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# A CONCEPT OF EGALITARIANISM UNDER PARTICIPATION CONSTRAINTS

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We define a new solution concept for transferable utility cooperative games in characteristic function form, in a framework where individuals believe in equality as a desirable social goal, although private preferences dictate selfish behavior. This latter aspect implies that the solution outcome(s) must satisfy core-like participation constraints while the concern for equality entails choice of Lorenz maximal elements from within the set of payoffs satisfying the participation constraints. Despite the Lorenz domination relation being a partial ranking, we show that the egalitarian solution is unique whenever it exists. Moreover, for convex games, the solution is in the core and Lorenz dominates every other core allocation.

KEYWORDS: Lorenz domination, core, consistency, convex games.

## 1. INTRODUCTION

CONSIDER A SOCIETY, represented as a coalition of $n$ individuals. Suppose that each member of this society subscribes to equality as a desirable end; that is, he upholds egalitarianism as a *social* value. However, his private preferences dictate selfish behavior in his daily actions. The problem is to use the social values to form a set of rules for the society, taking into account the incentive constraints imposed by self-seeking behavior once the society is operational and the rules are in place.

We should mention right away that our starting point presumes a tension between the social values of persons and their individual behavior. The recognition of such a tension is, of course, not new. In fact, a great deal of work in the idealist school of political philosophy is explicitly based on this position (see Arrow (1963) for a brief but illuminating summary).

We shall use the following approach to discuss the problem. Initially, we shall suppose that the society is in an "original position," where no member is aware of the endowment that he is going to receive.[2] Under this veil of ignorance, the society is to decide on a rule that will assign, to every vector of individual *endowments*, a subset of the class of all feasible *allocations*. Our question is: what assignment will the society choose?

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[2]The concept of an original position dates back to the "social contract" philosophers, as is well-known. Rawls (1974) is probably the most familiar modern use of this approach. It should be noted, though, that we do *not* employ the concept to build up a theory of social norms (e.g., justice) from individual values alone, as Rawls does. Rather, we use it to apply an *already existing* social ethic (egalitarianism) to the design of rules. The private values, as we shall presently see, act as limits to the undiluted application of the social ethic.

To be concrete, suppose that after endowments are revealed to the society, we shall have an *n-person transferable-utility cooperative game* in characteristic function form. In this framework, there are some obvious restrictions that the allocation rules must satisfy, so that the society as a whole remains viable.

Consider, for instance an extreme egalitarian rule that prescribes equal division of the aggregate worth of the society, *irrespective* of the initial distribution of individual worths and the worths attached to each subcoalition (or subsets of members). Such an allocation will inevitably run into incentive problems for some endowment distributions. The social consciousness of the individuals is manifested in the rules that they have framed for themselves in the original position, but in their private actions they will deviate from these rules if the benefits from doing so outweigh the costs. In particular, in the equal division example, some coalition may receive an allocation whose aggregate value falls short of the worth of that coalition. The coalition might then deviate from the proposed rule, and the society is no longer viable.

The above argument might suggest that the allocation rule, whatever it is, should yield feasible allocations that lie in the *core* of the cooperative game that follows the revelation of endowments. Specifically, a social concern with egalitarianism, coupled with a recognition of coalitional participation constraints, might prescribe a set of allocations *within* the core that are Lorenz-undominated by other *core* allocations.[3]

While this is a natural route to take[4] (and we do discuss it below[5]), it smacks of a certain degree of asymmetry in the following sense. Suppose that a coalition blocks a given allocation. What does it then do? The fact that its worth exceeds the value of the allocation assigned to it implies that there is *a* feasible allocation for this coalition that Pareto dominates the earlier proposals so that all members of this coalition are better off. However, there is no reason to suppose that this *particular* allocation will indeed be chosen by the deviating coalition.

For one thing, the core of the new game so induced (by treating the deviating coalition as the new grand coalition, ignoring nonmembers of this coalition, and retaining the old worths of all subsets of the deviating coalition) may well be empty. In what sense, then, is the deviating coalition a credible coalition? The problem is akin to that of subgame perfection in noncooperative games: one must know what a coalition can *credibly* do once it deviates.[6]

[3] We are going to use the Lorenz criterion as our (partial) ordering of unequal allocations. There is now a sizeable literature that deals with the characterization of the Lorenz ordering as a plausible concept of inequality (see e.g. Atkinson (1970), Dasgupta, Sen, and Starrett (1973), or Fields and Fei (1978)). For an excellent treatment of these and other issues, see e.g. Sen (1973). Briefly, the Lorenz criterion is widely accepted as embodying a set of minimal ethical judgements that "should" be made in carrying out inequality comparisons. Indeed, the small number of judgements implicit in the Lorenz criterion is responsible for the partial nature of the Lorenz ordering. However, additional ethical judgements needed to complete the ordering are not so widely agreed upon.

[4] One should point out that the problem of sharing the benefits from cooperation is a central theme in cooperative game theory, and there are different approaches one might adopt; see e.g. Moulin (1985, 1987) and Young (1984, 1988).

[5] See especially Section 5.

