Incidence problems

(Lecture Notes)

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1 Formulation

Line-point incidences. Consider a set P of n points and a set L of m lines in the plane. What is the maximum possible number of their *incidences*, i.e. pairs (p, ℓ) such that $p \in P$, $\ell \in L$ and $p \in \ell$? Denote the number of incidences for specific L and P by $I(L, P)$, and let $I(n, m)$ be the maximum of $I(L, P)$ over all choices of an n-element L and an m-element P.

A trivial upper bound is $I(n, m) \leq nm$, but it is easy to see that it can never be attained unless $n = 1$ or $m = 1$. In fact, if n has a similar order of magnitude as m then $I(n, m)$ is asymptotically much smaller than nm. The order of magnitude is known exactly, and is given by the following theorem, conjectured by Erdős and first proved by Szemerédi and Trotter in 1973:

1.1 Theorem (Szemerédi–Trotter theorem). *For all n, m, $I(m, n) = O(n + m + n^{2/3}m^{2/3})$, and this is asymptotically tight.*

The original proof of the upper bound was quite complicated. Then a simpler proof was found, and recently a very simple one. We give two proofs in the sequel (one simpler, and one including important techniques useful also in more general situations). To simplify calculations, we will mostly consider the case $m = n$, which is the most interesting one (the general case uses no new ideas, only a bit more calculation).

Of course, the problem of line–point incidences can be generalized in many ways. We can consider incidences between points and hyperplanes in higher dimensions, or between points and curves of some family in the plane, etc. A particularly interesting case is that of unit circles vs. points, which is closely related to so-called *unit distances problem.*

Unit distances and distinct distances. Let $U(n)$ denote the maximum possible number of pairs of points with unit distance in an n-point set in the plane. For $n \leq 3$ we have $U(n) = \binom{n}{2}$ (all distances can be 1), but already for $n = 4$ at most 5 of the 6 distances can be 1, i.e. $f(4) = 5$. Erdős asked about the asymptotic behavior of the function $U(n)$ for $n \to \infty$.

This can also be re-formulated as an incidence problem. Namely, consider and n-point set P, and draw a unit circle around each point of p, thereby obtaining a set C of n unit circles. Each pair of points at unit distance contributes two circle–point incidences, and hence $U(n) \leq \frac{1}{2}J(n,n)$, where $J(n,m)$ denotes the maximum possible number of incidences between n unit circles and m points.

Unlike to the case of line–point incidences, the correct order of magnitude of $U(n)$ is not known. An upper bound of $O(n^{4/3})$ can be obtained by modifying the proofs of the Szemerédi–Trotter theorem. But the best known lower bound is $U(n) \geq n^{1+c_1/\log\log n}$, for some positive constant c_1.

A related quantity is the minimum possible number of distinct distances determined by n points in the plane; formally

$$g(n) = \min_{P \subset \mathbf{R}^2:\, |P|=n} |\{\mathrm{dist}(x,y) \colon x, y \in P\}|\,.$$

Clearly $g(n) \geq \binom{n}{2}/U(n)$, so the above mentioned bound $U(n) = O(n^{4/3})$ gives $g(n) = \Omega(n^{2/3})$. This has been improved to a lower bound of $\Omega(n^{4/5})$. The best known upper bound is $O(n/\sqrt{\log n})$.

Many cells in arrangements. A problem related to line–point incidences is, what is the maximum total number of vertices of m distinct cells in an arrangement of n lines in the plane. Let us denote this number by $K(n,m)$. A simple construction shows that the maximum number of incidences, $I(n,m)$, is asymptotically upper-bounded by $K(n,m)$; more exactly, we have $I(n,m) \leq K(2n,m)/2$. To see this, consider a set L of n lines and a set P of m points realizing $I(n,m)$, and replace each line $\ell \in L$ by a pair of lines ℓ', ℓ'' parallel to ℓ and lying at distance ε from ℓ:

If $\varepsilon > 0$ is small enough, then a point $p \in P$ incident to k lines in the original arrangement now lies in a tiny cell with $2k$ vertices in the modified arrangement.

It turns out that $K(n,m)$ has the same order of magnitude as $I(n,m)$ does, and the upper bound can be obtained by methods similar to those used for $I(n,m)$. Generalizations of this "many cells problem" to higher dimensions with, say, bounded-degree algebraic surfaces instead of lines becomes considerably more difficult. Even for the complexity of a single cell in an arrangement of bounded-degree algebraic surfaces in $\mathbf{R}^3$, near-tight bounds have been obtained only very recently.

Exercises

1. Let $J(n,m)$ and $U(n)$ be as in the text.

(a) Prove that $J(2n) = O(J(n))$.

(b) We have seen that $U(n) \leq \frac{1}{2}J(n,n)$. Prove that $J(n,n) = O(U(n))$.

2. Show that in dimension 4, an n-point set may determine $\Omega(n^2)$ unit distances. [4 points]

2 Lower Bound for Line–Point Incidences

Here we show that the bound $I(n,n) = O(n^{4/3})$ for the maximum number of incidences of n lines and n points is asymptotically optimal (the lower bound for $I(n,m)$ with $n \neq m$ is obtained similarly). For simplicity, we suppose that $n = 4t^6$ for a natural number t, and we put $m = \sqrt{n} = 2t^3$. For the point set P, we choose the $m \times m$ grid, i.e. we set $P = \{(i,j)\colon i,j = 0,1,2,\ldots,m-1\}$. Let S denote the set of all pairs (a,b), where $a,b = 1,2,\ldots,t$ and a,b are co-prime (their greatest common divisor is 1). For each point $p \in P$ we consider all lines passing through p with slope a/b for some pair $(a,b) \in S$. For L we take the union of all the lines thus obtained for all points $p \in P$.

