beta(0))) < f(\beta(0))$ which violates that $\beta(0)$ was minimal). Thus $\beta(0) < f(\beta(0))$ and we put $f(\beta(0)) = \beta(1)$ and $\beta(n) = f^n(\beta(0))$ (the n-times iterated mapping f). Let $\beta = \sup\beta(n)$. β is also the limit of the sequence $\{\beta_n\}$. However as the sequences $\beta(n)$ and β_n are interlacing and as f maps the set $\{\beta(n)\}$ into a subset we get by monotonicity that that the sets $\{\beta(n)\}$ and $\{f(\beta_n)\}$ are interlacing again and thus by the definition of the graph G we get $f(\beta)=\beta$ (as β is the only vertex joined to the set $\{\beta'_n\}$ by directed paths). But then $f(\beta_n)=\beta_n$ for every n (as in our situation the mapping f has to preseve the length of paths between β and β'_n. This is a contradiction as if we choose m and n such that $\beta(m) < \beta_n < \beta(m+1)$ then $f(\beta_n) > \beta(m+1)$, a final contradiction. ♡
♡

Remark The existence of a rigid graph on every set is an important result which lies in the heart of several combinatorial and non-combinatorial embeddings (see [75]).

Let us also remark that the problem is a finite one, for finite sets a directed path is a rigid graph. These graphs served as building blocks of our construction. It is important that one can prove the existence of a rigid graph on every set in ZFC. This is in a sharp (and surprising) contrast with difficulties when one wants to construct a proper class of *mutually rigid* graphs (see [75] for a disscussion of this).

Proof.[of Theorem 2.13] Let X be a set of cardinality k and let (X,R) be a rigid relation. Using an undirected rigid indicator we get a rigid undirected graph G again on the set X. Considering of all possible 2^k orientetions of G we obtain 2^k mutually rigid relations on the set X and if we want we can apply again (the same) undirected rigid indicator to obtain 2^k mutually rigid graphs on the set X. ♡

Particularly, there are continuum many countable graphs which are mutually rigid.This usefull fact is sometimes referred to as *Ulam's problem*, see [75].

Let us return to finite sets. We note that the above techniques have some further corollaries and for example one can costruct an infinite independent set of finite graphs G_i which have the following properties:

1. each of the graphs G_i is planar;

2. each of the graphs G_i has all vertices ≤ 3.

This should be compared with results mentioned in the section 2.3 (where we stated that neither planar, and more generally graphs with bounded genus, nor bounded degree graphs fail to represent all finite categories-even finite groups and finite monoids). See also the problems stated in the following section.

2.5 Density

Let us continue our study of the properties of the poset $\mathcal{C}$ induced by all finite graphs and the existence of homomorphisms between them. As we have seen this is a very rich and in indeed universal poset (see Theorems 2.10, 2.11 and Appendix 1).

Here we are going to proceede in yet another direction. As we are going to disscuss order-theoretic notions we shall denote graphs by capital letters $A, B, \ldots$.

The key to this section is the following definition:

Definition 3 *A pair (A, B) of graphs is said to be a gap in $\mathcal{C}$ if $A < B$ and there is no graph C such that $A < C < B$.*

Similarly, for a subset $\mathcal{K}$ of $\mathcal{C}$, a pair (A, B) of graphs of $\mathcal{K}$ is said to be a gap in *K if $A < B$ and there is no graph $C \in \mathcal{K}$ such that $A < C < B$.*

The *Density Problem* for a class $\mathcal{K}$ is ask for the description of all gaps of the class $\mathcal{K}$. This is a challenging problem even in the simplest case of all undirected graphs. This question has been asked first by [55] in the context of the structure properties of classes of languages and grammar forms. The problem has been solved by E. Welzl [81]:

Theorem 2.15 (Density Theorem for Undirected Graphs)
The pairs (K_0, K_1) and (K_1, K_2) are the only gaps for the class of all undirected graphs. Explicitely, given undirected graphs G_1, G_2, $G_1 < G_2$, $G_1 \neq K_0$ and $G_1 \neq K_1$ there is a graph G satisfying $G_1 < G < G_2$.

In this surwey we give three proofs of the Density Theorem 2.15 which were recently found and which put this result in new context.

2.5.1 Probabilistic proof of undirected graph density

The proof is based on the following *Sparse Incomparability Lemma* first isolated in [66]:

Lemma 1
Let G, H be fixed graphs, H non-bipartite, ℓ a positive integer. Assume $G \to H$ and $H \not\to G$. Then there exists a graph G' with the following properties:
(i) $G' \to H$,
(ii) $G' \not\to G \not\to G'$, $H \not\to G'$,
(iii) G' has girth $> \ell$.

This of course strengthens the classical Erdős result [11] on high chromatic graphs with given girth (say, for $H = K_k$ and $G = K_{k-1}$). Sparse Incomparability Lemma seems to be an useful tool and [66] originally applied this result to graphs without given symmetries and endomorphisms (the so called *rigid graphs*).

First, we show that Density of Undirected graphs follows easily from the Sparse Incomparability Lemma:

Proof.

Let $G_1 < G_2$ be given. Apply Sparse Incomparability Lemma for $\ell = |V(G_2)|$, $H = G_2$, $G = G_1$ to get a graph G' with stated properties. Put $G = G' \cup G_1$. Then G has all the desired properties: $G_1 \hom G$ obviously and $G \hom G_2$ by (i) of Sparse Incomparability Lemma. On the other hand $G_2 \not\to G$ by girth and $G \not\to G_1$ by (ii). $\heartsuit$

An interested reader observed that we did not use condition (ii) of Sparse Incomparability Lemma in full. We used only the fact that H was non bipartite while G' contained no short *odd* cycles. This is much easier to guarantee and we shall return to this in the next section.

Proof. (of Sparse Incomparability Lemma)

Let graphs G, H be given, let ℓ be a given positive integer. Let t denotes the number of vertices of the graph G and, without loss of generality, let the set of vertices of H be $\{1, 2, ..., k\}$. For a (large) positive integer n consider pairwise disjoint sets $V_1, V_2, ..., V_k$, each of size n.

Let $\mathbf{G}$ be a random graph with vertex set $V = \cup_{i=1}^k A_i$ where the edges are chosen independently from the family $\{\{x, y\};\ x \in V_i,\ y \in V_j,\ \{i, j\} \in E(H)\}$, each with the probability $p = n^{\delta-1}$, where $0 < \delta < 1/\ell$.

A set $A \subset V$ is said to be *large* if there are $i, j,\ 1 \leq i < j \leq k$, such that $|A \cap V_i| \geq n/t$ and also $|A \cap V_j| \geq n/t$. For evey large set A we consider all such pairs $\{i, j\}$ and we call them *good pairs* of A. For a large set A denote by $|\mathbf{G}/A|$ the minimum number of edges of $\mathbf{G}$ which lie in the set $\{\{x, y\};\ x \in V_i,\ y \in V_j\}$ for a good pair of A.

We first estimate probability

$$\alpha = Prob[A \text{ large implies } |\mathbf{G}/A| \geq n].$$

We have

$$1 - \alpha \leq \sum_{A \text{ large}} Prob[|\mathbf{G}/A|] < n] \leq 2^{kn} \cdot \binom{\binom{kn}{2}}{n} \cdot (1-p)^{\frac{n^2}{t^2} - n}.$$

Now bounding very roughly

$$\binom{\binom{kn}{2}}{n} \leq \binom{k^2 n^2}{n} \leq k^{2n} n^{2n} < e^{cn \log_2 n}$$

and

$$(1-p)^{\frac{n^2}{t^2}-n} \leq e^{-p(\frac{n^2}{t^2}-n)}$$

we obtain

$$1-\alpha < e^{cn\log_2 n - c'n^{1+\delta}}$$

for some positive constants c and c' which are independent on n.

Thus we get $Prob[A$ large implies $|\mathbf{G}/A| \geq n] = 1 - o(1)$.

On the other hand if we denote by $c(\mathbf{G})$ the number of edges contained in all cycles of length $3, 4, \ldots, \ell$ in $\mathbf{G}$ then by the linearity of expected value we have

$$E(c(\mathbf{G})) \leq 3!\frac{kn}{3}p^3 + 4!\frac{kn}{4}p^4 + \ldots + \ell!\frac{kn}{\ell}p^\ell = \ell \cdot \frac{k^l n^\ell n^{\delta\ell}}{n^\ell} < n^{\delta\ell} = o(n).$$

Thus there exists a graph G'' (an instance of the random graph $\mathbf{G}$) such that
(i) If i, j is a good pair of a large set A then G'' has at least n edges in the set $\{\{x, y\};\ x \in V_i,\ y \in V_j\}$,
(ii) There exists $n-1$ edges $e_1, e_2, \ldots, e_{n-1}$ such that the graph G' which we obtain from G'' by deleting edges $e_1, e_2, ..., e_{n-1}$ has girth $> \ell$.

We prove that the graph G' satisfies the above conditions of Sparse Incomparability Lemma. Properties (i) and (iii) are evident from the construction of G'. To prove (ii) let us suppose that f is a homomorphism $G' \operatorname{hom} G$. Define a mapping $g : V(H) \operatorname{hom} V(G)$ by $g(i) = y$ if $|f^{-1}(y) \cap V_i| \geq n/t$ (we could call g a *majority mapping*). Clearly for every i one can choose $g(i)$ (if there are more possibilities we choose one arbitrarily). It follows from the properties (i) and (ii) of graph G' that the majority mapping g is a homomorphism $H \operatorname{hom} G$ which is a final contradiction.

This is the only non-arrow which one has to prove for (ii) (the remaining non-arrows follow from the girth of graph G'). ♡

See Appendix 2 where this proof is put in yet another setting.

2.5.2 Constructive proof of undirected graph density via products

This proof is due to M. Perles and J. Nešetřil (see e.g. [60], the proof is implicite in [66]) and it is particularly simple. It uses product and the construction of high chromatic graphs without short odd cycles (construction for existence).

Proof. (of Theorem 2.15)

Let G_1 and G_2 be given undirected graphs, let $f : G_1 \operatorname{hom} G_2$ be a homomorphism, and suppose there is no homomorphism $G_2 \to G_1$. As this pair is not equivalent to the gap (K_1, K_2), at least one component of the graph G_2 has chromatic number greater than 2. Also, at least one component of G_2 fails to be homomorphic to G_1, and this component may be assumed to be non-bipartite; let it contain an odd cycle of length k. Now choose a graph H with the following properties: H contains no odd cycle of length k or less, and the chromatic number of H is greater than $n_1^{n_2}$, where n_1 and n_2 denote the number of vertices of the graphs G_1 and G_2 respectively. Such a graph exists by the celebrated theorem of Erdős [11] but the existence follows much more easily and one can give also an easy construction of such graphs (shift graphs).

Now let $G = G_1 \cup (H \times G_2)$. Here $\times$ denotes the direct product of two graphs and $\cup$ means the disjoint union. We shall prove that G has the desired properties. Obviously $G_1 \hom G$ and $G \hom G_2$ follows as the second projection of $H \times G_2$ is a homomorphism into G_2. On the other hand there is no homomorphism from G_2 into G, as homomorphisms preserve odd cycles and they cannot increase the length of the shortest of them. Thus it suffices to prove that there is no homomorphism $G \to G_1$. Let us suppose for the contradiction that there is a homomorphism $f : H \times G_2 \to G_1$. Thus for any vertex x of H we have an induced mapping $f_x : V(G_2) \hom V(G_1)$ defined by $f_x(y) = f(x, y)$. (This mapping need not be a homomorphism.) As there are at most $n_1^{n_2}$ such mappings there are vertices x and x' forming an edge of H such that the mappings f_x and f'_x are identically equal, say to g. However in this case g is a homomorphism of G_2 into G_1, contrary to our assumption. ♡

Note that this construction of graph G given in the proof can be used to prove that Sparse Incomparability Lemma holds for large *odd girth* (and in this setting this proof is implicite in [66]).

This is a good place to review another construction:

Given two graphs G and H one can define G *power of* H, denoted by H^G, as the following graph: $V(H^G) = \{f : V(G) \to V(H)\}$ and a pair (f, g) forms an edge if $(f(x), g(y)) \in E(H)$ for every edge $(x, y) \in E(G)$. (We define the G power of H by the same formula for both undirected or directed graphs.)

This construction was isolated in the graph theoretic concept in [51], however this is also a standard category theory construction (where it is called "map object" or exponentiation) and this led to the notion of *cartesian closed category*. The power graph construction plays an important role in the study of *Hedetniemi conjecture*, [83].

The following is the crucial property which we use (and which in fact defines the power construction):

$$\text{for every graph } K,\ K \to H^G \text{ holds if and only if } K \times G \to H.$$

This is easy to see: given $f : K \to H^G$, define $g : K \times G \to H$ by $g(x, y) = f(x)(y)$. Conversely, given g we may define f by the same formula. One can easily check that f is a homomorphism $K \to H^G$ if and only if g is a homomorphism $K \times G \to H$.

Thus in the above proof we have $H \times G_2 \to G$ if and only if $H \to {G_1}^{G_2}$. Thus in the above proof we may assume that the chromatic number of H is greater than the chromatic number of ${G_1}^{G_2}$. As $G_2 \not\to G_1$ there are no loops in ${G_1}^{G_2}$ and it is also clear that the chromatic number of ${G_1}^{G_2}$ is at most the number of vertices of ${G_1}^{G_2}$. (However we do not try to optimize at this point.)

Neither of these proofs solves the density problem for oriented graphs which remained the main open problem for several years. Until recently the best result in this direction was [71] where all the gaps were characterized for the class of all oriented paths. This result may look modest on the first glance but even the following problem is presently open:

Problem 2 *(Tree Problem) Describe all the gaps of finite oriented trees.*

This problem is particularly interesting in view of the gap characterization which we state bellow as Theorem xx

But even for classes of undirected graph the density presents challenging problems. On such problem (due to Welzl on the circulants, [81]) was recently solved by C. Tardif [76].

Let us list two more problems which probably call for a new method.

Problem 3 *Describe all gaps for (undirected) planar graphs.*

Problem 4 *Describe all gaps for undirected graphs with maximal degree bounded by a fixed number k (i.e. for k-bounded graphs).*

In both cases the only gap presently known is the trivial (K_1, K_2) gap.

3 Paradoxes of Complexity

We consider here the following decision problem called *H-coloring problem*:

Problem 5 *Instance: A graph G*
Question: Does there exists a homomophism $G \longrightarrow H$

This problem covers many concrete problems which were and are studied :

(i) For K_k we get a k-coloring problem;

(ii) For graphs G_k^d we get *circular chromatic numbers*' see e.g. [84];

(iii) For Kneser graphs $K\begin{pmatrix} n \\ k \end{pmatrix}$ we get to called *multicoloring.*

Further examples include so called T-colorings, see e.g.[84],[74].