[6] A similar credibility problem—though easily resolved—underlies the concept of the core itself. This observation is made in Ray (1988).

More important in the present context, *the deviating coalition is* a *potential society of its own* and is therefore subject to the same rules that its egalitarian minded members have laid down for themselves in the original position. If egalitarian rules are to be considered at all, these should be considered by all coalitions which would be (or might be) societies in their own rights, and cannot be viewed as a feature of the grand coalition *alone*.[7]

The argument is best illustrated by an example. Consider a society of labor managed firms. Each firm combines the joint skills of its employees into a final output. After all the nonlabor costs are paid for, the surplus is to be shared. Let us suppose that a rule exists that assigns to each *conceivable* labor-managed firm a method of allocating its surplus among the employees. Think of such a rule as built into the "constitution" or "charter" of the economy.

Now consider a *particular* firm with $n$ members. Suppose that $m$ of its members ($m < n$) decide to break away and form a new firm. Now, the rule for surplus-sharing in this *new* firm has already been laid down, so that *what* the $m$ breakaway members can achieve is limited. In this example, not only are we applying the rule to the large firm, *but also to each of its "deviating" subsets*. It illustrates the general approach that we shall take.

The problem of finding an egalitarian rule must therefore be reformulated. In the original position we find, now, not only the grand coalition of $n$ persons, but also all conceivable (nonempty) subcoalitions drawn from these $n$ persons. Each such coalition is potentially a society (if and when it deviates). As a society, it, too, must adhere to the egalitarian rule that will be formed by its members.

We take it that the primary goal of this group of persons is to create a single society of $n$ people and allocate its wealth among the members in the most equal way possible. The caveat "possible" manifests itself in two parts. First, no allocation can be prescribed that is "blocked" by some viable subcoalition (defined below), where the term "block" is used in the sense that the subcoalition can find an allocation according to *its* egalitarian rule that makes all its members just as well off, and some member strictly better off. Second, if an allocation can be Lorenz dominated by another allocation which too cannot be "blocked" in the way just described, the former allocation cannot be prescribed by the egalitarian rule.

The construction of an egalitarian allocation has, therefore, a recursive structure to it. This is made explicit when we recognize that *each* coalition is subject to these constraints vis-a-vis its subcoalitions when formulating its egalitarian allocation, just as the grand coalition is.[8]

The reader who wishes to study further the basic assumptions underlying our construct is invited to study the definitions in Section 4 and then turn to Section 7. Here, we make two broad sets of remarks. The first set deals with interpersonal

[7]Indeed, we would apply this general principle (one of consistency if you will) to *any* norm that the grand coalition might wish to set for itself. Subcoalitions are also subject to precisely the same norms in the case of a deviation. Their members in the original position should logically see to that, if they are agreeing to impose their social ethics on the grand coalition.

[8]This recursive notion is, of course, applicable to many different solution concepts. See e.g. Bernheim, Peleg, and Whinston (1987) in the context of noncooperative games.

comparability and the consequent failure of our solution concept to satisfy a classical tenet of invariance—strategic equivalence. The second set embeds our solution concept into a broader class of situations, which necessitates neither transferable utility nor interpersonal comparability. The broader class brings out the essential methodological feature of our approach. This is in the *consistent* use of norms, not only in the overall choice of society, but also in determining the coalitional constraints which this choice must respect.

## 2. SUMMARY OF RESULTS

In this paper we discuss the construction and some properties of egalitarian allocations and examine two applications. Our first result (Theorem 1) is that for each realization of individual and coalitional worths, the egalitarian rule prescribes at most *one* feasible allocation for the grand coalition. That is, *every* other feasible allocation in the grand coalition is either blocked (in our sense), or is Lorenz dominated by some allocation which, in turn, is not blocked.

It is worth emphasizing that this result is extremely strong. Observe that the Lorenz criterion generates a *partial* ordering on the set of allocations.[9] In particular, within a given set, there will be in general *many* allocations that are not Lorenz dominated by other allocations within the set.[10] Examples are plentiful and easy to construct. Our result states that *this is not possible* for sets of allocations that are achieved as "participation-constrained" solutions to cooperative games in the way we have just described. The egalitarian rule must therefore indicate at most one allocation, which we shall call the egalitarian allocation. It follows that every "equality conscious" welfare function[11] defined on the domain of unblocked allocations and achieving a maximum will pick out the same unique point.

The egalitarian allocation respects *core-like* constraints, but bears no necessary relationship to the core. In Example 1, we describe a game where the core is nonempty but the egalitarian allocation does not exist.[12] In Example 2, the egalitarian allocation exists but the core is empty. Example 3 is a game where both the core is nonempty and the egalitarian allocation exists, but the egalitarian allocation does not lie in the core. However, it is of interest that in all *three*-person games, the egalitarian allocation lies in the core whenever the former exists and the latter is nonempty. (A complete treatment of the egalitarian allocation for all three-person games is given in the Appendix.)

[9] The phrase "the Lorenz curves cross" is often used when the criterion fails to compare two allocations.