First we bound the total number of lines in L. A line with slope a/b, with $(a,b) \in S$, passing through a point (i,j) also passes through all points $(i+qa, j+qb)$ for all integers q, and from this we see that any line of L passes through some point $(i,j) \in P$ with either $i \in \{0,1,\ldots,t-1\}$ or $j \in \{0,1,\ldots,t-1\}$. These are altogether at most $2tm$ points, and there are at $|S| \leq t^2$ lines passing through each such point. Altogether we get $|L| < 2t^3 m = n$.

The number of incidences between the points of P and the lines of L equals $|P| \cdot |S|$ (since exactly $|S|$ lines are passing through each point). Hence we need to lower-bound $|S|$, which is the number of pairs of co-prime numbers not exceeding t. Let us remark that with a bit of elementary number theory one can show that $|S| \approx \frac{6}{\pi^2}t^2$, but we will only do a weaker and easier estimate of the form $|S| \geq ct^2$, for a constant $c > 0$, which is enough for our purposes.

The total number of pairs (a,b) is t^2; we will subtract the number of pairs with a nontrivial common divisor. The number of pairs divisible by a prime $p \leq t$ is $\lfloor t/p \rfloor^2 \leq t^2/p^2$, and hence the number of pairs with a nontrivial common divisor is no more than $t^2 \sum_{p \leq t} 1/p^2$, where the sum extends over all primes p not exceeding t. We have

$$\sum_{p \leq t} \frac{1}{p^2} \leq \frac{1}{4} + \sum_{n=3}^{\infty} \frac{1}{n^2} \leq \frac{1}{4} + \int_2^{\infty} \frac{\mathrm{d}x}{x^2} = \frac{3}{4}\,.$$

In this way we have proved the promised estimate $|S| \geq t^2/4$, and hence $I(n,n) \geq .t^2/4 = \Omega(n^{4/3})$. $\square$

3 Lower Bound for the Number of Unit Distances

3.1 Theorem. *There exists a configuration of n points in the plane with at least $n^{1+c_1/\log\log n}$ unit distances, with c_1 a positive constant.*

A configuration with the largest known number of unit distances (asymptotically) is a $\sqrt{n} \times \sqrt{n}$ grid (with a suitably chosen step). Here unit distances are related to the number of possible representations of an integer as a sum of two squares. For the proof we need some results from number theory. We begin with the following claim:

3.2 Lemma. *Let $p_1 < p_2 < \ldots < p_r$ be primes of the form $4k+1$, and put $M = p_1 p_2 \ldots p_r$. Then M can be expressed as a sum of two squares of integers in at least 2^r ways.*

Proof. We need a fact from number theory: each prime p_i can be written as a sum of two squares, $p_j = a_j^2 + b_j^2$. In the sequel, we will work with the ring $\mathbf{Z}[\mathrm{i}]$, the so-called Gauss integers, consisting of all complex numbers $u + \mathrm{i}v$, where $u, v \in \mathbf{Z}$. We need that each element of $\mathbf{Z}[\mathrm{i}]$ can be uniquely factored into primes. From algebra, we recall that a *prime* in the ring $\mathbf{Z}[\mathrm{i}]$ is an element $\gamma \in \mathbf{Z}[\mathrm{i}]$ such that whenever $\gamma = \gamma_1\gamma_2$, with $\gamma_1, \gamma_2 \in \mathbf{Z}[\mathrm{i}]$, then $|\gamma_1| = 1$ or $|\gamma_2| = 1$. Both existence and uniqueness of the prime factorization follows from the fact that $\mathbf{Z}[\mathrm{i}]$ is a Euclidean ring (see an introductory course of algebra for explanation of these notions).

Let us put $\alpha_j = a_j + \mathrm{i}b_j$, and let $\bar{\alpha}_j = a_j - \mathrm{i}b_j$ be the complex conjugate of α_j. We have $\alpha_j\bar{\alpha}_j = (a_j + ib_j)(a_j - ib_j) = a_j^2 + b_j^2 = p_j$. Let us choose an arbitrary subset $J \subseteq I = \{1, 2, \ldots, r\}$ and define $A_J + \mathrm{i}B_J = \left(\prod_{j\in J} \alpha_j\right)\left(\prod_{j\in I\setminus J} \bar{\alpha}_j\right)$. Then $A_J - \mathrm{i}B_J = \left(\prod_{j\in J} \bar{\alpha}_j\right)\left(\prod_{j\in I\setminus J} \alpha_j\right)$, and hence $M = (A_J + \mathrm{i}B_J)(A_J - \mathrm{i}B_J) = A_J^2 + B_J^2$. This gives one expression of the number M as a sum of two squares. It remains to show that for two sets $J \neq J'$, $A_J + \mathrm{i}B_J \neq A_{J'} + \mathrm{i}B_{J'}$. To this end, it suffices to show that all α_j and $\bar{\alpha}_j$ are primes in $\mathbf{Z}[\mathrm{i}]$. Then the numbers $A_J + \mathrm{i}B_J$ and $A_{J'} + \mathrm{i}B_{J'}$ are distinct since they have distinct prime factorizations (no α_j or $\bar{\alpha}_j$ can be obtained from another one by multiplying it by a unit of the ring $\mathbf{Z}[\mathrm{i}]$ — the units are only the elements $1, -1, \mathrm{i}$, and $-\mathrm{i}$).

So suppose that, e.g., $\alpha_j = \gamma_1\gamma_2$, $\gamma_1, \gamma_2 \in \mathbf{Z}[\mathrm{i}]$. We have $p_j = \alpha_j\bar{\alpha}_j = \gamma_1\gamma_2\bar{\gamma}_1\bar{\gamma}_2 = |\gamma_1|^2|\gamma_2|^2$. Now $|\gamma_1|^2$, $|\gamma_2|^2$ are both integers, and since p_j is a prime, we get that $|\gamma_1| = 1$ or $|\gamma_2| = 1$. □

We still need to know that the primes of the form $4k+1$ are sufficiently dense. The relevant result comes from Dirichlet; we state it without proof. The proof is not simple, but it is very nice; we can only recommend the reader to read it in some number theory textbook.