Equivalently, the H-coloring problem may be considered as a decision problem related to the following class of graphs:

$$\longrightarrow H = \{G; G \longrightarrow H\}$$

Such classes (sometimes denoted by $\mathcal{C}_H$) are called *color classes* and their structure is one of the leitmotifs of this paper. For example in the previous section we proved that with the unique exception the partial ordering defined by the inclusion of color classes is dense (for undirected graphs). We also know that we can restrict ourselves to those color classes $\longrightarrow H$ where H is a core.

3.1 Hard Cases

Here we deal with complexity issues. The situation is well understood for complete graphs: For any fixed $k \geq 3$ the K_k-coloring problem (which is equivalent to the deciding $\chi(G) \leq k$) is NP-complete. On the other hand K_1 - and K_2-coloring problems are easy. Thus, in the undirected case, we will always assume that the graph H is not bipartite.

Some other problems are easy to solve. For example, if $H = C_5$ then we can consider the arrow construction which we introduced in the previous section :

Let the indication (I, a, b) be path of length 3 with ith end vertices called a and b. It is then easy to prove that for any undirected graph G the following two statements are equivalent:

(i) $G \longrightarrow K_5$

(ii) $G * (I, a, b) \longrightarrow C_5$

(In fact in this case $G * (I, a, b)$ takes a very simple form : we subdivide every edge by two points.)

This example is not as isolated (the same trick may be used e.g. for any odd cycle). Using similar indicators (and subindicators, and edge - subindicators) The following has been proved by Hell and Nesetril in 1987 [30]:

Theorem 3.1. *For a graph H the following two statements are equivalent :*

1) *H is non-bipartite ;*

2) *H-coloring problem is NP-complete.*

And this theorem (and its proof) have some particular features which we are now going to explain:

(a) The result claimed by the theorem is expected. In fact the result has been conjectured in [55], and elsewhere, but it took nearly 10 years before the conjecture had been verified.

(b) How the statement is expected the proof is unexpected.

What one would expect in this situation? Well we should prove first that C_{2k+1}-coloring is NP-complete (which is easy and in fact we sketched this above) and then we would "observe" that the problem is hereditary :
If H-coloring problem is NPC and $H' \supseteq H$ then also H'-coloring problem is NPC.

This statement may sound plausible but there is not known a direct proof of this statement. It is certainly a true statement (by virtue of Theorem) but the only known proof is via proof of Theorem. In fact there may be here more than meets an eye : for oriented graphs the analogy of this statement does not hold (Gutjahr was first to give a counter example) .

(c) Having said that, we should point out that (as it stands) Theorem 3.1 fails to the true. We have to assume that all graphs are undirected.

One can construct easily an orientation $\vec{H}$ of bipartite graph H such that $\vec{H}$ - coloring problem is NP-complete. Even more so, one can construct a balanced oriented graph H with the same property (That H-coloring problem is NP-complete; an oriented graph is called balanced if every cycle has the same number of forwarding and backwarding arcs).

This can be done using again the indicator technique: Let (I, a, b) be the indicator which is a path oriented in such a way that

1) I has an automorphism which exchanges a and b;
2) Every homomorphism $G*(I, a, b) \longrightarrow H*(I, a, b)$ induces a homomorphism $G \longrightarrow H$. (G, H are undirected graphs).

Once these conditions are spelled out it is easy to satisfy them. In fact an example such an indicator is depicted on Fig. xx above.

But then as stated $G \longrightarrow H$ iff

$$G * (I, a, b) \longrightarrow H * (I, a, b)$$

and thus e.g. $K_3 * (I, a, b)$ is an NP-complete problem.

Now one can go further and in this way one can ommit in H all cycles (not a necessarily oriented) of short lengths. But it is perhaps bit surprising that one can ommit all cycles. Namely, one has the following proved in [21] and [34]:

Theorem 3.2. *There are oriented trees T (i.e. T is an orientation of an undirected tree) such that T-coloring problem is NP-complete.*

In [21] such a tree with 258 vertices has been constructed while in [34] a tree T_0 with 45 vertices with the same property has been found. This examples is depicted on Fig.

We see that T_0 has a very simple structure (and it is called triad in [34], [33]). We should remark that 45 is not such a large number here. T_0 has to be a core and this already implies that T_0 has at least 15 vertices and this is just a very first estimate.

(d) Thus for oriented graphs we face a much more complicated situation. Even for special classes, very special classes indeed. For example, the following are presently open problems :

Problem 6 *Characterise oriented trees T for which T-coloring problem is NP-complete.*

(This is open even for triads. Triads are in a way minimal examples as P-coloring problem is polynomial for every oriented path P, see [35] and the following section.)

It seems that the problem lies in "sparse" graphs. On the other side of the spectrum, it has been shown the following

Theorem 3.3. *For a tournament T (i.e. T is an orientation of a complete graph) The following two statements are equivalent :*

(1) *T-coloring problem is NP-complete;*

(2) *T contains two directed cycles.*

(Bang- Jensen, Hell and MacGilliwray in fact prove the same result for "semi-complete graphs".)

(e) But in general the H-coloring problem seems to be a very hard problem. Presently there is no conjecture which should capture NP-completeness instances of H-coloring problem.

But may be there is even no such conjecture.

There is some evidence for this. For examples, as was shown in [17] the H-coloring problem for relational systems (i.e. we allow more relations on the same set) is reducible to the H-coloring problem for oriented graphs. (Motivation to [17] research commes from data base - theory.)

[17] also posed the following problem :

Problem 7 *Is it true that the H-coloring problem for any graph H is either polynomially solvable or NP-complete ?*

A promising line of the research has been started with [9] where they characterized those coloring problems for which the counting of the number $\mathbf{h}(\mathbf{G}, \mathbf{H})$ of homomorphisms $G \to H$ is a hard problem.

Let us finish with that and let us turn to polynomially solvable instances.

3.2 Polynomial Cases and Homomorphism Dualities

Coloring problems have to be solved, we usually do not walk away. But how to approach them?

A standard approach in a combinational setting is to look for obstacles, for configurations which are obstructing our goal, in our case the desired homomorphism $G \longrightarrow H$. These obstructions can be special subgraphs as we have it in the bipartite case : $G \longrightarrow K_2$ iff G does not contain an odd cycle.

However as we are interested in homomorphism $G \longrightarrow H$ these forbidden subgraphs (obstructions) are closed on homomorphism too:

If $F \not\longrightarrow H$ and $F \longrightarrow F'$ then $F' \not\longrightarrow H$.

Let us approach this more formally: We introduced already the class $\longrightarrow H = \{G; G \longrightarrow H\}$. The complementary class $\{G; G \not\longrightarrow H\}$ will be denoted by $\not\longrightarrow H$. As we just observed $\not\longrightarrow H$ is closed on homomorphism:

$F \in (\not\longrightarrow H)$, $F \longrightarrow F'$ implies $F' \in (\not\longrightarrow H)$.

Thus there exists a set $\mathcal{F}$ of graphs such that $\not\longrightarrow H = \{G; F \longrightarrow G$ for some $F \in \mathcal{F}\}$. The latter class will be denoted by $\mathcal{F} \longrightarrow$. Explicitely, $\mathcal{F} \longrightarrow$ consists from all graph G for which there exists a graph $F \in \mathcal{F}$ which is homomorphic to G.

Similarily, we denote by $\mathcal{F} \not\longrightarrow$ the class of all graph G for which no member $F \in \mathcal{F}$ is homomorphic to G.

Thus we have equality

$$\mathcal{F} \not\longrightarrow = \longrightarrow H$$

.

We observe that for any H there exists a family $\mathcal{F}$ such that (2) holds. Simply take $\mathcal{F} = \not\longrightarrow H$. But our goal is more demanding : we would like to find a simple family $\mathcal{F}$, if possible such a family where the membership of the class $\mathcal{F} \longrightarrow$ would be easy to prove.

Theorems which have structure as in (2) are called *Homomorphism Duality Theorems.*

We shall give some examples to make the duality point of view more explicit:

A typical example is the case of oriented paths. According to [35], a digraph G is homomorphic to an oriented path P if and only if each oriented path P' homomorphic to G is also homomorphic to P. Thus in this case the obstructions are oriented paths P' homomorphic to G but not to P. To make this obstruction point of view more explict we restate the characterization (for the case when P is an oriented path) as follows: *A diagraph G is not homomorphic to P if and only if there exists an oriented path P' which is homomorphic to G but not to P.*

In the other words we have *Path Duality* (proved in [35])

Theorem 3.4. *Let P be an oriented path. Then $\mathcal{P} \not\longrightarrow = \longrightarrow P$ where $\mathcal{P}$ is the family of all paths P' which are not homomorphic to P.*

Another class of digraphs with a similar characterization theorem is the class of unbalanced cycles. An unbalanced cycle is an oriented cycle in which the number of forward edges is different from the number of backward edges (with respect to some fixed traversal of the cycle). According to [36], *a digraph G is not homomorphic to an unbalanced cycle C if and only if there is an oriented cycle C' homomorphic to G but not homomorphic to C.*

In the other words we have *Cycle Duality* (proved in [36]):

Theorem 3.5. *Let C be an unbalanced cycle. Then $\mathcal{C}\dagger \not\longrightarrow = \longrightarrow C$*

where $\mathcal{C}\dagger$ is the family of all cycles C' which are not homomorphic to C.

We know that H-coloring problems can be hard even when H is an oriented tree T. However, there are also many oriented trees T for which there is structure to the T-coloring problem, which can be exploited to find a polynomial algorithm. The class of digraphs homomorphic to such a "nice" oriented tree T can be often characterized by the absence of certain subtrees.

Specifically, we say that H-coloring problem has *tree duality* if the following property holds for all digraphs G: *A digraph G is not homomorphic to H if and only if there exists an oriented tree homomorphic to G but not to H.* Tree duality seems to be a surprisingly useful property. In particular, one can prove that if H has tree duality then the H-coloring problem is polynomial [33].

This fits to our scheme (2): H has a tree duality iff

$$\mathcal{T} \not\longrightarrow = \longrightarrow H$$

where $\mathcal{T}$ is the set of all trees F which are not homomorphic to H.

The class of digraphs H with polynomial H-coloring problems can be further enlarged by generalizing tree duality to treewidth-k duality and to Bounded Tree Width Duality.

First, let us give some definitions:

An undirected graph is a k-tree is a k-tree if its maximal clique is of size $k+1$ and it does not contain an induced cycle of length > 3. (It follows from basic graph theory that k-trees have indeed a tree structure;a k-tree can be obtained from a $(k+1)$-clique. by repeatedly adding a vertex joined to existing vertices which form a k-clique. (Thus a tree is a 1-tree.) An undirected graph is said to *have treewidth k*, or to be a *partial k-tree*, if it is a subgraph of a k-tree. This is denoted by $tw(G) \leq k$. Partial k-trees have small separating sets and as a consequence they admit efficient algorithms for many hard computational problems see e.g. [43]. We say that an oriented graph has *treewidth k* (or is an oriented partial k-tree) if its underlying undirected graph has treewidth k.

Definition 4 *We say a digraph H has treewidth-k duality if the following property holds for all digraphs G: A digraph G is not homomorphic to H is and only if there exists an oriented partial k-tree homomorphic to G but not to H.*

An H-coloring problem is said to have *bounded treewidth duality* if there exists a positive integer $k = k(H)$ such that the following holds:

G is homomorphic to H if and only if every graph F homomorphic to G with treewidth $\leq k$ is also homomorphic to H.

This fits to our scheme (2):
H has k-treewidth duality iff $\mathcal{T}_k \not\longrightarrow = \longrightarrow H$
where $\mathcal{T}_k$ is the set of all partial k-trees F which are not homomorphic to H. Explicitely: For every graph G the non-existence of a homomorphism $G \longrightarrow H$ is equivalent to the existence of an F, $tw(F) \leq k$ such that $F \to G$.

The following has been proved independently and in different context in [33] and [17]:

Theorem 3.6. *Every H-coloring problem with bounded treewidth duality is polynomial time decidable.*

Presently Theorem 3.6 is the strongest tool for proving the polynomial time decidability of H-coloring problems. In fact, presently all known polynomial time decidable H-coloring problems are covered by Theorem x.

On the other hand, if we assume $P \neq NP$, then all NP-complete H-coloring problems do not possess bounded treewidth duality.

This line was pursered in [72] where the following has been allowed directly (i.e. without the assumption $P \neq P$) :

Theorem 3.7. *For an undirected graph H, the H-coloring problem has no bounded treewidth duality if and only if H contains an odd cycle.*

Also for some directed graphs H one can obtain similar results. For example one can prove the following (see [72]) :

Theorem 3.8. *There exists an oriented cycle C such that C-coloring problem has no bounded tree width duality.*

However the following is still open:

Problem 8 *Without $P \neq NP$ prove that for every k there exists an oriented tree T such that T-coloring problem has no bounded tree width duality?*

Let us finish this part with another problem.

First let us recall another results proved in [72]:

Theorem 3.9. *Given two positive integers k and m. If G is a graph of girth $n > 2^{k+2}(4km)^{4km-1} + 2(k+1)$ then any partial k-tree homomorphic to G is also homomorphic to the odd cycle C_{2m+1}.*

In the language of circular chromatic number χ_c(or star chromatic number) this implies $\chi_c(G) \leq 2 + \varepsilon$ for any large girth graph with bounded tree width. A related result in this direction is that any large girth planar graph is homomorphic to a given odd cycle. Quite surprisingly the similar results do not hold for bounded degree and even cubic graphs. We have the following (proved recently in [46]).

Theorem 3.10. *For any $g \geq 3$ and any $l \geq 10$ there exists a cubic graph G with the following properties:*

1) *G has girth $\geq g$*

2) *$G \not\longrightarrow C_{2e+1}$*

The non-constructive proof of [46] leaves the following open:

Problem 9 *Is it true that any large girth cubic graph G is homomorphic to C_5?*

(This problem was also discussed in [18].)

It is proved in [42] that the answer to this problem is negative for 4-regular graphs.

3.3 Gaps and Dualities

Let us return to our main theme:

All the above results fall into the framework of Homomorphism Dualities. The following is perhaps the simpler instance of such homomorphism duality:

A *singleton homomorphism duality* is a pair of graphs (F, H) satisfying

$$F \not\longrightarrow = \longrightarrow H$$

.

Similarly, *finitary homomorphism duality* is a pair $(\mathcal{F}, H)$, where $\mathcal{F}$ a finite set of graphs, satisfying

$$\mathcal{F} \not\longrightarrow = \longrightarrow H$$

.

In this case we also say that H has *finitary homomorphism duality.*

At the first glance this scheme seems to be too restrictive and indeed it is, at least for undirected graphs. The following result was proved essentially in [65].

