5. DÚ

1. $\sqrt{x}$ je spojitá v $x \in [0, \infty)$

Cíl: $\forall x\ \forall \varepsilon > 0\ \exists \delta;\ |x-a| < \delta \Rightarrow |\sqrt{x} - \sqrt{a}| < \varepsilon$

• $x = 0$ zvolíme $\delta := \varepsilon^2$. Potom $|0-a| = a < \delta \Rightarrow |\sqrt{0} - \sqrt{a}| = \sqrt{a} < \sqrt{\delta} = \varepsilon$

• $x \in (0, \infty)$ $\quad \delta > |x-a| = |(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})| = |\sqrt{x}-\sqrt{a}|(\sqrt{x}+\sqrt{a})$

$$|\sqrt{x}-\sqrt{a}| < \frac{\delta}{\sqrt{x}+\sqrt{a}} \leq \frac{\delta}{\sqrt{x}}$$

$$\delta := \varepsilon\sqrt{x}$$

2. $$\lim_{x\to 0} \frac{\arcsin(\sqrt{x})}{\ln(1+\sqrt{x})} = \lim_{x\to 0} \frac{\arcsin(\sqrt{x})}{\sin(\arcsin\sqrt{x})} \cdot \frac{\sin(\arcsin(\sqrt{x}))}{\ln(1+\sqrt{x})} =$$

$$= \lim_{x\to 0} \frac{1}{\frac{\sin(\arcsin\sqrt{x})}{\arcsin(\sqrt{x})}} \cdot \frac{1}{\frac{\ln(1+\sqrt{x})}{\sqrt{x}}} = 1.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$$

$f(x) = o(g(x))$ pro $x \to a$ znamená, že $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$

$f(x) = x^2$ je $o(x^3)$ pro $x \to \infty$ $\qquad \ln(x) = o(x)$ atd. pro $x \to \infty$

$x^3 = o(x^2)$ pro $x \to 0$

$$\lim_{x\to 0} \frac{e^x\sin(x) - x(1+x)}{x^3} = \lim_{x\to 0} \frac{\left(1+x+\frac{x^2}{2}+o(x^2)\right)\left(x - \frac{x^3}{3!} + o(x^3)\right) - x(1+x)}{x^3} =$$

$$= \lim_{x\to 0} \frac{\left(1+x+\frac{x^2}{2}+o(x^2)\right)\left(1-\frac{x^2}{3!}+o(x^2)\right) - (1+x)}{x^2} = \lim \frac{\cancel{1}+\cancel{x}+\frac{x^2}{2}-\frac{x^2}{3!}+o(x^2)\cancel{-1}-\cancel{x}}{x^2} =$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2} - \frac{x^2}{3!} + o(x^2)}{x^2} = \frac{1}{2} - \frac{1}{6} + \overset{\to 0}{o(1)} = \frac{1}{3}$$

$$\lim_{x\to\infty} \left(x - x^2\ln\left(1+\frac{1}{x}\right)\right) =$$

Použijeme Taylorův rozvoj pro $\ln\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)$

$$= \lim_{x\to\infty} x - x^2\left(\frac{1}{x} - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right) \overset{\lim_{x\to\infty}}{=} x - x + \frac{1}{2} - \underbrace{o\left(\frac{1}{x^2}\cdot x^2\right)}_{= o(1)} = \frac{1}{2}$$

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alternativně l'Hospital

$$\lim_{x\to\infty} \left(x - x^2\ln\left(1+\frac{1}{x}\right)\right) = \lim x^2\left(\frac{1}{x} - \ln\left(1+\frac{1}{x}\right)\right) = \lim \frac{\frac{1}{x} - \ln\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}}$$ a $2\times$ l'H.

Průběh funkce $\frac{x^3}{(x-2)^2}$ $\qquad$ DO $\mathbb{R}\setminus\{2\}$

$$\lim_{x\to 2^+} \frac{x^3}{(x-2)^2} = \text{"}\frac{8}{0^+}\text{"} = \infty = \lim_{x\to 2^-} \frac{x^3}{(x-2)^2}$$

$$\lim_{x\to\infty} \frac{x^3}{(x-2)^2} = \lim_{x\to\infty} \frac{x}{1-\frac{4}{x}+\frac{4}{x^2}} = \infty$$

$$\lim_{x\to-\infty} \frac{x^3}{(x-2)^2} = \lim_{x\to-\infty} \frac{x}{1-\frac{4}{x}+\frac{4}{x^2}} = -\infty$$

$(0,0)$ průsečík s osami

Asymptota se směrnicí $kx+q$; $\lim_{x\to\infty} f(x) - kx - q = 0$

Pokud $$\lim_{x\to\infty} \frac{\frac{x^3}{(x-2)^2}}{x} = k \in \mathbb{R}$$ pak je to směrnice asymptoty

$$\lim_{x\to\infty} \frac{\frac{x^3}{(x-2)^2}}{x} = 1 \qquad q = \lim_{x\to\infty} \frac{x^3}{(x-2)^2} - x = \lim \frac{x^3 - x^3 + 4x^2 - 4x}{x^2 - 4x + 4} = 4$$

asymptota pro $x \to \infty$ je $g(x) = x + 4$

-||- $\quad x \to -\infty \quad$ -||-

1. derivace

$$\left(\frac{x^3}{(x-2)^2}\right)' = \frac{x^2(x-6)}{(x-2)^3}$$

$(-\infty, 2)$ rostoucí $\qquad$ OH $\mathbb{R}$

$(2, 6)$ klesající

$(6, \infty)$ rostoucí

v bodě 6 je lokální minimum.

2. derivace

$$\left(\frac{x^3}{(x-2)^2}\right)'' = \frac{24x}{(x-2)^4}$$

$(-\infty, 0)$ konkávní

$(0, \infty)$ konvexní