3.3 Theorem (Dirichlet's theorem, a special case). *Let $\pi_{4,1}(n)$ be the number of primes of the form $4k+1$ not exceeding n. For a certain constant $c > 0$ and all sufficiently large n, we have $\pi_{4,1}(n) \geq cn/\log n$.* □

The well-known *prime number theorem* asserts that

$$\pi(n) = (1 + o(1))\frac{n}{\ln n},$$

where $\pi(n)$ denotes the number of primes not exceeding n. The proof is quite complicated. On the other hand, weaker bounds of the form $c_1 n/\log n <$

$\pi(n) < c_2 n/\log n$ (for suitable positive constants c_1, c_2) have quite simple proofs. Dirichlet's theorem implies that the distribution of primes in the arithmetic progression $(1, 5, 9, 13, \ldots)$ is similar to the distribution of primes in the set of all natural numbers, and similarly for any infinite arithmetic progression whose members are not obviously all composite numbers (that is, the first term and the difference are relatively prime).

Proof of Theorem 3.1. Let us suppose that n is a square, and for the set P we choose the points of the $\sqrt{n} \times \sqrt{n}$ grid with step $1/\sqrt{M}$, where M is the product of the first $r-1$ primes of the form $4k+1$, and r is chosen as the largest number such that $M \le \frac{n}{4}$. It is easy to see that each point of the grid participates in at least as many unit distances as there are representations of M by a sum of two squares of nonnegative integers. Since one representation by a sum of two squares nonnegative integers corresponds to at most 4 representations by a sum of two arbitrary integers (the signs can be chosen in 4 ways), there are at least $2^{r-1}/4$ representations by Lemma 3.2, and we have at least $2^{r-1}n/16$ unit distances.

By the choice of r, we have $n < 4p_1p_2\ldots p_r$, and hence $p_r > (\frac{n}{4})^{1/r}$. Further we get by Theorem 3.3, for large enough n, $r = \pi_{4,1}(p_r) \ge cp_r/\log p_r \ge \sqrt{p_r} \ge n^{1/3r}$,and thus $r^{3r} \ge n$. Taking logarithms, we get $3r \log r \ge \log n$, and hence $r \ge \log n/(3\log r) \ge n/(3 \log\log n)$. The number of unit distances is at least $n\, 2^{r-4} \ge n^{1+c_1/\log\log n}$ as Theorem 3.1 claims. Let us remark that for a sufficiently large n the constant c_1 can be made as close to 1 as desired. □

4 Line–Point Incidences via Crossing Numbers

Here we present a very simple proof of the Szemerédi–Trotter theorem 1.1. The proof is based on a result concerning graph drawing.

4.1 Definition. *Let G be a graph (possibly with multiple edges). The* crossing number *of a drawing of G in the plane is the number of (unordered) pairs of edges that intersect in the considered drawing (two edges ending in a common vertex is not considered as an intersection—so a drawing is planar if and only if its crossing number is 0). The* crossing number *of the graph G is the smallest possible crossing number of a drawing of G; we denote it by $cr(G)$.*

As is well-known, for $n > 2$, a planar graph with n vertices has at most $3n-6$ edges. This can be rephrased as follows: if the number of edges is $\ge 3n-5$ then $cr(G) > 0$. The following theorem can be viewed as a generalization of this fact.

4.2 Theorem (Crossing number theorem). *Let G be a simple graph (no multiple edges) with n vertices and t edges. Then*

$$cr(G) \ge c\left(\frac{t^3}{n^2} - 100n\right)$$

where c is a positive constant (the proof given below gives an estimate $c \ge \frac{4}{243} \approx 0.016$.)

Remarks. The lower bound in this theorem is asymptotically tight, i.e. there exist graphs with n vertices, t edges and crossing number $O(t^3/n^2)$ (see Exercise 1). The assumption that the graph be simple cannot be omitted (but see Lemma 5.2 below).

For a proof of this theorem, we need a simple lemma:

4.3 Lemma. *The crossing number of any graph (without multiple edges) with n vertices and t edges is at least $t - 3n$.*

Proof. If $t \leq 3n$ there is nothing to prove; otherwise proceed by induction on t with n fixed. Let G have $t > 3n$ edges, and consider some drawing of G. This is non-planar, hence there exists an edge e crossing some other edge. By deleting e from the considered drawing we get a drawing of a graph G' with n vertices and $t-1$ edges. By the inductive hypothesis, there at least $t-1-3n$ pairs of intersecting edges in this drawing, and hence the original drawing of G had at least $t - 3n$ such pairs. $\square$

Proof of Theorem 4.2. It can be done by induction, but a probabilistic proof seems more elegant. We may assume $t \geq \frac{9}{2}n$, for otherwise the claimed bound is negative. Consider some drawing of a graph $G = (V, E)$ with n vertices, t edges, and crossing number x. Let $p \in (0,1)$ be a parameter; later on we set it to a suitable value. Choose a random subset $V' \subseteq V$, picking each vertex $v \in V$ into V' independently with probability p. Let G' be the subgraph of G induced by the subset V'. Put $n' = |V'|$, $t' = |E(G')|$, and let x' be the crossing number the graph G' in the drawing "inherited" from the considered drawing of G. The expectation of n' is $\mathbf{E}[n'] = np$. The probability that a given edge appears in $E(G')$ is p^2, and hence $\mathbf{E}[t'] = tp^2$, and similarly we get $\mathbf{E}[x'] = xp^4$. At the same time, by Lemma 4.3 we always have $x' \geq t' - 3n'$, and so this relation holds for the expectations as well: $\mathbf{E}[x'] \geq \mathbf{E}[t'] - 3\mathbf{E}[n']$. From this we have $xp^4 \geq tp^2 - 3np$. Setting $p = 9n/2t$ (this is valid since we assume $t \geq \frac{9}{2}n$) and a simple calculation yield