Theorem 3.11. *For an undirected graph H the following two statements are equivalent:*

1) *H has finitary homomorphism duality*

2) *Either $H = \phi$ or $H = K_1$.*

Thus for undirected graphs only trivial finitary homomorphism dualites exist:

$$K_1 \not\longrightarrow = \longrightarrow \phi \text{ and } K_2 \not\longrightarrow = \longrightarrow K_1$$

.

We include the proof (given in [65]) as it uses one of the basic trics in this area:
Proof.

Clearly it suffices to prove that there are no other dualites. Assume the counary, so let $\mathcal{F} \not\longrightarrow = \longrightarrow H$ be a finitary duality for a non-bipartite H. Put $\mathcal{F} = \{F_1, \ldots, F_t\}$. Let l be larger than the shortest length of an odd cycle in any of graphs $F_1, \ldots, F_t$ (of course $F_i \not\longrightarrow H$). Now let $G = G_{k,l}$ be the Erdös graph with the following properties: $\chi(G) > k = |V(H)|$ and girth of $G > l$. Then both $G \not\longrightarrow H$ and $F_i \not\longrightarrow H$, which is a contradiction. ♡

However for other structures the singleton homomorphism dualities present more complex patterns and capture interesting theorems. For example for oriented matroids and (convenient version of) strong maps one can prove that singleton morphism duality give Farkas Lemma and one can even prove that no other such dualities (in this framework) exist, see [38].

Similarily, for ports with strong port mappings the only singleton duality is equivalent to Menger theorem, see [39].

Even for directed graph this situation is far more complicated and a full characterisation was achived (in a special case) by Komárek [45] and (in the full generality) by Nesetril, Tardif [68, 70].

Theorem 3.12. *(Characterisation of Singleton Dualities)*

Up to a homomorphic equivalence the only singleton dualities for oriented graphs have the following form:

$$T \not\longrightarrow = \longrightarrow H_T$$

where is an oriented tree and the graph H_T is uniquelly determined by T.

Theorem 3.13. *(Characterisation of Finitary Dualities)*

Up to a homomorphic equivalence the only finitary hommomorphism dualites for oriented graphs have form

$$\mathcal{F} \not\longrightarrow = \longrightarrow H$$

where $\mathcal{F}$ is a finite set of trees $\{F_1, \ldots, F_t\}$ and $H = \prod_{i=1}^{t} H_{F_i}$ where $F_i \not\longrightarrow = \longrightarrow H_{F_i}$.

We prove both of these results bellow in the next section. The proof involves (nearly) all tools which we developed in this paper.

3.4 Duality and Density (and Gaps)

Nešetřil and Tardif found in [68] and [70] the following (perhaps surprising) connection of duality pairs. This provided the key to the characterization not only of the dualities but also to the characterization of gaps for classes of directed (and undirected) graphs.

We say that a gap (G, H) is *connected* if H is a connected graph. Observe that if (F, H) is a duality pair then F is necessarily connected (for if F is a core and F_1 and F_2 are distinct components of F then $F \not\to F_i$, and thus $F_i \to H$ for $i = 1, 2$. Thus $F \to H$, which a contradiction).

Theorem 3.14 *There is a one-to-one correspondence between duality pairs and gaps for the class of directed (and also undirected) graphs. Explicitely, given a duality pair (F, H) then $(F \times H, F)$ is a gap. Conversely, given a connected gap (G, H) then (H, G^H) is a duality pair.*

Proof. First, suppose that $G \to H \not\to G$ is a gap. We prove that for any graph K holds $H \not\to K$ iff $K \to G^H$. Thus let a graph K be such that $H \not\to K$ and suppose for the contrary that $K \not\to G^H$. Then obviously $G \to G \cup (G \times H) \to H$. If $H \to G \cup (K \times H)$ then (by the connectivity and the assumption $H \not\to G$) $H \to K \times H$ and thus $H \to K$,contrary to our assumpiton. If $G \cup (K \times H) \to G$ then $K \times H \to G$ and $K \to G^H$ again contrary to our assumption. Thus the class $H \not\to$ is a subclass of the class $\to G^H$. In order to prove the reverse inclusion let K be a graph satisfying $K \to G^H$ and $H \to K$. This implies $H \to H \times G^H \to G$ (as $H \times G^H \to G$ is equivalent to $G^H \to G^H$ and thus it always holds). Thus (H, H^G) is a duality pair.

Conversely, let (F, H) be a duality. We may clearly assume that F is a core (i.e. every homomorphism $F \to F$ is an automorphism) and further F is connected. Thus $F \times H \to F$ and $F \not\to F \times H$ (as $F \to F \times H$ would imply $F \to H$). We claim that there is no graph K satisfying $F \times H \to K \to F$ and $F \not\to K \not\to F \times H$: If $K \to F \not\to F \times H$ then the duality implies $K \to H$ and thus $K \to F \times H$ which contradicts our assumptions. This completes the proof of theorem. ♡

This Theorem 3.14 leads to yet another proof of Density Theorem 2.15:

Proof. By 3.11 (K_2, K_1) and (K_1, K_0) are the only duality pairs. Thus $(K_2 \times K_1)$ (which is equivalent to (K_1, K_2)) and $(K_0 \times K_1, K_1)$ (which is equivalent to (K_0, K_1) are the only gaps $\heartsuit$

Thus, modulo the above "arrow calculus" involved in the above proof of 3.14, the density theorem has been known even before it has been formulated.

Finally, let us remark that the above also shows that Theorem 3.13 gives all non-gap pairs for directed graphs: Let (T, H_T) be all singleton duality pairs for oriented graphs (characterized by Theorem 3.12). Then $(T \times H_T, T)$ are exactly all gaps.

Theorem 3.13is in neither direction an easy result:

The construction of the graph H_T is quite complicated and it has been approached from different sides in [45] and in [70].

Here we show the opposite direction (that there are no other gaps). This in fact solves the density problem for oriented graphs by a remarkably easy construction which we have already introduced in a different context - it is our indicator construction again.

Theorem 3.15
Let G, H be directed graphs which are cores. Let H be connected and assume that H fails to be an orientation of a tree. Further assume that $G \to H \not\to G$ holds. Then there exists a directed graph K with $G \to K \to H$ and $H \not\to K \not\to G$.

We shall make use of the following obvious (but key) property of the arrow construction:

Lemma 2
Let G and H be directed graphs with $\chi(G) > |V(H)|$ and let every homomorphism $f : I \to H$ satisfies $f(a) \neq f(b)$. Then $G \star (I, a, b) \not\to H$.

Proof. [Third proof of Density Theorem for Undirected Graphs]
Let G, H be undirected graphs, H non bipartite, with $G \to H \not\to G$. Clearly we may assume that G and H are cores. Let $e = \{a, a'\} \in E(H)$ belong to a circuit in H. Put $I = H - e + \{a', b\}$ where $b \notin V(H)$. (Thus I arises from H by deleting the edge e, adding a new vertex $b \notin V(H)$ together with the edge $\{a', b\}$.)

It is clear that $I \to H$ (identifying vertices a and b) but any homomorphism $f : I \to G$ satisfies $f(a) \neq f(b)$ (for otherwise we get a contradiction with $H \not\to G$). Now let F be any graph satisfying $\chi(F) > |V(G)|$ and let F' be any orientation of F. Consider the arrow construction $F' \star (I, a, b)$ and define the graph K by $K = (F' \star (I, a, b)) \cup G$.

We prove that K has properties claimed by Undirected Graph Density: Clearly $G \to K$. We also have $K \to H$ as the mapping f defined by $f([e, x]) = x$ for

$x \in V(H)$ and $e \in E(F')$ and $f([e,b]) = a$ is a homomorphism $K \to H$ (we preserve the above notation concerning the arrow construction $F' \star (I, a, b)$). Further, by the above Lemma 2, $K \not\to G$ (as $\chi(F) > |V(G)|$). Thus it remains to be shown that $H \not\to K$. Suppose the contrary and let $g : H \to F \star (I, a, b)$ be a homomorphism. Then $f \circ g : H \to H$ where f is the above defined homomorphism $F \star (I, a, b) \to H$. As H is a core $f \circ g$ is a homomorphism. Put $h = (f \circ g)^{-1}$. Then $f \circ g \circ h(x) = x$ for every $x \in V(H)$. Put $g \circ h(a) = [(e, a)]$ with $e = (u, v)$. Then the image $g \circ h(H)$ of H is a connected subgraph of $F \star (I, a, b)$ which is (by the injectivity of the mapping $f \circ g \circ h$) contained in the set of all $[(e', x)]$ where e' is incident with u and $x \in V(I)$ (this set is the "star" induced by those edges of F' which are incident with the vertex u). But then the edge $\{[g \circ h(a)], [g \circ h(a')]\}$ is a cut edge in the graph $g \circ h(G)$ which is the final contradiction as a, a' was contained in a cycle of H. ♡

Proof. [of Theorem 3.15]

Let G, H satisfy the assumption of the theorem. Let H be a core and let $(a, a') \in E(H)$ belong to a cycle in H. Put $I = H - (a, a') + (b, a')$ where $b \notin V(H)$ (i.e. we first delete arc (a, a') and then add a new vertex b together with the arc (b, a')). Let F be an oriented graph with $\chi(F) > |V(G)|$ and consider the arrow construction $F \star (I, a, b)$. Put $K = G \cup (F \star (I, a, b))$. Then we have:

- $G \to K$ (by the inclusion map);
- $K \to H$ (by the same mapping as in the above proof);
- $K \not\to G$ (by the chromatic number assumption);
- $H \not\to K$ (as above in the proof for undirected graphs).

Thus the graph K has the desired properties. ♡

3.5 Final View

Our approach to H-coloring problem may be put in various context. We list in these closing remarks three such approaches.

3.5.1 Good Characterizations

Finitary good characterizations are examples of *Good Characterizations* in the sense of Edmonds [12]: Given a finitary duality

$$\mathcal{F} \not\to = \to H \tag{5}$$

we can proof easily that a given graph G is not H-colorable. We simply check which graph F of the finite set F permits a homomorphism $F \to G$. This obviously takes polynomially many steps (and in fact one can do so in $o(n^{k\omega/3})$ steps where ω is the fast matrix multiplication constant and $k = max\{|V(F)|; F \in \mathcal{F}\}$, see [64]). On the other side the existence of an H-coloring is easy to verify.

In the previous sections (Theorem 3.13) we characterized all finitary dualities for coloring problems for graphs. Note that the main result may be extended to relational systems and even to the relational structures of a given type (i.e. to finite models of a given type).

Finitary good characterizations are examples of *Good = Characterizations* in the sense of Edmonds [12]: Given a finitary duality

$$\mathcal{F} \not\to = \to H, \tag{6}$$

we can prove easily that a given graph G is not H-colorable. We simply check which graph F of the finite set $\mathcal{F}$ permits a homomorphism $F \to G$. This obviously takes polynomially many steps (and in fact one can do so in $O(n^{k\omega/3})$ steps, where ω is the fast matrix multiplication constant and $k = \max\{|V(F)|; F \in \mathcal{F}\}$; see [50]). On the other side, the existence of an H-coloring is easy to verify.

In Theorem 3.13, we characterized all finitary dualities for coloring problems for graphs. Note that the main result may be extended to relational systems and even to the relational structures of a given type (i.e., to finite models of a given type).

Despite this generality, we see that Theorem 3.13 is very special (as one can "forbid" relational trees only). This is in a sharp contrast with the abundance of finitary dualities if other "morphisms" are allowed. For example, (as follows from Robertson - Seymour - Thomas project) every minor closed property has "finitary duality" (with morphisms being minors). However, note that most of these results are related to undirected graphs only. Other examples for matroids are given in [32, 33].

3.5.2 hom - Universal Graphs

One of the fundamental results of P. Erdös [11] can be formulated as follows:

Theorem 3.16. *For a finite graph F, the following two statements are equivalent*:

(1) *There exists a $k = 3Dk(F)$ such that any graph G with $\chi(G) \geq k$ contains F as a subgraph.*

(2) *F is a forest.*

This can be expressed also as a weaker form of homomorphism duality:
For a finite graph F, the following two statements are equivalent:

(1′) *There exists a graph H such that $\{G; F \not\subset G\} \subset \{G; G \to H\}$.*

(2′) *F is a forest.*

If condition ($1'$) is valid, then we say that the graph H is *hom-universal* for the class of all F-free graphs.

In this setting, Theorem 3.12 presents an extension of the Erdös result to forbidden homomorphisms: In the case of a tree F, the class $\{G; F \not\to G\}$ has not only an universal graph but it can be *defined* by = homomorphisms *into* a fixed graph. More precisely, there exists a graph H_F such that

$$F \not\to = \{G; F \not\to G\} = \{G; G \to H\} = \to H.$$

(Let us stress at this moment that this holds in the full generality for finite structures. It is perhaps a bit surprising that by forbidding homomorphisms, i.e., by forbidding a graph F together with all its homomorphic images, we get so much more structure.)

3.5.3 Bounded Degree Graphs

Universal graphs obviously exist for bounded degree graphs: If $\Delta(G) \leq k$, then $G \to K_{k+1}$. It has been proved by R. Häggkvist and P. Hell in [19] that for *any* graph F, there exists a graph $U_{F,k}$, $F \not\to U_{F,k}$, such that $G \to U_{F,k}$ for any graph G with $\Delta(G) \leq k$ and $F \not\to G$.

This has been extended recently [9] as follows:

Theorem 3.17. *For every choice of graphs F, H, $F \not\to H$, there exists a graph $U_{F,k,H}$ such that*

1) *$F \not\to U_{F,k,H}$ and $U_{F,k,H} \to H$, and*
2) *if G is a graph with $\Delta(G) \leq k$, $F \not\to G$, and $G \to H$, then $G \to U_{F,k,H}$.*

Particularly, there exists a 3-chromatic triangle free graph U such that $G \to U$ for every traingle free, cubic, 3-chromatic graph G.

The universal graphs for cubic graphs are related to several interesting problems. The following one attracted recently some attention.

Problem 10 (Pentagon Problem). *Is it true that there exists a constant k such that for any cubic graph with girth $\geq k$ there is a homomorphism $G \to C_{2k+1}$?*

In this context, one should note that this problem has a negative solution for homomorphisms into C_{11} (instead of C_5) as showed in [39] and also for graphs with maximal degree 4 as showed in [35].

The Pentagon Problem is motivated both by some complexity considerations [16] and by attempts to solve the Density Problem for cubic graphs. Particularly, even the following seems to be presently open:

Given a cubic graph G, $C_5 < G$, prove that there exists a cubic graph H such that $C_5 < H < G$.

From the negative solution of the Pentagon Problem follows the existence of H easily (along the lines of the Second Proof of Density of Undirected Graphs).