$$x \geq \frac{4}{243}\frac{t^3}{n^2}\,.$$

$\square$

Proof of the Szemerédi–Trotter theorem 1.1. We consider a set L of n lines and a set P of m points in the plane realizing the maximum number of incidences $I(n,m)$. We define a certain topological graph G, that is, a graph together with its drawing in the plane. The vertex set is P. By deleting the points of P, each line of L is partitioned into several segments and two semilines (if a line contains no points of P it can be dropped anyway). We delete the semilines, and each of the remaining segments is an edge connecting two points of P, namely its endpoints. So if a line $\ell \in L$ contains k points of P it contributes $k-1$ edges to P, and hence $I(n,m) = t + m$, t being the number of edges of G. Since the edges are parts of the n lines, at most $\binom{n}{2}$ pairs may cross: $cr(G) \leq \binom{n}{2}$. On the other hand, from Theorem 4.2 we get $cr(G) \geq c(t^3/m^2 - 100m)$. So $c(t^3/m^2 - 100m) \leq cr(G) \leq \binom{n}{2}$, and calculation gives $t = O(n^{2/3}m^{2/3} + m)$. This proves Theorem 1.1. $\square$

The best known upper bound on the number of unit distances, $U(n) = O(n^{4/3})$, can be proved along similar lines; see Exercise 2.

Exercises

1. Show that for any n and t, $5n \le t \le \binom{n}{2}$, there exist graphs with n vertices, t edges and crossing number $O(t^3/n^2)$. [2 points]
2. In a manner similar to the above proof for line–point incidences, prove the bound $J(n,n) = O(n^{4/3})$, where $J(n,m)$ denotes the maximum possible number of incidences between n points and m unit circles in the plane (be careful in handling possible multiple edges in the considered topological graph!). [3 points]
3. Let $v_1, \ldots, v_n$ be nonzero vectors in $\mathbf{R}^3$. Prove that the number of pairs (v_i, v_j) such that v_i and v_j are orthogonal is $O(n^{4/3})$. [3 points]
4. Using the Szemerédi–Trotter theorem, show that the maximum possible number of distinct lines such that each of them contains at least k points of a given m-point set P in the plane is $O(m^2/k^3 + m/k)$. [2 points]
5. Let $K(n,m)$ denote the maximum total number of edges of m distinct cells in an arrangement of n lines in the plane. Prove $K(n,m) = O(n^{2/3}m^{2/3} + n + m)$ using the method of the present section (it may be convenient to classify edges into top and bottom ones and bound each type separately). [4 points]

5 Distinct Distances via Crossing Numbers

Here we present Székely's proof of the current best bound on the number of distinct distances in the plane:

5.1 Theorem (Distinct distances in $\mathbf{R}^2$). *The minimum number $g(n)$ of distinct distances determined by an n-point set in the plane satisfies $g(n) = \Omega(n^{4/5})$.*

First, we need a simple extension of Theorem 4.2 for multigraphs:

5.2 Lemma. *Let G be a multigraph with n vertices, t edges, and maximum edge multiplicity k. Then*

$$cr(G) = \Omega\left(\frac{t^3}{kn^2}\right) - O(k^2 n)\,.$$

Proof. Consider a fixed drawing of G. We choose a subgraph G' of G by the following random experiment. In the first stage, we consider each edge of G independently, and we delete it with probability $1 - 1/k$. In the second stage, we delete all remaining multiple edges, and this gives G', which has n vertices, t' edges, and x' crossing pairs of edges. Consider the probability p_e that a fixed edge $e \in E(G)$ remains in G'. Clearly $p_e \le 1/k$. On the other hand, if e was one of $k' \le k$ parallel edges, then the probability that after the first stage it remains and all other edges parallel to it are deleted is

$$\frac{1}{k}\left(1 - \frac{1}{k}\right)^{k'-1} \ge \frac{1}{3k}$$

(since $(1-1/k)^{k-1} \geq \frac{1}{3}$). We get $\mathbf{E}[t'] \geq t/3k$ and $\mathbf{E}[x'] \leq x/k^2$. Applying Theorem 4.2 for the graph G' and taking expectations, we have

$$\mathbf{E}[x'] \geq c \left(\frac{\mathbf{E}[t'^3]}{n^2} - 100n \right) .$$

By convexity (Jensen's inequality), we have $\mathbf{E}[t'^3] \geq (\mathbf{E}[t'])^3 = \Omega(t^3/k^3)$. Plugging this plus the bound $\mathbf{E}[x'] \leq x/k^2$ into the above formula, we get

$$\frac{x}{k^2} = \Omega\left(\frac{t^3}{k^3 n^2}\right) - O(n) ,$$

and the claim follows. □

Proof of Theorem 5.1. Fix an n-point set P, and let g be the number of distinct distances determined by P.

We may assume $g \leq n/3 \log n$ (otherwise we are done). Around each point of p, we draw g concentric circles whose radii are all the distances occurring in P. Since all possible distances are included, this set of circles has $n(n-1)$ incidences with P.

Next, we delete all circles containing at most 2 points of P. This destroys at most $2n$ incidences, so still almost n^2 incidences remain. Define a multigraph G similarly as we did in the proof for line–point incidences; that is, the points of P are vertices and the arcs of the circles are edges. The number of edges of G equals the number of remaining incidences, i.e. $\Omega(n^2)$.

Since any two of the gn considered circles intersect at most twice, $cr(G) \leq g^2n^2$. We would like to apply Lemma 5.2 but we have no good bound on edge multiplicity. In fact, the edge multiplicity may in principle be as high as g, and if we substituted this into Lemma 5.2 directly we would only get $g = \Omega(n^{2/3})$. To do better, edges with very high multiplicity have to be handled separately.

Consider two points $u, v \in P$ and edges $e_1, \ldots, e_k$ connecting them in our multigraph G. These edges are circular arcs. The centers of these arcs are points $p_1, \ldots, p_k$ of P, and they all lie on the symmetry axis ℓ_{uv} of the segment uv:

v

ℓ_{uv}

u

Thus multiple edges between u and v produce a line ℓ_{uv} with many incidences with the points of P. The idea is now to apply the line–point incidence bound for these lines.