Let us finish this survey by the following three recent problems (raised in:

Problem 11. *Does there exist a triangle free graph U_3 such that any triangle free planar graph G is homomorphic to U_3: $G \to U_3$?*

Problem 12. *Does there exist a K_4-free graph U_4 such that any K_4-free planar graph G is homomorphic to U_4: $G \to U_4$?*

Problem 13. *Does there exist a K_5-free graph U_5 such that any planar graph G is homomorphic to U_5: $G \to U_5$?*

Well, this is as far we can go. The answer to the Problem 13 is already positive: U_5 certainly exists - take $U_5 = K_4$. This is 4-Color Theorem. Still there is an interesting problem even here: can one construct U_5 without using (involved and "insecure") 4-Color Theorem? The Problem 13. seems somehow to be closer to the 5-Color Theorem, but this may be an illusion.

4 Appendix 1 - On Extendability and Universality Propoerties of $\mathcal{C}$.

When the first version of this paper has been written in 1999 the universality of the coloring poset $\mathcal{C}$ has been a complicated result without a combinatorial proof (or combinatorialy satisfactory proof).

By now we can present such a proof and we shall include it in this appendix. This proof put the question of the universality of $\mathcal{C}$ in yet another setting. This proof is taken from [63].

Let us state explicitely the universality property of $\mathcal{C}$:

Theorem 4.1 *Every countable poset is an induced subposet of $\mathcal{C}$.*

More formally, for every countable poset (X, R) we can find an injective mapping which assigns to every $x \in X$ a finite graph G_x such that $(x, y) \in R$ if and only if $G_x \leq G_y$.

In the other words $\mathcal{C}$ is countable *universal* poset (for all countable posets).

Thus the poset $\mathcal{C}$ plays a similar role as the *Rado graph* $\mathcal{R}$ which is the countable universal graph. This property (of Rado graph) is easy to prove as the Rado graph has the *graph extension property*: for every finite graph G and any $x \in V(G)$ holds: any subgraph of $\mathcal{R}$ isomorphic to $G - x$ may be extended to a subgraph isomorphic to G. By the same token we can construct universal countable poset $\mathcal{U}$: we start with the singleton poset and having constructed a finite portition $\mathcal{U}_n$ of $\mathcal{U}$ we let $\mathcal{U}_{n+1}$ be the poset which is formed by all possible singleton extensions of of $\mathcal{U}_n$. $\mathcal{U}$ will be formed as the union of all $\mathcal{U}_n$. (This is a bit sletchy but this is fairly standard, see e.g. [14]. The class of all posets has (unbounded) extension property.)

However the situation is far from beeing so simple for poset $\mathcal{C}$. The poset $\mathcal{C}$ has substantially more algebraical structure than $\mathcal{U}$ (e.g. it is a lattice with respect to products and sums; it has also powers; all this will be explained in a greater detail below) and as a result of this the extension property does not hold for $\mathcal{C}$ in a very strong sense.

The proof of Theorem 4.1 is complicated. In fact it took several papers (see e.g. [23]) to achieve it (and the validity of Theorem 4.1 was a long-standing open problem, see e.g. [28, 75]). The solution was achieved only in the broader context of embedding of categories where one introduces several (many) intermediate steps. The whole proof takes cca 25 pages in the monography [75] and it is one of the main results of this book (and also [75] is the only available proof in print).

Here we give an independent and direct (and we believe a simpler) proof of Theorem 4.1 which grew out of the study of extension properties of $\mathcal{C}$.

Perhaps more importantly this proof does not use many ad hoc ideas and instead (as we are going to demonstrate, see e.g. Section 2) it is related to some of the fundamental properties of $\mathcal{C}$ - *density, product conjecture and extendability* of the class $\mathcal{C}$. It follows that the class of graphs which represent $\mathcal{C}$ can be built from general ("Erdős-type graphs") which can be generated at random.

4.1 Extension and Cancelation Properties of $\mathcal{C}$

The proof relies on the new approach to embedding theorems where we want to extend (on-line) the existing representation. This is called extendability (or extension property) and as we shall see this includes several problems and results which we studied earlier.

A poset $\mathcal{P}$ is said to be *k-extendable* (or to have *k-extension property*) if any its subposet P with k elements and for any poset P' with $k+1$ elements which contain P as an induced subposet (i.e. P' is a single point extension of P) one can find a copy of P' in $\mathcal{P}$ which contains P (i.e. we can get P' from P by addition of one element of $\mathcal{P}$). $\mathcal{P}$ is said to be *extendable* (or to have *k-extension property*) if it is k-extendable for every positive k.

The k-extension properties of coloring poset $\mathcal{C}$ are interesting already for small values of k and they were studied under different names extensively. Perhaps the k-extendability is a convenient classification for this type of results. However here we give only examples which are related to our main theme.

Extending any poset given by graphs G_1, G_2 to a poset with graphs G_1, G_2, H where H is homomorphic to both G_1 and G_2 is easy: let H be categorical (or direct) product $G_1 \times G_2$. Similarly, extending any poset given by graphs G_1, G_2 to a poset with graphs G_1, G_2, H where both G_1 and G_2 are homomorphic to H is easy: let H be the disjoint union $G_1 + G_2$.

However 2-extendability also includes the case of an extension of poset $G_2 \to G_1 \not\to G_1$ to a poset with graphs G_1, G_2, H where $H \to G_1$ while there is no homomorphism between G_2 and H.

H always exists and in fact it can be chosen with arbitrarily large girth. This result is called *Sparse Incomparability Lemma* and it was proved in [66] (and independently in [85], see [60]).

Let us consider 2-extendability for the case of an extension of poset $G_2 \to G_1 \not\to G_1$ to a poset with graphs G_1, G_2, H where $G_2 \to H \to G_1$. This is called *density problem* and the answer can be negative: consider $G_1 = K_1$ and $G_2 = K_2$. In such a case we say that the pair (G_1, G_2) forms a gap. Welzl [82] proved that for undirected graphs there are no other gaps. However gaps are abundant for oriented graphs (and relational structures) and they have been completely characterized only recently C. Tardif and the author [69], see also [68].

3-extendability includes the following situation: Given graphs G_1, G_2, G_3 such that $G_i \not\to G_3$, $i = 1, 2$, does there exists a graph H such that $H \to G_i$, $i = 1, 2$, while $H \not\to G_3$. In the other words we want $H \to G_1$, $H \to G_2$, and $H \not\to G_3$. However $H \to G_1$, $H \to G_2$ implies $H \to G_1 \times G_2$ and thus this is the same question as to ask whether $G_1 \times G_2 \not\to G_3$ providing that $G_i \not\to G_3$, $i = 1, 2$. Such graphs G_3 are called *productive* [65] (as the class of all graphs G which are not homomorphic to G_3 is productive; sometimes the name *multiplicative* graphs is used). The explicite characterization of productive graphs is a very difficult problem already in the "simplest" instances $G_3 = K_k$. This particular case is known as the *Hedetniemi - Lovász problem.* There are infinitely many non-productive graphs: these include of course graphs $G_1 \times G_2$ where G_1 and G_2 are not homomorphic to each other, but also particular instances of Kneser graphs (e.g $K\binom{3k}{k}$; see [77]), and infinitely many further examples. For example we can take graph $K_k \times G$ where G is any graph vertex critical graph with $\chi(G) > k+1$. In the graph $K_k \times G$ collapse the set $V(K_k) \times \{x\}$ into a single vertex, say x, for all x in an independent set A of the graph G. Call the resulting graph H. H is not productive.

Among 3-extension properties the productivity is the only essential problem.

The categorical (or direct) product is denoted by $\times$, the product of more factors is denoted by $\prod_{i=1}^{n} G_i$ or $\prod_{i\in I} G_i$. The disjoint union of graphs G and H is denoted by $G+H$ and for more factors we use $\sum_{i=1}^{n} G_i$ or $\sum_{i\in I} G_i$.

Let us consider the following instance of $(m+n)$-extension property:

Given graphs $G_1, G_2, \ldots, G_m, H_1, H_2, \ldots, H_n$, $G_i \to H_j$ while $H_j \not\to G_i$ for every $1 \le i \le m, 1 \le m \le n$, we want to find a graph F such that $G_i \to F \to H_j$ simultaneously fro all $1 \le i \le m, 1 \le m \le n$.

Clearly for $m = n = 1$ this is the density problem and thus we can call the above problem (m,n)-*density problem.* The (m,n)-density problem is possible to reduce to $(1,1)$-density problem by simply considering graphs $G = \sum_{i=1}^{n} G_i$ and $H = \prod_{i=1}^{n} H_i$ for it is easy to see that a necessary condition for the existence of graph F is $\prod_{i=1}^{n} H_i \not\to \sum_{i=1}^{n} G_i$ (and this relates the (m,n)-density problem to productive properties of graphs H_i).

We state the (m,n)-density result explicitely as it inspired the proof of the main result (Theorem 6.1) of this paper. (Another motivation came from unpatience that such a fundamental result as Theorem 4.1 did not have an accessible proof which we badly needed for a forthcomming book with P. Hell and X. Zhu).

Proposition 1 *((m,n)-density)*

Given graphs $G_1, G_2, \ldots, G_m, H_1, H_2, \ldots, H_n$, $G_i \to H_j$ for all $1 \le i \le m, 1 \le m \le n$. while $\prod_{i=1}^{n} H_i \not\to \sum_{i=1}^{n} G_i$. Then there exists a graph F such that $G_i \to F \to H_j$ and $H_j \not\to F \not\to G_i$ simultaneously for all $1 \le i \le m, 1 \le m \le n$.

Proof. Put $a = \sum_{i=1}^{m} |G_i|$ and $b = \prod_{j=1}^{n} |V(H_j)|$ and let H be any graph with $\chi(H) > a^b$ and without odd cycles of length $\le b$.

Define F by the following formula:

$$F = (\sum_{i=1}^{m} G_i) + ((\prod_{j=1}^{n} H_j) \times H) \tag{6}$$

Then the existence of homomorphism $G_i \to F \to H_j$ is clear while $H_j \not\to F$ follows by the girth assumption. $F \not\to G_i$ may be obtained as follows: To the contrary assume that $h : ((\prod_{j=1}^{n} H_j) \times H) \to \sum_{i=1}^{m} G_i$ and define the mappings $h_y : V(\prod_{j=1}^{n} H_j)) \to V(\sum_{i=1}^{m} G_i)$ by $h_y(x) = h(x,y)$. The mapping h_y need not be a homomorphism but if we interpret the mapping h_y as a color of the vertex y we get that there are vertices y, y' which form an edge in H such that $h_y = h_{y'}$. However then the mapping $g = h_y = h_{y'}$ is necessary a homomorphism $g : \prod_{j=1}^{n} H_j \to \sum_{i=1}^{m} G_i$ (as $\{g(x), g(x')\} = \{h_y(x), h_{y'}(x')\} \in E(F)$ for any edge $\{x,y\} \in E(\prod_{j=1}^{n} H_j)))$. This is a contradiction with our assumption. ♡

The following result seems to be a usefull generalization of the previous Proposition 1 and of particular cases which appeared in e.g [20, 66, 68, 72]. A graph G is said to be *pointed* for a graph F if for any tho homomorphisms $f, f' : F \to G$ and for any vertex $x \in V(G)$ holds:

If $f(y) = f'(y)$ for every $y \neq x$ then also $f(x) = f'(x)$.

We have:

Theorem 4.2 *(Homomorphism Cancelation Property)*

Let F, G be graphs, let H be any graph with $\chi(H) > |V(F)|^{|V(G)|}$. Then

i. If there exists a homomorphism $h : G \times H \to F$ the there is a homomorphism $g : G \to F$.

ii. If G is connected and F is pointed for G then the homomorphism g is uniquelly determined by h.

Proof. The proof of *i.* is similar to the proof of Proposition 1.

To prove *ii.* it suffices to prove the following: if h_y and h_z are homomorphisms and $\{y, z\} \in E(H)$ then $h_y = h_z$.

Thus assume to the contrary that there are homomorphisms $h_y, h_z : G \to F$ such that $h_y(x_0) \neq h_z(x_0)$ for a vertex $x_0 \in V(G)$. Define mapping $h : V(G) \to V(F)$ as $h(x) = h_y(x)$ for $x \neq x_0$ and $h(x_0) = h_z(x_0)$. Then h is a homomorphism $G \to H$. (It suffices to check edges of G incident with x_0: If $\{x, x_0\} \in E(G)$ then $\{(x, y), (x_0, z)\} \in E(G \times H)$ and thus $\{h_y(x), h_z(x_0)\} = \{h(x), h(x_0)\} \in E(F)$.) However this is a contrary with the fact that F is G-pointed. ♡

Note that if all degrees of G are > 2 and F has no C_4 then F is pointed for G, so the conditions of Theorem 4.2 are easy to satisfy.

We shall use Homomorphism Cancelation Property (HCP for short) 4.2 many times in this paper.

Also the following notion and property will be needed in the proof of Theorem 6.1.

A (finite or infinite)set $\{S_i : i = 1, 2, \ldots, n, \ldots\}$ of graphs is said to be *inverse set* if for any two distinct indices i, j holds: For $i < j$ there is no homomorphisms $S_i \to S_j$ while for $j > i$ here is a homomorphisms $S_j \to S_i$.

Proposition 2 *For every k and every increasing sequence $(l_i; i = 1, 2, \ldots)$ of integers ≥ 3 there exists an inverse set $\{S_i : i = 1, 2, \ldots\}$ of connected graphs and vertices $s(i) \in V(S_i)$ such that each S_i satisfies:*

i. $\chi(S_i) = k$;

ii. The odd girth of S_i is l_i;

iii. For every $j > i$ there exists a homomorphism $f : S_j \to S_i$ such that $f(s(j)) = s(i)$.

Proof.

We construct S_i on-line as follows.

Let $S_1, S_2, \ldots, S_{n-1}$ with vertices $s(1), s(2), \ldots, s(n-1)$ be already constructed inverse set. Put $a = \sum_{i=1}^{n-1} |G_i|$ and let H be any graph with $\chi(H) > a^a$ and without odd cycles of length $\leq a$. Define S_n by the formula:

$$S_n = S_{n-1} \times H, \tag{6}$$

Let $s(n)$ be any vertex in the set $\{s(n-1)\} \times V(H)$.

Then for $i < n$ there is no homomorphism $S_i \to S_n$ (by the odd girth assumption) and for $n > j$ the projection to S_{n-1} together with induction assumption gives a homomorphism $S_n \to S_j$. ♡

5 On - Line Representations

Whole America can be On - Line. Our goal is a bit more modest and it is captured by the following definiton:

By an *on-line representation* of a poset $\mathcal{P}$ we mean that one can construct a representation of $\mathcal{P}$ under the circumstances that the elements of $\mathcal{P}$ are revealed one by one (without a priori knowledge about the whole poset $\mathcal{P}$). (The on-line representability by paths was considered in [71].