Claim. *For each k, the number of edges in G having at least k edges parallel to them is at most $O(n^2g/k^2 + gn\log n)$.*

Proof. Let M_i be the set of pairs $\{u,v\}$ of vertices of G such that there are at least 2^i and fewer than 2^{i+1} parallel edges connecting u and v in G, and let E_i be the set of all edges connecting the pairs from M_i. What we are looking for is $\sum_{k\le 2^i<n}|E_i|$.

Let L_i be the set of lines ℓ_{uv} corresponding to the pairs $\{u,v\}\in M_i$ (i.e. the axes of symmetry of all these pairs). As was noted above, each edge connecting u and v determines one incidence of the line ℓ_{uv} with P. On the other hand, we note that one such incidence $p\in\ell$ may be determined by at most $2g$ edges (since there are g circles centered at p, and each circle may contribute at most two edges intersecting the line ℓ). Therefore, the size of E_i is at most $2g$-times the number of incidences of L_i with P.

Each line of L_i has at least 2^i incidences with the n points of P, and the Szemerédi–Trotter theorem implies

$$|L_i| = O\left(\frac{n^2}{2^{3i}}+\frac{n}{2^i}\right)$$

(see Exercise 4 in section 4). So the lines of L_i have altogether $O(n^2/2^{2i}+n)$ incidences with P. Hence $|E_i| = O(n^2g/2^{2i}+ng)$, and the claim follows by summing over i.

To finish the proof of Theorem 5.1, we set $k=C\sqrt{g}$ for a sufficiently large constant C, and we delete all edges of multiplicity larger than k from the multigraph G. By the claim, we have deleted at most $O(n^2/C^2+gn\log n)\le n^2/2$ edges (recall that we have assumed $g\le n/3\log n$ and that C is chosen as large as needed), so still $\Omega(n^2)$ edges remain. The resulting multigraph G' has n vertices, $\Omega(n^2)$ edges, maximum edge multiplicity $k=O(\sqrt{g})$, and crossing number $\le g^2n^2$. Plugging all this into Lemma 5.2 and simplifying, we finally obtain $g=\Omega(n^{4/5})$ as desired. □

Exercises

1. Let $\bar{J}(n,m)$ be the maximum number of incidences between n arbitrary circles and m points in the plane. Try the following approach to bounding $\bar{J}(n,n)$. Let K be a set of n circles, C the set of their centers, and P a set of n points.

   (a) First, assume that the centers of the circles are mutually distinct, i.e. $|C|=|K|$. Remove circles with ≤ 2 incidences, and let the others define a drawing of a multigraph G with vertex set P and arcs of the circles as edges. Divide the edges into groups $E_0,E_1,\ldots$, where E_i containins edges of multiplicity at least 2^i and smaller than 2^{i+1}. Each edge connecting points u and v defines an incidence of a point $c\in C$ with the line ℓ_{uv}, the symmetry axis of u and v. Argue that by Szemerédi–Trotter, $|E_i|=O(n^2/2^{2i}+n)$. Use this for large i, and for small i apply Lemma 5.2; balance the threshold suitably. What bound is obtained for the total number of incidences? [3 points]

   (b) Extend the argument to handle concentric circles too. [2 points]

2. This exercise provides another bound for $\bar{J}(n,n)$, the maximum possible number of incidences between n arbitrary circles and n points in the plane. Let P be the

points and K the circles. Let P_i be the points with at least $d_i = 2^i$ and fewer than 2^{i+1} incidences; we will argue for each P_i separately.

Define the multigraph G on P_i as usual, with arcs of circles of K connecting neighboring points of P_i (circles with ≤ 2 incidences with P_i are deleted). Let E be the set of edges of G. For a point $u \in P_i$, let $N(u)$ be the set of its neighboring points, and for a $v \in N(u)$, let $\mu(u,v)$ be the number of edges connecting u and v. For an edge e, define its *partner edge* as the edge following after e clockwise around its circle.

(a) Show that for each $u \in P_i$, $|\{v \in N(u)\colon \mu(u,v) \geq 4\sqrt{d_i}\}| < \sqrt{d_i}/2$. [1 point]

(b) Let $E_h \subseteq E$ be the edges of multiplicity at least $4\sqrt{d_i}$. Argue that for at least $\frac{1}{4}$ of the edges in E_h, their partner edges do not belong to E_h, and hence $|E \setminus E_h| = \Omega(|E|)$. [3 points]

(c) Delete the edges of E_h from the graph, and apply Lemma 5.2 to bound $|E \setminus E_h|$. What overall bound does all this give for $\bar{J}(n,n)$? [2 points]

6 Line–Point Incidences via Cuttings

Here we give another proof of the upper bound for line–point incidences, namely $I(n,n) = O(n^{4/3})$. The technique is quite different, leads to an efficient algorithm, and currently it seems more generally applicable.

6.1 Lemma (A worse but useful bound).

$$I(n,m) = O(n\sqrt{m} + m) \tag{1}$$
$$I(n,m) = O(m\sqrt{n} + n)\,. \tag{2}$$

First proof. We define a suitable bipartite graph. One class of vertices are the given points, the second class are the given lines. An edge (p,ℓ) means that the point p lies on the line ℓ. So the number of incidences is just the number of edges. Since two distinct points determine a line uniquely, we get that this graph contains no $K_{2,2}$ as a subgraph (two distinct lines both containing the same two distinct points). In extremal graph theory, there is a theorem saying that a bipartite graph with class sizes m and n, respectively, and containing no $K_{2,2}$ has at most $O(m\sqrt{n} + n)$ edges. Since the situation is symmetric with respect to m and n, we also get the bound $O(n\sqrt{m} + m)$. □

For arbitrary bipartite graphs without $K_{2,2}$ (not necessarily coming from a geometric situation like the one we discussed above), the just mentioned upper bound on the number of edges is asymptotically tight. So, to do better for line-point incidences, one has to use some more geometry.