The on-line representation of can be thought as a game of two players A and B (with usual names Bob and Alice). Bob (as usual - the destroyer) selects $\mathcal{P}$ and reveals the elements of $\mathcal{P}$ one by one to Alice. Whenever an element x of $\mathcal{P}$ is revealed, the relation among x and previously revealed elements is also revealed. Alice is required to contstruct a finite graph G_x. Alice wins if the graphs G_x which she constructed during the game represent poset $\mathcal{P}$.

Clearly this make sense both for finite and infinite poset $\mathcal{P}$. However we are assuming that infinite poset $\mathcal{P}$ is revealed in such a way that at each step the number of previously revealed points is finite. Thus we allow at each step only finite time - however time may run forever; clearly this amounts to saying that the points of $\mathcal{P}$ are indexed by natural numbers. It is easy to see that the Density Theorem [82, 68] implies that Alice can win for every linearly ordered set (she just has to avoid selecting the single jump). In fact the extension property (more precisely, k-extension property for every $k > 0$) guarantees that Alice can win. In fact she can even let Bob to choose the first graph! In fact she can win even if Bob is choosing all graphs G_x providing the choice is consistent with $\mathcal{P}$; there cannot be any trap. However, unlike graphs, $\mathcal{C}$ does not have extension property, and thus this formulation of the on-line representability (one which gives Alice more chances) is justified.

The advantage of on-line representability is that it reduces the infinite problem to finite case by the following theorem (compare [71]):

Proposition 3 *The following three statements are equivalent:*
i. Every countable poset is representable by $\mathcal{C}$;
ii. Every countable poset is on-line representable by $\mathcal{C}$;
iii. Every finite poset is on-line representable by $\mathcal{C}$.

Proof. Obviously *ii.* implies *i.*. Also *iii.* implies *ii.*, as at each step the revealed part of $\mathcal{C}$ is finite. Also *i.* implies *iii.* by an easy argument: Let $\mathcal{U}$ be universal countable poset (for countable posets). Let us consider a representation $x \to G_x$ of $\mathcal{U}$ and we let Alice to play at each step according to the representation of $\mathcal{U}$. As $\mathcal{U}$ has k-extension property for every k this procedure cannot end in a deadlock. ♡

Advancing the proof of Theorem 6.1 we introduce the following notation:

We shall deal with disconnected graphs with many components. These graphs will be built recursively ("on-line") and we shall have to preserve their hierarchical structure. We use the following model:

A *hierarchical structure* of a multiset $(G_i; i \in I)$ of graphs is the graph $\sum_{i\in I} G_i$ together with a system of subsets $(I_x; x \in \mathcal{S})$ which satisfy

i. Every $i \in I$ and I itself belongs to an $I(x)$ for $x \in \mathcal{S}$;

ii. For every $x, x' \in \mathcal{S}$ the sets $I(x)$ and $I(x')$ are either disjoint or in the inclusion (which means that the sets $(I(x); x \in \mathcal{S})$ may be visualized by a tree).

Such a hierarchical structure will be denoted by $((G_i; i \in I), \mathcal{S})$

For a vertex x of $\mathcal{S}$ we define *height* $h(x)$ inductively as follows: $h(x) = i+1$ if all sets $I(y) \subset I(x)$, $y \in \mathcal{S}$, satisfy $h(y) \leq i$ while there is a predecessor of y of height i.

If $((G_i; i \in I), \mathcal{S})$ and $((G'_i; i \in I'), \mathcal{S}')$ are hierarchical structures, then a homomorphism from $((G_i; i \in I), \mathcal{S})$ to $((G'_i; i \in I'), \mathcal{S}')$ is a pair (f, f_H) where f is a homomorphism

$$\sum_{i\in I} G_i \to \sum_{i\in I'} G'_i$$

and f_H is a mapping $f_H : \mathcal{S} \to \mathcal{S}'$ which satisfies the following condition:

i. h does not increase the heights of vertices (i.e. $h(f_H(x)) \leq h(x)$ for every $x \in \mathcal{S}$);

ii. For every point x of $\mathcal{S}$ f maps $\sum_{i\in I(x)} G_i$ to the graph $\sum_{i\in I(f_H(x))} G'_i$.

(In the other words, the homomorphism f preserves the hierarchical structure of the multiset $(G_i; i \in I)$.)

It is easy to code a hierarchical structure by *suspension graphs* and we do this inductively as follows:

First, let for every positive d be given a graph S_d with a specified vertex $s(d)$. (We already know by Proposition 2 how to construct inverse set on-line.)

If $\mathcal{S}$ is of height 1 then we let $(\sum(G_i : i \in I))_{HS}$ denote the graph which arises from $(\sum_{i\in I} G_i) + S_1$ adding disjoint paths of length 2 joining every vertex $x \in \cup_{i\in I} V(G_i)$ with the fixed vertex $s(1)$ of S_1.

If $\mathcal{S}$ is of height $d > 1$ then denote by C_- the set of predecessors of I (this is a deliberately clumsy notation; it is in accordance with the notation which we have to introduce below). We let

$$(\sum(G_i; i \in I))_{HS}$$

to denote the graph which arises from $(\sum_{x\in C_-}(\sum(G_i : i \in I_x))_{HS}) + S_r$ by adding disjoint paths of length $2d$ joining every vertex $x \in \cup_{i\in I} V(G_i)$ with the fixed vertex $s(d)$ of S_d.

(We may call $(\sum(G_i : i \in I))_{HS}$ the *hierarchical suspension graph*, the dependence on the graphs S_d is supressed as these graphs will be clear from the context.)

The following technical statement establishes the fact that homomorphisms between hierarchical structures are coded by homomorphisms between corresponding hierarchical suspension graphs:

Proposition 4 *Let $((G_i; i \in I), \mathcal{S})$ and $((G'_i; i \in I'), \mathcal{S}')$ be hierarchical structures, let every graph in $G_i, i \in I_x$ (or every graph in $G_i, i \in I_x$) with $h(x) = d$ has odd girth $\leq 2d + 1$. Consider the hierarchical suspension graphs $(\sum(G_i : i \in I))_{HS}$ and $(\sum(G'_i : i \in I'))_{HS}$. Suppose that there is no homomorphism between any G_i and S_d (for $i \in I$ and $d > 0$) and any G'_i and S_d (for $i \in I'$ and $d > 0$) , and suppose that graphs S_d ,$d = 1, 2, \ldots$ form an inverse set. Then for every homomorphism*

$$f : (\sum(G_i : i \in I))_{HS} \to (\sum(G'_i : i \in I'))_{HS}$$

there exists a mapping $f_H : \mathcal{S} \to \mathcal{S}'$ such that (f, f_H) is a homomorphism of hierarchical structures $((G_i; i \in I), \mathcal{S}) \to ((G'_i; i \in I'), \mathcal{S}')$.

Proof. The proof is easier than the statement. It follows from the assumption that the homomorphism f mapps every set $\cup_{i\in I(x)} V(G_i)$ to the set $\cup_{i\in I'(y)} V(G'_i)$ for some $y \in \mathcal{S}'$. Further, vertices of any copy of S_d are mapped to a copy of $S_{d'}$ for a $d' \leq d$. This implies that any suspension graph $(\sum(G_i : i \in I(x)))_{HS}$ is mapped to a suspension graph $(\sum(G'_i : i \in I'(y)))_{HS}$ where x and y are the vertices of T and T' respectively. This defines a mapping $f_H : \mathcal{S} \to \mathcal{S}'$ and it is easy to chcek that (f, f_H) is a homomorphism of hierarchical structures $((G_i; i \in I), \mathcal{S}) \to ((G'_i; i \in I'), \mathcal{S}')$. $\heartsuit$

6 Posets On-Line

6.0.1 Construction

Let $\mathcal{P}$ be a given poset with points $\{1,2,3,\ldots\}$ and relation R. By induction (on line) we shall construct three sets of graphs: $G_1, G_2, \ldots, G_n, \ldots$, $G'_1, G'_2, \ldots, G'_n, \ldots$, and $S_1, S_2, \ldots, S_n, \ldots$ and a sequence of hierarchical structures $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n, \ldots$. This means that in the construction of G_n, G'_n, S_n we are using only gre graphs G_i, G'_i, S_i and structures $\mathcal{S}_i$ for $i < n$ and the properties of the poset $\mathcal{P}$ restricted to the set $\{1,2,3,\ldots,n\}$; this poset will be denoted $\mathcal{P}_n$.

The graphs S_n will be chosen on-line as the inverse set with the following properties:

1. The odd girth of S_n is $2n+3$;

ii. The chromatic number $\chi(S_n) = k+1$.

By Proposition 2 we know how to do that on-line.

Put $G'_1 = C_5 \oplus K_{k-3}$ ($\oplus$ denotes the complete join of two graphs). We put $\mathcal{S}_1 = \{1\}$ (hierarchical structure of height 1) and form a hierarchical suspension graph $G_1 = (C_5 \oplus K_{k-3})_{HS}$ using the graph S_1 (and paths of length 2; see also above construction of a hierarchical suspension graph – we preserve all the above notation).

In the inductive step let graphs $G_1, G_2, \ldots G_{n-1}, G'_1, G'_2, \ldots G'_{n-1}$ and $S_1, S_2, \ldots, S_{n-1}$ and a sequence of hierarchical structures $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{n-l}$ be given.

Consider the point $n \in \mathcal{P}$ and denote by $C_-(n)$ the set of all points i of $\mathcal{P}_n$ which are covered by n (i.e. $C_(n)$ is the set of all i such that $(i,n) \in R$ and there is no j with $(i,j) \in R, (j,n) \in R$). Similarly, denote by $C_+(n)$ the set of all points of $\mathcal{P}_n$ which cover n.

If both sets $C_-(n)$ and $C_+(n)$ are empty (which corresponds to the case that the point n is isolated in $\mathcal{P}_n$) then let G'_n be any graph G which has girth $= 3$ and with $\chi(G) = k$ and for which there are no homomorphisms $G_i \to G$ and $G \to G_i$ for any $i < n$. This is easy to achieve: First, we take any graph H with both its chromatic number and odd girth $> \max|V(G_i)|$. We then let $G = G'_n$ to be the graph $H\star(I,a,b)$ which arises form H by replacing every edge of H by a copy of the graph $P_3 \oplus K_{k-3}$. As in the definition of G_1 we put $\mathcal{S}_n = \{n\}$ (hierarchical structure of height 1) and form a hierarchical suspension graph $G_1 = (G'_n)_{HS}$ using the graph S_n (and paths of length $2n$).

Thus let $|C_-(n)| + |C_+(n)| > 0$

Denote by $a = |V(G_1)| \times |V(G_2)| \times \ldots \times |V(G_{n-1})|$ and put $b = a^a$. Let H_n be any graph H with the following properties:

1. The odd girth of H is $2n+1$ and every edge of H belongs to an odd cycle of length $2n+1$;

ii. The chromatic number $\chi(H) > b$.

We do not try to optimize at this point. However note that one can construct easily examples of graphs H_n and S_n (for example by the iteration of oriented line graphs - so called *shift* graphs).

Define graphs G'_n by the formula:

$$G'_n = \left(\sum_{i \in C_-(n)} G'_i + (\prod_{j \in C_+(n)} G'_j) \times H \right) \tag{6}$$

In order to construct G_n we visualize the formula defining G'_n componentwise (as indeed G'_n has many components) as multi-set $(K_i; i \in I)$ (i.e. K_i are all the components of G'_n). This set has two parts: $I = I' + I"$, where $I' = \sum_{i \in C_-(n)} I'(i)$. Denote by $(K_i; i \in I'(i), \mathcal{S}')$ the hierarchical structure inherited from the graph $G_i, i \in C_-(n)$.

The hierarchical structure $\mathcal{S}_n$ of G_n will be defined as disjoint sum of hierarchical structures $((K_i; i \in I'(i)), \mathcal{S}')$ and further by the set I (at height n). We let $s(n) \in V(S_n)$ be joined by disjoint paths of length $2n$ to every vertex of G'_n.

This finishes the construction of the graph G_n.

Symbolically, this can be written as

$$G_n = \left(\sum_{i \in C_-(n)} G_i + (\prod_{j \in C_+(n)} G'_j) \times H \right)_{HS} \tag{6}$$

This single formula expresses the on-line definition of G_n and ends this construction.

We shall prove the following

Theorem 6.1 *Let graphs $G_1, G_2, \ldots G_{n-1}$ represent a poset $\mathcal{P}_{n-1} = (\{1, 2, \ldots, n-1\}, R')$. Let $\mathcal{P}_n = (\{1, 2, \ldots, n\}, R)$ be any poset which extends P' (i.e. for which $P_n - \{n\} = P_{n-1}$). Then the graph G_n together with the graphs $G_1, G_2, \ldots G_n$ represent the poset $\mathcal{P}_n$.*

Proof.

Before disscussing the representability of $\mathcal{P}_n$ let us first remark that it follows from the properties of the graph S_i that $S_i \not\to G'_j$ for every $1 \leq i, j < n$ (by *ii.*). We have also $G'_j \not\to S_i$ for every $1 \leq i, j < n$; this follows for $j \leq i$ by the girth assumption and for $j > i$ by Homomorphisms Cancelation Property of graphs H_i: If $G'_j \not\to S_i$ for $j > i$ then take that component of G'_j which has form $G'_l \times H_{l+1} \times H_{l+2} \times \ldots \times H_j$ and then by HCP applied consecutively for $j, j-1, \ldots, l$ we obtain that there exists a homomorphism $G'_l \to S_i$ which a contradiction with the girths assumption on G'_l

and S_i (the graph G'_l was chosen at the moment when both sets $C_-(l)$ and $C_+(l)$ were empty and thus G'_l has girth 3).

This means that any homomorphism $f : G_i \to G_j$, $1 \le i, j \le n$ maps vertices of the graph G'_i to vertices of the graph G'_j and all suspension graphs isomorphic to S_l for $l \le i$ in the hierarchical structure of the graph G_i to those suspension graphs in the hierarchical structure of G_j which are isomorphic to the graphs S_l for $l \le i$ (this follows from the fact that suspension graphs form an independent family). As the homomorphism f when restricted to the suspension graphs maps a graph isomorphic to S_l to a graph isomorphic again to S_l, we see that the homomorphism f preserves the hierarchical structure of (components of) G'_i.

After these preparations we prove that the graphs $G_1, G_2, \dots G_{n-1}, G_n$ represent $\mathcal{P}_n$.

Obviously $G_i \to G_n$ for every $i \in C_-(n)$ and thus also $G_i \to G_n$ for every i with $(i, n) \in R$.

Similarly $G_n \to G_i$ for every $i \in C_+(n)$ and thus also $G_n \to G_i$ for every i with $(n, i) \in R$.