In general, if r and s are constants, it is known that a bipartite graph with class sizes m and n containing no $K_{r,s}$ (where the r vertices would sit in the m-point class and the s vertices in the n-point class) has no more than $O(\min(m^{1-1/r}n + m, n^{1-1/s}m + n))$ edges. Note that for $r \neq s$, the situation is not symmetric anymore: by forbidding the "reverse" placement of $K_{r,s}$, we get a different bound in general. This is suspected to be tight, but a matching lower bound is only known for few special values of r and s.

Second proof of Lemma 6.1. There are $\binom{n}{2}$ crossing pairs of lines in total. On the other hand, a point $p_i \in P$ with d_i incidences "consumes" $\binom{d}{2}$ crossing pairs (their intersections all lie in p_i). Therefore $\sum_{i=1}^{m} \binom{d_i}{2} \leq \binom{n}{2}$.

We want to upper-bound $\sum d_i$, which can be done in various ways. For instance, we may assume $d_i \geq 1$ for all i, and then we have $\binom{d_i}{2} \geq (d_i - 1)^2/2$. By Cauchy–Schwarz, we get

$$\sum_{i=1}^{m}(d_i - 1) \leq \sqrt{m\sum_{i=1}^{m}(d_i-1)^2} \leq \sqrt{2\binom{n}{2}m},$$

and hence $\sum d_i = O(n\sqrt{m} + m)$. The other inequality can be proved similarly by looking at pairs of points on each line, or by noting that $I(n,m) = I(m,n)$ by geometric duality.

What we did in this second proof was just a direct geometric interpretation of the usual proof of the bound for the number of edges of $K_{2,2}$-free graphs used in the first proof. So both proofs are really the same, more or less. □

The bound from Lemma 6.1 is weaker than the tight Szemerédi–Trotter bound, but it is tight if $n^2 \leq m$ or $m^2 \leq n$.

Proof of Theorem 1.1 for $n = m$. The idea is to convert the "balanced" case (n points, n lines) into a collection of "unbalanced" subproblems, for which the above lemma is good enough. We apply the following important result:

6.2 Lemma (Cutting lemma). *Let L be a set of n lines in the plane, and let r be a parameter, $1 < r < n$. Then the plane can be subdivided into t generalized triangles (this means, intersections of three halfplanes) $\Delta_1, \Delta_2, \ldots, \Delta_t$ in such a way that the interior of each triangle Δ_i is intersected by at most n/r lines of L, and we have $t \leq Cr^2$ for a certain constant C.*

A proof will be discussed later, now we continue the proof of the Szemerédi–Trotter theorem.

Let P be the considered n-point set, L the set of n lines, and $I(P,L)$ the set of their incidences. We fix a "magic" value $r = n^{1/3}$ and by the cutting lemma we divide the plane into $t = O(r^2) = O(n^{2/3})$ generalized triangles $\Delta_1, \ldots, \Delta_t$, such that the interior of each is intersected by at most $n/r = n^{2/3}$ lines of L.

Let P_i denote the points of P lying inside Δ_i or on its boundary but not at the vertices of Δ_i, and let L_i be the set of lines of L intersecting the interior of Δ_i. We have

$$\begin{array}{rcll} |I(L,P)| & \leq & \sum_{i=1}^{t} |I(L_i,P_i)| & (A) \\ & & +I(n,3t) & (B) \\ & & +I(3t,n)\,. & (B') \end{array}$$

The term (B) subsumes the cases of points of P lying at vertices of some of the triangles Δ_i, which were included in no P_i. These are at most $3t$ points which contribute no more than $I(n,3t)$ incidences with the lines of L. The term (B') covers all cases when a line of L contains an edge of a triangle Δ_i (but does not intersect its interior, so it is not included in L_i). There are at most $3t$ lines containing edges, and these lines have no more than $I(3t, n)$ incidences with the points of P. By Lemma 6.1, the terms (B) and (B') are both bounded by $O(t\sqrt{n} + n) = O(n^{7/6}) \ll n^{4/3}$.

It remains to estimate the main term (A). We have $|L_i| \le n/r = n^{2/3}$ and $\sum_{i=1}^{t} |P_i| \le 2n$ (since each point of P goes into at most two P_i). Using the bound (1) for each $I(L_i, P_i)$ we get

$$\sum_{i=1}^{t} |I(L_i, P_i)| \le \sum_{i=1}^{t} I(n^{2/3}, |P_i|) = \sum_{i=1}^{t} O(|P_i| n^{1/3} + n^{2/3}) =$$

$$O(n^{1/3}) \left(\sum_{i=1}^{t} |P_i| \right) + O(tn^{2/3}) = O(n^{4/3}) .$$

This finally shows $|I(n, n)| = O(n^{4/3})$. □

Exercises

1. In this exercise we aim at bounding $K(n, m)$, the maximum total number of edges of m distinct cells in an arrangement of n lines in the plane. Let L be a set of n lines in general position.
   (a) Prove the bound $K(n, m) = O(n\sqrt{m} + m)$. [3 points]
   (b) Prove $K(n, n) = O(n^{4/3})$ using the cutting lemma. [4 points]
2. Let $J(n, m)$ be the maximum number of incidences between n unit circles and m points in the plane. Prove $J(m, n) = O(m\sqrt{n} + n)$ by the method of Lemma 6.1. [2 points]
3. Let $\bar{J}(n, m)$ be the maximum possible number of incidences between n arbitrary circles and m points in the plane. Prove $\bar{J}(n, m) = O(n\sqrt{m} + n)$ and $\bar{J}(n, m) = O(mn^{2/3} + n)$. [2 points]

7 A Weaker Cutting Lemma

In order to express ourselves more economically, we introduce the following definition. A *cutting* is a finite collection of generalized triangles with disjoint interiors covering the plane. A given cutting is a $\frac{1}{r}$*-cutting for* a set L of n lines if the interior of each triangle of the cutting is intersected by at most n/r lines of L.

A lower bound. First let us consider a lower bound for the number of (generalized) triangles which are necessary to form a $\frac{1}{r}$-cutting for a set of n lines. Consider n lines in general position. Their arrangement has, as we know, $\binom{n}{2} + n + 1 \ge n^2/2$ cells. On the other hand, if we consider a triangle Δ_i whose interior is intersected by $k \le n/r$ lines ($k \ge 1$), we see that the interior of Δ_i is divided into at most $\binom{k}{2} + k + 1 \le 2k^2$ cells. Since each cell of the arrangement has to show up in the interior of at least one triangle Δ_i, we get that the number of triangles is at least $n^2/4k^2 = \Omega(r^2)$. Hence the cutting lemma is asymptotically best possible.

Proof of a weaker version of the cutting lemma 6.2. Here we prove a weaker version of the cutting lemma, namely that for any given n lines a $\frac{1}{r}$-cutting consisting of $O(r^2 \log^2 n)$ triangles exists. This would give us correspondingly weaker results in applications of the cutting lemma. For instance, we would only get the bound $I(n, n) = O(n^{4/3} \log^{2/3} n)$ for line–point incidences.

Let us select a random sample $S \subseteq L$ of the given lines; we make s independent random draws, drawing a random line from L each time. These are draws with replacement: one line can be selected several times, so S may have fewer than s lines.

Consider the arrangement of the lines in S. Partition the cells that are not triangles into triangles by adding some suitable diagonals, as is illustrated below:

This creates (generalized) triangles $\Delta_1, \Delta_2, \ldots, \Delta_t$ where $t = O(s^2)$ (since we have a drawing of a planar graph with $\binom{s}{2}$ vertices; see also Exercise 2).

7.1 Lemma. *For $s = 6r \ln n$, the following holds with a positive probability: The Δ_i form a $\frac{1}{r}$-cutting for L; that is, the interior of no Δ_i is intersected by more than n/r lines of L.*

This implies the promised weaker version of the cutting lemma: since the probability of the sample S being good is positive, there exists at least one good S which yields the desired collection of triangles.

Proof of Lemma 7.1. Let us say that a triangle T is *dangerous* if its interior is intersected by at least $k = n/r$ lines of L. We fix some arbitrary dangerous triangle T. What is the probability that no line of the sample S intersects the interior of T? We select a random line s times. The probability that we never hit one of the k lines intersecting the interior of T is at most $(1 - k/n)^s$. Using the well-known inequality $e^x \geq 1 + x$, we can upper-bound this probability by $e^{-ks/n} = e^{-6 \ln n} = n^{-6}$.

Call a triangle T *interesting* (for L) if it can appear in a triangulation for some sample $S \subseteq L$. Any interesting triangle has vertices at some three vertices of the arrangement of L, and hence there are fewer than n^6 interesting triangles. Therefore, with a positive probability, a random sample S intersect the interiors of all the dangerous interesting triangles simultaneously. In particular, none of the triangles Δ_i appearing in the triangulation of such a sample S may be dangerous. This proves Lemma 7.1. $\square$

The proof just shown can be generalized to deal with hyperplanes in $\mathbf{R}^d$. In such a situation, we want to partition $\mathbf{R}^d$ into "generalized simplices", the interior of each being intersected by at most n/r hyperplanes. The minimum necessary number of simplices is of the order r^d, and this can actually be achieved, although the simple approach just outlined only gives $O(r^d \log^d n)$. A further generalization of the problem is possible; hyperplanes may be replaced by more general surfaces, typically by algebraic surfaces of constant-bounded degree (then we do not partition into simplices, but into suitable more complicated "elementary cells"). This more general problem is not yet solved satisfactorily; only some partial results and weaker bounds are known.

Returning to our specific situation (lines in the plane), how can a better bound than the $O(r^2 \log^2 n)$ given above be proved? A more sophisticated probabilistic reasoning (e.g., the so-called ε-net theorem) shows that it is in fact sufficient to choose $s = \mathit{const} \cdot r \log r$ in Lemma 7.1, instead of $\mathit{const} \cdot r^2 \log^2 n$, and still no interesting dangerous triangle will be missed by S. This improvement is important for small r (say constant); it shows that the number of triangles in a $\frac{1}{r}$-cutting can be bounded independent of n.

To prove the asymptotically tight bound $O(r^2)$ seems considerably more complicated. Currently, there are two methods, both based on random sampling, that achieve this tight bound and that generalize for hyperplanes in $\mathbf{R}^d$. For the case of lines in the plane, there is one more method, deterministic and quite elementary, which we demonstrate next.

Exercises

1. Calculate the exact expected size of S, a sample drawn from n elements by s independent random draws with replacements. [3 points]

2. Calculate the number of (generalized) triangles arising by triangulating an arrangement of n lines in the plane in general position. (First, specify how exactly are the unbounded cells triangulated.) [2 points]

3. (Cutting lemma for circles) Consider a set K of n circles in the plane. Select a sample $S \subseteq K$ by s independent random draws with replacement. Consider the arrangement of S, and construct its *vertical decomposition*, that is, from each vertex extend a vertical segment upwards and downwards until it hits some circle of S. Similarly draw vertical segments from the leftmost and rightmost point of each circle.

   (a) Show that this partitions the plane into $O(s^2)$ "circular trapezoids" (shapes bounded by at most two vertical segments and at most two circular arcs). [2 points]

   (b) Show that for $s = Cr \ln n$ for a sufficiently large constant C, there is a positive probability that the sample S intersects all dangerous interesting circular trapezoids, where "dangerous" and "interesting" are defined analogously to the above proof of the weaker version of the cutting lemma. [3 points]

4. Using Exercises 3 and 6.2, show that the number of unit distances determined by n points in the plane is $O(n^{4/3} \log^{2/3} n)$. [3 points]

5. Using Exercises 3 and 6.3, show $\bar{J}(n,n) = O(n^{7/5} \log^c n)$ (for some constant c), where $\bar{J}(n,m)$ is the maximum possible number of incidences between n arbitrary circles and m points in the plane. [3 points]

8 Cutting Lemma: a Tight Bound

Here we give a simple and elementary proof of the cutting lemma (Lemma 6.2), which does not seem to generalize to higher-dimensional situations.