Now suppose $G_{\check{\imath}} \to G_n$. As H_n has odd girth $> 2n + 1$ and as every edge of $G_{\check{\imath}}$ is in an odd cycle of length $< 2n - 1$, we know that no edge of $G'_{\check{\imath}}$ maps to

$$(\prod_{i \in C_+(n)} G'_i) \times H$$

and thus

$$G'_{\check{\imath}} \to \sum_{i \in C_-(n)} G_i$$

But then by the hierarchical suspension property (Proposition 4) we have that $(\check{\imath}, i) \in R$ for some $i \in C_-(n)$ and thus $(i, n) \in R$.

Finally suppose that $G_n \to G_{\check{\imath}}$. Put $C_+(n) = \{i_1^+ < i_2^+ < \dots < i_s^+\}$ and let $i_j < \check{\imath} < i_{j+1}$ (we allow $j = s$ to cover the case $i_s < \check{\imath}$ and $j = 0$ to cover the case $\check{\imath} < i_1$).

By HCP 4.2 (as $|V(G_{\check{\imath}})| < a$ and $\chi(H) > b$) we get that

$$(\prod_{i \in C_+(n)} G'_i) \times H \to G'_{\check{\imath}}$$

implies

$$\prod_{i \in C_+(n)} G'_i \to G'_{\check{\imath}}$$

.

Now if $j = s$ then we are done as $i_s < \check{\imath}$ and thus the odd girth of $\prod_{i \in C_+(n)} G'_i$ is less than the odd girth of the graph $G'_{\check{\imath}}$ and we get a contradiction.

If $j < s$ then we proceede similarly. We shall illustrate it on the case $j = s - 1$: We have

$$\prod_{i \in C_+(n)} G'_i \to G'_{\check{\imath}}$$

(and $|V(G_{\check{\imath}})| < a_s$). Now we apply HCP for graphs G'_i) to the product $\prod_{i \in C_+(n)} G'_i = \prod_{i=1}^{s} G'_i$ and we obtain

$$\prod_{i=1}^{s-1} G'_i \to G'_{\check{\imath}}$$

.

If $s - 1 < \check{\imath}$ we are done as we get a contradiction as above. If $s - 1 > \check{\imath}$ we can continue in the same way for $k = s - 1, s - 2, ..., j$ all the time applying HCP to the product $\prod_{i=1}^{k} G'_i$), finally obtaining

$$\prod_{i=1}^{s-1} G_i \to G'_{\check{\imath}}$$

which is a contradiction with the girth assumption. (If $\check{\imath} < i_1$ then we get a contradiction of $G_{i_1} \to G_{\check{\imath}}$ and odd girth.

Note that every edge of the graph G'_n belongs to an odd cycle of length $\leq 2n + 1$. This proves the Theorem 6.1. ♡

Corollary 1 *The coloring poset $\mathcal{C}$ is universal (for countable posets).*

6.1 Concluding Remarks

Let us at the end of this appendix several comments which relate the main result Theorem 6.1 to oether results.

The above proof of the universality of finite graphs and homomorphism order can be modified to get the following:

Theorem 6.2 *$\mathcal{C}$ contains universal poset $\mathcal{R}$ ($\mathcal{R}$ for rigid) such that for any two graphs $G, G' \in \mathcal{R}$ there exists at most one homomorphism $G \to G'$.*

Proof. (**a sketch**) We want to proof that all our graphs G_n (in the notation of Theorem 6.1) can be made rigid and there is at most one homomorphism between them. However, the inverse family $S_i, i = 1, 2, \ldots$ can be assumed to be rigid (and this was proved in [66]) and the unicity of homomorphisms $S_j \to S_i$ is claimed by HCP. Also our starting building blocks (i.e. graphs G_i correspnding the case $C_-(n) = C_+(n) = \emptyset$ can be made rigid (providing we take a rigid H then the graph $H \star (I, a, b)$, $I = P_3 \oplus K_{k-3}$, is also rigid. The graphs $G_i \times H$ can be made rigid (by suspension) The unicity of mappings between G_i and G_j is taken care by HCP and rigidity of S_i. This is all using we known techniques and the details will appear elsewhere. ♡

Corollary 2 *Every thin category may be represented by the category of graphs and homomorphisms.*

Our proof of the universality of $\mathcal{C}$ does not use any special (ad hoc) constructions. The building blocks are graphs with high chromatic number and with large odd girth (which can be changed to the large girth), and these building blocks can be chosen arbitrarily at random. (This yields that one can prove the universality of the coloring poset $\mathcal{C}$ for many restricted families of graphs.) In fact presently when proving some steps in the proof of Theorem 6.2 one relies on the probabilitic method.

For example it is not known how to construct uniquelly H-colorable graph with a large girth (see [85, 72].

However it is possible that even very simple families of graphs (such as bounded degree graphs, or even oriented paths; compare [71]) could represent every countable posets. Presently, there are no negative results in this direction.

Another related research is motivated by Müller's Theorem [58] where one tries to control homomorphism into *all* small graphs. This approach to [58] is taken in [73].

Finally, let us remark, that the problem of representing of infinite posets (of arbitrary cardinality) by infinite graphs and their homomorphisms has a positive solution only under certain set-theoretic axioms. Even the existence of a proper class of independent graphs (i.e. a representation of the discrete poset indexed by ordinals) is unknown, say, in ZFC. (See [75] for a disscussion of this.) Our method has also some implications for representation by infinite graphs. However this is a paper on finite combinatorics and we leave it at that.

7 Appendix 2: Sparse Graphs with Given Homomorphisms

We add here another recent result which not only complements previous text but presents perhaps the proper understanding to celbrated theorem of V. Müller [58] and Erdős about uniquelly colorable graphs. This appendix is based on recent paper [73].

(Recall just one definition which was contained in Appendix 1: A graph H is said to be *pointed for* G (or shortly G-pointed) if for any two homomorphisms $g, g' : G \to H$ which satisfy $g(x) = g'(x)$ for all $x \neq x_0$ (for some fixed vertex $x_0 \in V(G)$) holds also $g(x_0) = g'(x_0)$. Note that any core graph H is H-pointed.

The following is the main result of this chapter:

Theorem 7.1 *For every graph F and every choice of positive integers k and l there exists a graph G together with a homomorphism $c : G \to F$ with the following properties:*

i. $g(G) > l$;

ii. For every graph H *with at most* k *vertices and there exists a homomorphism* $g : G \to H$ *if and only if there exists a homomorphism* $f : F \to H$.

iii. For every F*-pointed graph* H *with at most* k *vertices and for every homomorphism* $g : G \to H$ *there exists a unique homomorphism* $f : F \to H$ *such that* $g = f \circ c$.

The conditions *ii.* and *iii.* may be expressed by the following diagram:

$$\begin{tikzcd} F \arrow[r, "f"] & H \\ G \arrow[u, "c"] \arrow[ur, "g"'] & \end{tikzcd}$$

It is easy to give an example which shows that the statement analogous to *iii.* can not be true for all (i.e. not necessarily pointed) small graphs H: Given two homomorphisms $f', f'' : F \to H$ satisfying $f'(x) = f''(x)$ for all $x \neq x_0$ and $f'(x_0) \neq f''(x_0)$ the set $c^{-1}(x_0)$ may be splitted in two set A and B and we can define $g : G \to H$ by $f' \circ c(x)$ for all x with $c(x) \neq x_0$ and $g(x) = f' \circ c(x)$ for all $x \in A$ and $g(x) = f'' \circ c(x)$ for all $x \in B$. The homomorphism g cannot be written as $g = f \circ c$ for a homomorphism $f : F \to H$.

Theorem 7.1 is a generalization of the study of graphs with large chromatic number and large girth. The existence of such graphs is established by the following landmark result (and landmark proof):

Theorem 7.2 (Erdős [11]) *let* k *and* l *be positive integers. Then there exists a graph* $G = G_{k,l}$ *with the following properties:*

i. $\chi(G) > k$;

ii. $g(G) > l$.

Theorem 7.1 may look like a technical lemma, however, it has several interesting corollaries which prove structural extensions of Erdős theorem.

We say a graph G is uniquely H-colorable if there is an onto homomorphism c from G to H, and any other homomorphism from G to H is the composition $\sigma \circ c$ of c with an automorphism σ of H.

The problem of the existence of uniquely k-colorable graphs with large girth has an interesting history: [59] settled triangle-free case (i.e. $l = 3$) and this was improved by Greenwell and Lovász [20] to a given odd girth. Meanwhile, in 1973, Erdős claimed the general case by probabilistic method (see [57]) and the construction was provided in the full generality by Müller [57, 58].

Müller's proof used a constructive proof of Theorem 7.2. (Later a non-constructive proof has been published by Bollobás and Sauer [5].) Finally, Zhu [85] proved the following result which is a particular case (choose $F = H$) of our Theorem 7.1

Corollary 3 *For every core H and positive integer l there exists uniquely H-colorable graph G with girth $\geq l$.*

A constructive proof of this result is presently open. This we state as

Problem 10 *Find a constructive proof of Corollary 3.*

In fact we prove that G is *strongly uniquely H-colorable* in the sense that any homomorphism $G \to H'$ to any small pointed graph H' is induced by a homomorphism $F \to H'$.

The existence of sparse uniquely k-colorable graphs was generalized in [57, 58] in another direction, where the following remarkable strengthening of the unique colorability was proved:

Theorem 7.3 (Müller [57, 58]) *Let k, l, t be positive integers, $k > 2$. Let A be a finite set and let $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_t$ be distinct partitions of the set A each into at most k classes. Then there exists a k-chromatic graph $G = (V, E)$ such that the following holds:*

i. $g(G) > l$;

ii. A is a subset of V;

iii. G has just t colorings $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_t$ by k-colors such that each of the coloring $\mathcal{B}_i$ restricted to the set A coincides with the coloring $\mathcal{A}_i$, $i = 1, 2, \cdots, t$.

It seems that this result is little known (although it is included in Jensen - Toft book [41]) and even the existence of construction of uniquely colorable graphs without short cycles was recently quoted as a problem.

Here we approach Müller's Theorem more generally and thus perhaps find a proper setting for it.

We now demonstrate that Müller's theorem 7.3 corresponds to the case $F = {K_k}^t, H = K_k$ of Theorem 7.1. For this purpose, we need the following particular property of complete graphs (which is established in [58]):

The only homomorphisms ${K_k}^t \to K_k$ are projections.

In a recent paper by B. Larose and C. Tardif [49, 48], this property was studied (in relationship to Hedetniemi's product conjecture) and called projectivity: A graph H is said to be *t-projective* if every homomorphism $f : H^t \to H$ which satisfies $f(x, x, \cdots, x) = x$ for every $x \in V(H)$ is a projection. A graph is *projective* if it is t-projective for every t (i.e. for projective graphs, up to an automorphism, the only homomorphisms $H^t \to H$ are projections).

(By a *product* we mean here *direct product* (sometimes called categorical product) defined as follows: Suppose G and H are simple finite graphs. The direct product $G \times H$ of G and H has vertex set $V(G \times H) = \{(x, y) : x \in V(G) \text{ and } y \in V(H)\}$ and edge set $E(G \times H) = \{\{(x, y)(x', y')\} : \{x, x'\} \in E(G) \text{ and } \{y, y'\} \in E(H)\}$.)

Thus the above mentioned result of Müller can be stated by saying that complete graphs are projective. Larose and Tardif proved some sufficient conditions for a graph to be projective. It is easy to derive from these conditions that many classes of graphs are projective, including Kneser graphs, graphs ${G_k}^d$, etc..

The notion of projective graphs leads to the following (which extends Müller's theorem to non-complete graphs):

Corollary 4 *Let H be projective core graph with k vertices. Let A be a finite set and let $f_1, f_2, \cdots, f_t$ be distinct mappings $A \to V(H)$. Then there exists a graph $G = (V, E)$ such that the following holds:*

i. A is a subset of V;

ii. For every $i = 1, 2, \cdots, t$ there exists unique homomorphism $g_i : G \to H$ such that g_i restricted to the set A coincides with the mapping f_i;

iii. For every homomorphism $f : G \to H$ there exists $i, 1 \le i \le t$ and a homomorphism $h : H \to H$ such that $h \circ f_i = f$;

iv. G has girth $> l$.

Proof. Consider the graph $F = H^t \times K_N$ where K_N is the complete graph with N vertices, $N > max\{k^t, |A|\}$. We apply 7.1 for the graphs F and H. Thus there exists a graph G and a homomorphism $c : G \to F$ (we preserve the notation of Theorem 7.1) such that any homomorphism $g : G \to H$ there exists a homomorphism $f : F \to H$ such that $g = f \circ c$. Now , up to automorphisms of H, all the homomorphisms $F \to H$ are induced by t projections $\pi_1, \pi_2, \cdots, \pi_t : H^t \to H$. In other words, every homomorphism $f : F \to H$ for which $f(x, \cdots, x, a) = x$ is of the form $f(x, a) = \pi_i(x)$ for every vertex (x, a) of F and some $i, 1 \le i \le t$. (Here we use $N > k^t$, see section 3 where this will be explained in a greater detail.) Now consider mappings $f_1, f_2, \cdots, f_t$ together with an injective mapping $f_0 : A \to V(K_N)$. Then the corresponding mapping $\phi = (f_0, f_1, f_2, \cdots, f_t) : A \to V(F)$ is injective and thus we can identify A with its image of $\phi(A)$. Clearly all homomorphisms $f : G_1 \to H$ coincide on $A = \phi(A)$ with one of the maps f_i $i = 1, 2, \cdots, t$. ♡

Theorem 7.1 has also the following corollary known as *Sparse Incomparability Lemma* [66].[85]:

Corollary 5 *For every pair H, H' of graphs such that H' is H-colorable and H fails to be H'-colorable there exists a graph G with the following properties:*
i. $g(G) \geq l$
ii. G is H-colorable and G fails to be H'-colorable.

(To obtain Corollary 5 we put $F = H$, $k = |V(H')|$ in Theorem 7.1.)

The chapter is organized as follows: In Subsection 2 we prove Theorem 7.1. As the proof is non-constructive we include in Subsection 3 a simple proof of a weaker statement with girth replaced by the odd girth. In Subsection 4 we prove that the analogy of Müller's theorem 4 holds for projective graphs only and add a few remarks and problems.

7.1 Proof of Theorem 7.1

Our proof uses probabilistic method and most of the calculations are fairly standard. But it is an indication of the proper setting of Theorem 7.1 that the proof is perhaps easier than the proofs of particular cases, [5, 85].