For simplicity, suppose that the given set L of n lines is in general position. (If not, perturb it slightly to get general position, construct the $\frac{1}{r}$-cutting, and perturb back; this gives a $\frac{1}{r}$-cutting for the original collection of lines — we omit the details.) First we need some definitions and observations concerning levels.

Levels and their simplifications. Recall that the level of a point in the plane is the number of lines of L lying strictly below it. Consider the set E_k of all edges of the arrangement of L having level k (where $0 \le k < n$). These edges form a monotone polygonal line (monotone means that each vertical line intersects it at exactly one point). This polygonal line makes a turn at each endpoint of its edges; schematically, it looks as follows:

Let $e_0, e_1, \ldots, e_t$ be the edges of E_k numbered from left to right; e_0 and e_t are the unbounded rays. Let us fix a point p_i in the interior of e_i. For an integer parameter $q \ge 2$, we define the *q-simplification of level k* as the monotone polygonal line containing the part e_0 up to the point p_0, the segments p_0p_q, $p_qp_{2q}, \ldots, p_{\lfloor (t-1)/q \rfloor q}p_t$, and the part of e_t from p_t on. Thus, the q-simplification has at most $t/q+2$ edges. Here is an illustration for $t=9$, $q=4$:

p_1 p_2 p_3 p_4 p_8 p_9 e_9 p_0 e_0

(We could have defined the q-simplification by connecting every qth vertex of the level, but the present way simplifies some future considerations a bit.)

8.1 Lemma.

(i) The portion Π of the level k (considered as a polygonal line) between the points p_j and p_{j+q} is intersected by at most $q+1$ lines of L.

(ii) The segment p_jp_{j+q} is intersected by at most $q+1$ lines of L.

(iii) The q-simplification of level k is contained in the strip between levels $k - \lceil q/2 \rceil$ and $k + \lceil q/2 \rceil$.

Proof. Part (i) is obvious: each line of L intersecting Π contains one of the edges $e_j, e_{j+1}, \ldots, e_{j+q}$. As for (ii), Π is connected, and hence all lines intersecting its convex hull must intersect Π itself as well. The segment $p_j p_{j+q}$ is contained in $\mathrm{conv}(\Pi)$.

Concerning (iii), imagine walking along some segment $p_j p_{j+q}$ of the q-simplification. We start at an endpoint, which has level k. Our current level may only change as we cross a line of L. Moreover, having traversed the whole segment we must be back to level k. Thus, to get from level k to $k+i$ and back to k we need to cross at least $2i$ lines on the way. From this and (ii), $2i \leq q+1$, and hence $i \leq \lfloor (q+1)/2 \rfloor = \lceil q/2 \rceil$. □

Proof of the cutting lemma 6.2. Let r be the given parameter. If $r = \Omega(n)$ then it suffices to produce a 0-cutting of size $O(n^2)$ by simply triangulating the arrangement of L. Hence we may assume that $r \ll n$.

Set $q = \lceil n/10r \rceil$. Divide the levels $E_0, E_1, \ldots, E_{n-1}$ into q groups: the ith group contains all E_j with j congruent to i modulo q ($i = 0, 1, \ldots, q-1$). Since the total number of edges in the arrangement is n^2, there is an i such that the ith group contains at most n^2/q edges. We fix one such i; from now on, we will only consider the levels $i, q+i, 2q+i, \ldots$, and we construct the desired $\frac{1}{r}$-cutting from them.

Let P_j be the q-simplification of the level $jq+i$. If E_{qj+i} had m_j edges then P_j has at most $m_j/q + 3$ edges, and the total number of edges of the P_j, $j = 0, 1, \ldots, \lfloor (n-1)/q \rfloor$, can be estimated by $n^2/q^2 + 3(n/q+1) = O(n^2/q^2)$. We note that the polygonal chains P_j never intersect properly: if they did, a vertex of some P_j, which has level $qj+i$, would be above P_{j+1}, and this is ruled out by Lemma 8.1(iii). Form the vertical decomposition for the P_j; that is, extend a vertical segments from each vertex of P_j upwards and downwards, until it hits P_{j-1} and P_{j+1}:

This subdivides the plane into $O(n^2/q^2) = O(r^2)$ trapezoids.

We claim that each such trapezoid is intersected by at most n/r lines of L, and hence that the desired $\frac{1}{r}$-cutting can be obtained by subdividing each trapezoid into two triangles (or, for most applications of $\frac{1}{r}$-cuttings, trapezoids are as good as triangles). So we look at some trapezoid in the strip between P_j and P_{j+1}. By Lemma 8.1, it lies between levels $qj + i - \lceil q/2 \rceil$ and $q(j+1) + i + \lceil q/2 \rceil$, and therefore each of its vertical sides is intersected by no more than $3q$ lines. The bottom side is a part of an edge of P_j, and consequently it is intersected by no more than $q+1$ lines; similarly for the top side. Hence the number of lines intersecting the considered trapezoid is certainly at most $10q \leq n/r$. (A more careful analysis shows that one trapezoid is in fact intersected by at most $2q + O(1)$ lines.) □

Exercises

1. (a) Verify that each trapezoid arising in the described construction is intersected by at most $2.5q + O(1)$ lines. Setting q appropriately, show that the plane can subdivided into $12.5r^2 + O(r)$ trapezoids, each intersected by at most n/r lines, assuming $1 \ll r \ll n$. [2 points]

   (b) Improve the bounds from (a) to $2q + O(1)$ and $8r^2 + O(r)$, respectively. [4 points]