Suppose that the graph F has a vertices and that the vertices are $\{1, 2, \cdots, a\}$, and the edge set $E(F)$ has cardinality q. Let $V_1, V_2, \cdots, V_a$ be disjoint n-sets. Let G_0 be the graph with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_k$, and $\{x, y\} \in E(G_0)$ if and only if $x \in V_i, y \in V_j$ and $(i, j) \in E(F)$. Then G_0 has qn^2 edges. Let $\mathcal{G}$ be the set of all subgraphs G of G_0 with $m = \lfloor qn^{1+\epsilon} \rfloor$ edges, where $0 < \epsilon < 1/l$. Put also $\delta = \min\{\epsilon l, 1/k\}$. Then $|\mathcal{G}| = \binom{qn^2}{m}$.

In the following, n is assumed to be sufficiently large. We consider $\mathcal{G}$ as a probability space with each member occurring with the same probability $1/|\mathcal{G}|$. This is asymptotically the same thing as the random graph $\mathbf{G}$ where we choose edges from the set $E(G_0)$ independently with the probability $n^{-1+\epsilon}$.

We shall make use of the fact that most graphs in $\mathcal{G}$ have few short cycles which are pairwise vertex disjoint. On the other hand, the edges are "dense"in some sense. These results are stated as Claims 1-4:

Claim 1 The expected number of cycles of length $\leq l$ in a graph $G \in \mathcal{G}$ is bounded by n^δ and thus asymptotically almost all graphs from $\mathcal{G}$ have at most n^δ cycles of length $\leq l$.

Claim 2 The expected number of pairs of cycles of length $\leq l$ in a graph $G \in \mathcal{G}$ which intersect in at least one vertex is bounded by n^δ and thus asymptotically almost all graphs from $\mathcal{G}$ have at most n^δ cycles of length $\leq l$, and these cycles are all vertex disjoint.

Claim 3 A set $A \subset V$ is said to be *large* if there are i, j, $1 \leq i < j \leq k$, $\{i, j\} \in E(F)$, such that $|A \cap V_i| \geq \delta n$ and also $|A \cap V_j| \geq \delta n$. We call an edge $\{i, j\}$ of F a *good edges* of A if $|A \cap V_i| \geq \delta n$ and $|A \cap V_j| \geq \delta n$. For a large set A denote by $|\mathbf{G}/A|$ the minimum number of edges of $\mathbf{G}$ which lie in the set $\{\{x, y\};\ x \in V_i,\ y \in V_j\}$ for a good edge of A.

Then the probability $Prob[A$ large implies $|\mathbf{G}/A| \geq n] = 1 - o(1)$.

Thus asymptotically almost all graphs from $\mathcal{G}$ have the property that all good edges (of F) of any large set induce at least n edges (of $\mathbf{G}$).

Claim 4 Almost all graphs from $\mathcal{G}$ do not contain two non empty sets $A \subset V_{i_0}, B \subset V_{j_0}, 1 \leq i_0 < j_0 \leq a$, $|A| + |B| \geq \delta n$ such that the set $A \cup B$ contains at most $\min\{|A|, |B|, n^\delta\}$ edges and these edges form a matching (i.e. a set of mutually disjoint edges).

For the completeness let us include at least short proofs of Claims 3 and 4:

Proof. (of Claim 3)

We first estimate probability

$$\alpha = Prob[A \text{ large implies } |\mathbf{G}/A| \geq n].$$

We have

$$1 - \alpha \leq \sum_{A \text{ large}} Prob[|\mathbf{G}/A|] < n] \leq 2^{kn} \cdot \binom{\binom{kn}{2}}{n} \cdot (1-p)^{\delta^2 n^2 - n}.$$

Now bounding very roughly

$$\binom{\binom{kn}{2}}{n} \leq \binom{k^2 n^2}{n} \leq k^{2n} n^{2n} < e^{cn \log_2 n}$$

and

$$(1-p)^{\delta^2 n^2 - n} \leq e^{-p(\delta^2 n^2 - n)}$$

we obtain

$$1 - \alpha < e^{cn \log_2 n - c' n^{1+\epsilon}}$$

for some positive constants c and c' which are independent on n.

Thus we get $Prob[A$ large implies $|\mathbf{G}/A| \geq n] = 1 - o(1)$. ♡

Proof. (of Claim 4)

This is similar, this time we give a counting version of the proof. We shall show that very few graphs in $\mathcal{G}$ contain subgraphs induced by $A \cup B$, $|A \cup B| = \delta n$ with at most $\min\{|A|, |B|, n^\delta\}$ edges (even without matching condition). Towards this end for $b \leq n^\delta, s \leq \min\{b, n^\delta\}$ we denote by $P(b, s)$ the expected number of pairs

$A \subset V_i, B \subset V_j$ such that $\{i,j\} \in E(H)$, $|A|+|B| = \delta n$, $|A| = b$ and there are exactly s edges between W_i and W_j.

Then

$$P(b,s) < 2q\binom{n}{\delta n - b}\binom{n}{b}\binom{b(\delta n - b)}{s}\binom{qn^2 - b(\delta n - b)}{m - s}\binom{qn^2}{m}^{-1}.$$

We have $b < \frac{\delta n}{2}$, and $b(\delta n - b) \geq \frac{\delta b n}{2}$ thus

$$\binom{qn^2 - b(\delta n - b)}{m - s}\binom{qn^2}{m}^{-1} < \binom{qn^2 - \delta bn/2}{m}\binom{qn^2}{m}^{-1} < e^{-\frac{bn^\delta}{2}}.$$

(Here we used the inequality $\binom{a-x}{b}\binom{a}{b}^{-1} \leq \left(\frac{a-b}{a}\right)^x < e^{-bx/a}$.)

Therefore

$$P(b,s) < 2qn^{\delta n - b}n^b(bn)^s e^{-\frac{bn^\delta}{2}} < 2q(bn)^s \exp\left(-\frac{bn^\delta}{2} + \delta n \log n\right) < n^{2s}e^{-\frac{bn^\delta}{2}}.$$

Let $P(b)$ be the sum of all $P(b,s)$ for which $s \leq \min\{b, n^\delta\}$.

If $b < n^\delta$, then $s \leq b$, and hence

$$P(b) < \exp\left(-\frac{bn^\delta}{2} + 3b\log n\right) < e^{-\frac{bn^\delta}{3}} < e^{-n^{\delta/2}}.$$

If $b \geq n^\delta$, then $s \leq n^\delta$, and hence

$$P(b) < \exp\left(-\frac{n^\delta}{2k} + 3n^\delta \log n\right) < e^{-\frac{n^\delta}{3k}} < e^{-n^\delta}.$$

Therefore

$$\sum_{1 \leq b \leq \frac{n}{2}} P(b) < e^{-n^{\delta/3}}.$$

This establishes proof of Claim 4. ♡

(Proofs of all these claims are by now a folklore and appear in various forms in the literature and mostly go back to Erdős.)

Thus let G' be an instance of the graph from $\mathcal{G}$ with all the properties claimed in Claims 1-4 for majority of graphs from $\mathcal{G}$.

Explicitly, let $G' \in \mathcal{G}$ be any graph with the following properties:

The graph G' contains at most n^δ cycles of length $\leq l$ and all these cycles are vertex disjoint.

Consequently, there exists a set M of edges of G' which forms a matching in G' of size at most n^δ such that the graph $G' - M = (V(G'), E(G') - M)$ has no cycles of length $\leq l$. Put $G = G' - M = (V, E)$. We prove that the graph G is our desired graph. It is clear that G has girth $> l$

The graph G has further properties which follow from Claims 3 and 4. For every good pair $i < j$ in any large set A there is an edge of G with one vertex in $A \cap V_i$ and the other in $A \cap V_j$ (this follows from Claim 3). Further, it follows from the Claim 4 that even for every pair of non empty subsets A, B, $|A| + |B| \geq \delta n$ there exists an edge of G with its end vertices in the set $A \cap B$.

Define the mapping $c : V \to V(F)$ by $c(x) = i$ iff $x \in V_i, i = 1, \cdots, a$. Clearly c is a homomorphism $G \to F$.

Let H be a fixed graph with at most k vertices and let $g : G \to H$ be a homomorphism.

We define a mapping $\phi : V(F) \to V(H)$ as follows: For each $i \in V(F)$, there exists a vertex $x \in V(H)$ such that $|V_i \cap g^{-1}(x)| \geq n/k \geq \delta n$ by pigeonhole principle. We let $\phi(i)$ be any (fixed) x with $|V_i \cap g^{-1}(x)| \geq n/k \geq \delta n$. (If there are more than one x satisfy the condition, we arbitrarily choose one.)

We prove that ϕ is a homomorphism $F \to H$. Thus let $\{i, j\}$ be an edge of F. Put $A_i = g^{-1}(\phi(i))$ and $A_j = g^{-1}(\phi(j))$. As there exists an edge e of G with its ends in $A_i \cup A_j$ (this follows from Claim 3) we have that $\phi(i)$ and $\phi(j)$ form an edge of H.

This proves part ii. of the Theorem 7.1. Thus from now on let H be a pointed graph.

It follows that we can further assume that $|V_i \cap g^{-1}(j)| < \delta n$ for all $j \neq i$. For otherwise there are i, j such that $j \neq i$ with $|V_i \cap g^{-1}(j)| \geq \delta n$. Then we can define another mapping ϕ' which agrees with ϕ on every other vertex, and $\phi'(i) = j$. However this is a contradiction as H is a F -pointed graph.

The set $V_i \cap g^{-1}(\phi(i))$ will be denoted by W_i, $i = 1, \cdots, a$.

Thus the homomorphism ϕ is uniquely determined by the homomorphism g. It remains to prove that $\phi \circ c = g$.

Assume to the contrary that $\phi \circ c \neq g$.

Thus there exists $x \in V(H), x \neq x_0$ and $i_0 \in V(F)$ such that $\phi(i_0) = x_0 \neq x$ while the set $W = g^{-1}(x) \cap V_{i_0}$ is non-empty.

Now for any $j \in V(F)$ with $\{j, i_0\} \in E(F)$ the set $W_j \cup W$ has at least δn vertices and thus (using Claim 4) there exists an edge $e \in E(G)$ with its ends in W_j and W. And this happens (by the same argument) if and only if there exists an edge $e \in E(G)$ with with its ends in W_j and W_{i_0}. Hence the mapping ϕ' defined by $\phi'(i) = \phi(i)$ for all $i \neq i_0$, $\phi'(i_0) = x$ is a homomorphism $F \to H$ and $\phi' \neq \phi$. This is contradicting the property that H is F-pointed.

Thus $g = \phi \circ c$.

This completes the proof of Theorem 7.1

7.2 Odd Girth, constructible sets and products

Here we prove the analogy of Theorem 7.1 for graphs of a given odd girth by giving a construction. The reason why we prove this weaker statement is that for general graphs H such a construction is not available (and this is stated as Problem 1.

Using a recent results of Larose and Tardif we also characterize all graphs H for which the Müller type theorem 7.1 is valid. Perhaps this more generall setting provides a proper setting to uniquelly colorability and Mïler's theorem.

Theorem 7.4 *For every graph F and every choice of positive integers k and l there exists a graph G_0 such that the graph $G = F \times G_0$ together with the projection $c : F \times G_0 \to F$ has the following properties:*

i. $oddg(G) > l$;

ii. For every graph H with at most k vertices, there exists a homomorphism $g : G \to H$ if and only if there exists a homomorphism $f : F \to H$.

iii. For every F-pointed core graph H with at most k vertices and for every homomorphism $g : G \to H$ there exists a unique homomorphism $f : F \to H$ such that $g = f \circ c$.

.

$$\begin{tikzcd} F \arrow[r, "f"] & H \\ F \times G_0 \arrow[u, "c"] \arrow[ur, "g"'] & \end{tikzcd}$$

Proof

Let G_0 be a graph with odd girth $> l$ and chromatic number $> k^{|V(F)|}$ (a construction is provided e.g. by iterated shift graphs). Put $G = F \times G_0$. It is well known (and easy to see) that $oddg(F \times G_0) = \max\{oddg(F), oddg(G_0)\}$.

Thus let H be a graph with at most k vertices and let $g : G \to H$ be a homomorphism.

For every $y \in V(G_0)$ define the mapping $f_y : V(F) \to V(H)$ by $f_y(x) = g((x, y))$. Note that the mapping f_y need not be a homomorphism $F \to H$ but as $\chi(G_0) > k^{|V(F)|}$ there exists an edge $\{y, y'\} \in E(G_0)$ such that $f_y = f'_y$. However in this case the mapping $f = f_y = f'_y$ is a homomorphism $F \to H$: given $\{x, x'\} \in E(F)$ we have $\{(x,y),(x',y')\} \in E(F \times G_0)$ and thus $\{g(x,y), g(x',y')\} = \{f_y(x), f'_y(x')\} = \{f_y(x), f_y(x')\} = \{f(x), f(x')\} \in E(H)$.

This proves *ii.*

So suppose that in addition the graph H is a F-pointed core. Under this assumption the validity of *iii.* follows readily from the following (which generalizes [20, 66, 86]):

Claim

Let f_y be a homomorphism $G \to H$, $\{y,z\} \in E(G_0)$. Then $f_z = f_y$.

Assume to the contrary that $f_z(x_0) \neq f_y(x_0)$ for a vertex $x_0 \in V(F)$. Define mapping $f : V(F) \to V(H)$ as $f(x) = f_y(x)$ for $x \neq x_0$ and $f(x_0) = f_z(x_0)$. Then f is a homomorphism $F \to H$ (It suffices to check edges of F incident with x_0: If $\{x, x_0\} \in E(F)$ then $\{(x,y),(x_0,z)\} \in E(F \times G_0)$ and thus $\{f_y(x), f_z(x_0)\} = \{f(x), f(x_0)\} \in E(H)$.) However this is a contrary with the fact that H is F-pointed.

Corollary 6 *Let H be projective core graph with k vertices. Let A be a finite set and let $f_1, f_2, \cdots, f_t$ be distinct mappings $A \to V(H)$. Then there exists a graph G_0 such that the graph $G = H^t \times G_0$ contains a subset A' together with 1-1 correspondence $\iota : A' \to A$ with the following properties:*

i. For every $i = 1, 2, \cdots, t$ there exists unique homomorphism $g_i : G \to H$ such that g_i restricted to the set A' coincides with the mapping $f_i \circ \iota$;

ii. For every homomorphism $f : G \to H$ there exists $i, 1 \leq i \leq t$ such that $f_i \circ \iota = f$;

iii. G has odd girth $> l$.

One can ask to what extend is the projectivity a necessary condition for a validity of Corollaries 4,6.

Recent work of Claude Tardif and Benoit Larose allows us to characterize all graphs H for which an analogy of Müller's Theorem is valid. This is non-trivial and it is based on the following notions which are introduced in [49, 48] and go back to [4]:

A set C of vertices of a graph H is said to be *constructible* if there exists a graph G, vertices $x_0, x_1, \ldots, x_n$ of G and vertices $y_1, \ldots, y_n$ of H such that C is the set of all $g(x_0)$ where g is any homomorphism from G to H such that $g(x_i) = y_i$ for all $i = 1, \ldots, n$.

Larose and Tardif [49] proved a remarkable Theorem which states that the graph H is projective if and only if every subset of its vertex set is constructible and this is equivalent to that every two element subset of $V(H)$ is constructible.

This is related to our notion of pointed graph H: If H is F-pointed then every vertex of $V(H)$ (considered as 1-element subset) is constructible. We have seen that constructibility of 1-element subsets of H is a necessary condition for a validity of Theorem 7.1.

Perhaps surprisingly, the constructibility of 2-element sets is equivalent with the validity of Müller's theorem. We state this as:

Theorem 7.5 *For a core graph H, the following statements are equivalent:*

I. *For any choice of a finite set A and distinct mappings $f_1, f_2, \cdots, f_t : A \to V(H)$ there exists a graph $G = (V, E)$ such that the following holds:*

i. A is a subset of V;

ii. For every $i = 1, 2, \cdots, t$ there exists unique homomorphism $g_i : G \to H$ such that g_i restricted to the set A coincides with the mapping f_i;

iii. For every homomorphism $f : G \to H$ there exists $i, 1 \leq i \leq t$ and a homomorphism $h : H \to H$ such that $h \circ f_i = f$;

iv. G has girth $> l$.

II. *The graph H is projective;*

III. *Every 2-subset of $V(H)$ is constructible.*

Proof. The equivalence of **II**. and **III**. was established in [49], **II**. implies **I**. by Corollary 4. We prove **I**. implies **III**.:

Assume H has vertex set $V = \{v_1, v_2, \cdots, v_n\}$. Let $\{v_i, v_j\}$ be a 2-element subset of V. We need to show that $\{v_i, v_j\}$ is constructible.

Consider i. for $A = V^2$ with projections $f_1 = \pi_1$ and $f_2 = \pi_2$. Let G be a graph satisfying **I**.. Let $x_0 = (v_i, v_j)$ and for $i = 1, 2, \cdots, n$, let $x_i = (v_i, v_i)$ and $y_i = v_i$. Then for any homomorphism $f : G \to H$ with $f(x_i) = y_i$ for $i = 1, 2, \cdots, n$, and $f(x_0) \in \{v_i, v_j\}$. Furthermore, there is a homomorphism f from G to H, namely the extension of f_1 to G, which satisfies $f(x_i) = y_i$ for $i = 1, 2, \cdots, n$ and $f(x_0) = v_i$; and there is a homomorphism f from G to H, namely the extension of f_2 to G, which satisfies $f(x_i) = y_i$ for $i = 1, 2, \cdots, n$ and $f(x_0) = v_j$. Thus $\{v_i, v_j\}$ is a constructible set. ♡

References

[1] L. Babai: Automorphism groups of graphs and edge contraction, Discrete Math. 8(1974), 13-22

[2] L. Babai, A. Pultr: Endomorphism Monoids and Topological Subgraphs of Graphs, J.Comb.Th.B 28,3 (1980), 278 - 283

[3] J. Bang-Jensen, P. Hell: On the effect of two cycles on the complexity of colouring, Discrete Applied. Math. 26(1990), 1-23

[4] V. G. Bodnarčuk, L. A. Kaluzhnin, V. N. Kotov, B. A. Romov, Galois theory for Post algebras I-II (russian), *Kibernetika*, **3** (1969), 1-10 and **5** (1969), 1-9. English version: *Cybernetics*, (1969), 243-252 and 531-539.

[5] B. Bollobás and N. Sauer, *Uniquely colorable graphs with large girth*, Can. J. Math. 28(1976), 1340-1344.

[6] G. Brightwell, P.Winkler: Graph Homomorphisms and Phase Transitions, J. Comb. Th.B 77, (1999), 221-262.

[7] G. J. Chang, D. D. - F. Liu, X. Zhu: Distance Graphs and T - coloring, J. Comb. Th. B, 75 (1999(, 259 - 269

[8] V. Chvátal, P. Hell, L. Kučera, J. Nešetřil: Every Finite Graph is a full Subgraph of a Rigid Graph, J. Comb. Th. B, 11 (1973), 239 - 251

[9] M. Dyer, C. Greenhill: The Complexity of Counting Graph Homomorphisms, Random Structures and Algorithms (to appear)

[10] P. A. Dreyer,Jr., C. Malon, J. Nešetřil: Universal H-colorable graphs without a given configurations, KAM-DIMATIA Series 99-428.

[11] P. Erdös: Graph Theory and Probability, Canad. J. Math. 11(1959),34-38

[12] J. Edmonds, Paths, trees and flowers, Canadian J. Math. 17 (1965), 449-467

[13] T. Feder, M. Vardi: Monotonne monadic SNP and constraint satisfaction. In: Proceedings of the 25th ACM STOC, ACM, 1993,pp. 612-622

[14] R. Fraïssée: Theory of Relations, Studies in Logic and the Foundation of mathematics 118, Elsevier, 1986.

[15] P. J. Freyd: Concreteness, J. Pure and Applied Alg. 3 (1973), 171-191

[16] R. Frucht: Herrstellung von Graphen mit vorgegebener abstracter Gruppe, Comp. Math. 6(1938), 230-250

[17] T. Feder, M. Vardi: Monotonne monadic SNP and constraint satisfaction. In: Proceedings of the 25th ACM STOC. ACM, 1993 612-622

[18] A. Galluccio,P. Hell, J.Nešetřil: The complexity of H-coloring of bounded degree graphs, KAM-DIMATIA Series 99-416; Discrete Math. (to appear)

[19] M. R. Garey, D. S. Johnson: Computers and Intractability. W. H. Freeman and Co., New York, 1979

[20] D. Greenwell and L. Lovász, Applications of product coloring, Acta Math. Acad. Sci. Hungar. 25(1974), 335-340.

[21] W. Gutjahr, E. Welzl, G. Woeginger: Polynomial graph colourings, Discrete Applied Math. 35(1992). 29-46

[22] R. Häggkvist, P.Hell: Universality of A-mote graphs, European J.Comb. (1993), 23-27

[23] Z. Hedrlín: On universal partly ordered sets and classes, J. Algebra 11 (1969), 503-509.

[24] Z. Hedrlín: Extensions of structures and full embedding of categories, Actes du Congrès Intern. de Math., Dunod, Paris, 1971, pp. 319-322.

[25] Z. Hedrlín, J. Lambek: How comprehensive is the category of semigroups?, J. Algebra 11 (1969), 195 - 212.

[26] Z.Hedrlín, A. Pultr:Symmmetric relations (undirected graphs) with given semigroups,Monatsh, f. Mathematik 69(1965), 318-322

[27] Z.Hedrlín, A. Pultr: O predstavljenii malych kategorij, Dokl. AN SSSR 160 (1965), 284-286

[28] P. Hell: Rigid graphs with a given numer of edges, Comm.Math.Univ.Carol. 9 (1968), 51-69

[29] P.Hell, J. Nešetřil:The core of a graph, Discrete Math. 109 (1992), 117-126

[30] P.Hell, J. Nešetřil: On the complexity of H-coloring, J. Comb. Th. B,48(1990),92-110

[31] P.Hell, J. Nešetřil: Graphs and k-Societies, Canad. Math. Bull. 13 (1970), 375-381

[32] P.Hell, J. Nešetřil: Homomorphisms of Graphs and their Orientations, Monatsh. Math. 85 (1978), 39-48

[33] P.Hell, J. Nešetřil, X. Zhu: Duality and polynomial testing of tree homomorphisms, Trans. Amer. Math. Soc.,1281-1297

[34] P.Hell, J. Nešetřil, X. Zhu:Complexity of Tree Homomorphism,

[35] P.Hell, X. Zhu: Homomorphisms to oriented paths, Discrete Math. 132 (1992),117-126

[36] P.Hell, X. Zhu: The Existence of Homomorphisms to Oriented Cycles, SIAM J. Disc. Math. 8,2 (1995), 208-222

[37] P.Hell,H. Zhou, X. Zhu: Homomorphisms to oriented cycles, Combinatorica 13(1993), 421-433

[38] W. Hochstätter,J. Nešetřil: Linear Programming Duality and Morphisms, Comm. Math. Univ. Carol. (1999)

[39] W. Hochstätter,J. Nešetřil: A note on MaxFlow-MinCut and Homomorphic Equivalence in Matroids, KAM-DIMATIA Series 99-417 (to appear)

[40] J. R. Isbel: Two set-theoretical theorems in categories, Fund. Math. 53(1963), 43-49

[41] T. R. Jensen and B. Toft: Graph coloring problems, Wiley 1995.

[42] J.H.Kim,J. Nešetřil: Coloring of bounded degree graphs

[43] T.Kloks: Treewidth-computations and approximations, Springer Verlag, Lecture Notes in Computer Science, 842,1994

[44] P.Komárek: Some new good characterizations of directed graphs, Časopis Pěst. Mat. 51(1984), 348-354

[45] P.Komárek: Good orientations of graphs (in Czech),Prague 1988 (doctoral thesis)

[46] A.Kostochka,J. Nešetřil, P.Smolíková: Coloring bounden and degenerated graphs, KAM-DIMATIA Series 98-406; Discrete Math. (to appear)

[47] V. Koubek, V. Rödl: On the Minimum Order of Graphs with Given Semigroup, J. Comb. Th. B 36 (1984), 135 - 155

[48] B. Larose, C. Tardif: Hedetniemi's conjecture and the retracts of products of graphs, 15 pages, submitted, 1999.

[49] B. Larose and C. Tardif, *Strongly rigid graphs and projectivity*, manuscript, 1999.

[50] F. W. Lawvere, F. H. Schanuel: Conceptual Mathematics, Cambridge Univ. Press, 1997

[51] L. Lovász: On the cancelation law among finite relational structures, Periodica Math. Hung. 1,2(1971),145 - 156

[52] L. Lovász: Operations wth structures, Acta.Math.Acad.Sci.Hung. 18 1967, 321 - 329

[53] S. MacLane: Categories for Working Mathematician, Springer

[54] J. Matoušek, J. Nešetřil: Invitation to Discrete Mathematics, Oxford University Press 1998

[55] H.A.Maurer, A. Salomaa, D.Wood: Colorings and interpretations: a connection between graphs and grammar forms, Discrete Applied Math. 3(1981), 119-135

[56] V. Müller: The edge reconstruction hypothesis is true for graphs with more than $n\log_2 n$ edges, J. Comb. Th. B,22(1977), 281-283

[57] V. Müller: On colorable critical and uniquely colorable critical graphs, In: Recent Advances in Graph Theory (ed. M. Fiedler), Academia, Prague, 1975.

[58] V. Müller: On coloring of graphs without short cycles, Discrete Math., 26(1979), 165-179.

[59] J. Nešetřil: On uniquely colorable graphs without short cycles, Časopis Pěst. Mat. 98(1973), 122-125.

[60] J. Nešetřil: The Homomorphism Structure of Classes of Graphs, Combinatorics, Probab. and Comp. 8 (1999), 177-184

[61] J. Nešetřil: Aspects of Structural Combinatorics, Taiwanese J. Math. 3, 4 (1999), 381 - 424.

[62] J. Nešetřil: A rigid graph for every set, KAM-DIMATIA Series 2000-456 (to appear)

[63] J. Nešetřil: The Coloring Poset and its On - Line Universality, KAM Series 2000-458 (submitted)

[64] J. Nešetřil, S. Poljak: Complexity of the Subgraph Problem, Comm. Math. Univ. Carol., 26,2 (1985), 415-420

[65] J. Nešetřil, A. Pultr: On classes of relations and graphs determined by subobjects and factorobjects, Discrete Math. 22(1978), 287-300

[66] J. Nešetřil, V.Rödl: Chromatically optimal rigid graphs, J. Comb. Th. B, 46(1989),133-141

[67] J. Nešetřil, E.Sopena: On Four Coloring Problems, KAM-DIMATIA Series, 98-407

[68] J. Nešetřil, C.Tardif:Density. In: Contemporary Trends in Discrete Mathematics (R.L.Graham, J. Kratochvíl, J. Nešetřil, F.S. Roberts, eds.), AMS, 1999, pp. 229-237

[69] J. Nešetřil, C.Tardif:Density via Duality, Theoret. Comp. Sci.(to appear) (submitted)

[70] J. Nešetřil, C.Tardif:Duality Theorems for Finite Structures (characterizing gaps and good characterizatios). KAM-DIMATIA Series, 98-407 (submitted)

[71] J. Nešetřil, X. Zhu: Path Homomorphisms, Proc. Camb. Phil. Soc., 120 (1996),207-220

[72] J. Nešetřil, X. Zhu: On Bounded Treewidth Duality of Graphs, J. Graph Th. 23,2(1996), 151-162

[73] J. Nešetřil, X. Zhu: On Sparse Graphs with Given Colorings and Homomorphisms, KAM-DIMATIA Series 2000-456.

[74] F. S. Roberts: T-colorings of Graphs Recent Results and Open Problems, Discrete Math. 93(1991),229-245

[75] A. Pultr, V. Trnková: Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories, North Holland, 1980.

[76] C. Tardif: Fractional multiples of Graphs andthe density of vertex transitive graphs, J. Algebraic Comb (to appear)

[77] C. Tardif, X. Zhu: The level of nonmultiplicativity of graphs (preprint 1999).

[78] J.Vinárek: A new proof of the Freyd's Theorem, J. Pure and Appl. Alg. 8(1976), 1-4

[79] P. Vopěnka, A. Pultr, Z.Hedrlín: A rigid relation exists on any set, Comm. Math. Univ. Carol. 6(1965), 149-155

[80] D. J. A. Welsh:Complexity: knots, colourings and counting, Cambridge Univ. Press, Cambridge, 1993

[81] E. Welzl: Symmetric graphs and interpretations. *J. Combin. Theory, Ser. B* **37** (1984), 235–244.

[82] E. Welzl: Color Families are Dense, J. Theoret. Comp. Sci. 17 (1982), 29-41

[83] X. Zhu:A survey on Hedetniemi conjecture, Taiwanese J. Math. 2(1998), 1-24

[84] X. Zhu:Circular chromatic number: a survey. In: Combinaotorics, Graph Theory, Algorithms and Applications (M.Fiedler, J. Kratochvíl, J. Nešetřil, eds.), North Holland (2000)

[85] X. Zhu: Uniquely H-colorable graphs with large girth, J. Graph Theory, **23** (1996), 33-41.

[86] X. Zhu: Construction of uniquely H-colorable graphs, J. Graph Theory, 30(1999), 1-6.

Department of Applied Mathematics
Faculty of Mathematics and Physics
Charles University
Malostranské nám. 25 11800 Praha 1 Czech Republic

nesetril@kam.ms.mff.cuni.cz