[10] This is especially reasonable in the present context, where the set of "unblocked" allocations is neither too small, i.e., a singleton (in fact our set is a superset of the core—when it exists), nor, because of incentive constraints, is it too large so that equal division is always possible.

[11] See Atkinson (1970), Dasgupta, Sen, and Starrett (1973), Sen (1973), and the remarks in Section 7.2. These papers provide characterizations of such welfare functions.

[12] The general question of existence is not addressed in this paper, though it is tackled for convex games. For a general computational algorithm that either determines the egalitarian allocation in a bounded number of steps, or indicates nonexistence, see Dutta and Ray (1987).

There is a strong connection between the concepts of core and egalitarian allocation in the class of *convex games*, and we take this up in Section 5 of the paper. There we prove (Theorem 2) that an egalitarian allocation exists and lies in the core. We also describe a computational algorithm for locating this allocation.

A further connection (in convex games) between the core and the egalitarian allocation is brought out in Theorem 3 where it is established that the egalitarian allocation Lorenz dominates *every* other point in the core. Put another way, every point in the core can be moulded into the egalitarian allocation by a sequence of progressive transfers. This provides a definitive solution to the alternative route we mentioned earlier, finding the most egalitarian outcomes within the core itself. When the game is convex, we find that there is a unique solution to this problem, that it coincides with the more general concept we have chosen to adopt, and that the solution has the additional merit of being Lorenz *comparable* (and *superior*) to *every* other core allocation. Among the other allocations is also the Shapley value, an allocation concept that has often been described as "equitable".[13]

One could try to argue that Theorem 3 can be strengthened. First, it might be felt that the egalitarian allocation always Lorenz dominates every other point in the core, whether or not the game is convex. But this is not true. In Example 4, we describe a game where the unique egalitarian allocation lies in the core. Yet it fails to Lorenz dominate every other point in the core. Needless to say, the game is not convex. Second, one might conjecture that in the class of convex games, the egalitarian allocation Lorenz dominates every other allocation that is unblocked in our sense, not only the core allocations. This, too, is generally not true, and the point is illustrated by Example 5.

In Section 6, we discuss two applications. The first concerns the familiar problem of sharing the cost of a public good. We characterize the egalitarian allocation in this problem, and compare it to the allocation recommended by the Shapley value. The second application addresses the issue of surplus-sharing in a labor managed firm. Here, too, we use the egalitarian allocation to fully describe a surplus-sharing rule.[14]

Section 7 concludes with some remarks on the foundations of the egalitarian allocation.

## 3. NOTATION

For any nonempty subset $S$ of $\{1, \ldots, n\}$, denote by $|S|$ the cardinality of $S$. We will write $\mathbb{R}^S$ for $\mathbb{R}^{|S|}$($|S|$-dimensional Euclidean space). For two vectors $x$ and $y$ in $\mathbb{R}^S$, we write $x = y$ if all their components are equal, and $x > y$ if $x_i \geqslant y_i$ for all $i = 1, \ldots, |S|$, with strict inequality for some $i$. For any $x \in \mathbb{R}^S$, we denote by $\boldsymbol{x}$ the vector obtained by permuting the indices of $x$ such that $\boldsymbol{x}_1 \geqslant \boldsymbol{x}_2 \geqslant \cdots$

[13] See e.g. Champsaur (1975). We provide some explicit comparisons between the egalitarian allocation and the Shapley value in the applications discussed in Section 6.

[14] In Dutta and Ray (1987), we examine in detail a third application of the egalitarian allocation—to a creditor problem from the Talmud.

$\geqslant \boldsymbol{x}_{|S|}$. Also, suppose that for some $S$ (a subset of $\{1,\ldots,n\}$), $x \in \mathbb{R}^S$ is given. We shall denote the projection of $S$ on $T$ (a subset of $S$) by $x(T)$. Finally, the notation $\subseteq$ denotes inclusion, $\subset$ denotes *proper* inclusion, and $\varnothing$ denotes the empty set.

## 4. EGALITARIAN ALLOCATIONS

### 4.1. *The Model*

We consider a transferable utility game in characteristic function form. There are $n$ players. A *coalition* is a nonempty subset of $N = \{1,2,\ldots,n\}$. A *subcoalition* of a coalition $S$ is a coalition that is a subset of $S$. $N$ is the *grand coalition*. The *worth* of a coalition $S$ is given by a scalar $v(S)$. An allocation $x \in R^S$ is *feasible* for $S$ if $\sum_{i \in S} x_i = v(S)$.

The notion of an egalitarian allocation will now be developed. First, the *Lorenz map E* is defined on the domain $\{A/A \subseteq \mathbb{R}^k$ for some $k$, and there exists $v \in R$ such that $\sum_{i=1}^k x_i = v$ for all $x \in A\}$. For each such set $A$, $EA$ is the set of all allocations in $A$ that are Lorenz undominated within $A$. Formally,

(1) $$EA = \left\{ x \in A \mid \text{there is no } y \in A \text{ such that } \sum_{i=1}^{j} \boldsymbol{y}_i \leqslant \sum_{i=1}^{j} \boldsymbol{x}_i \text{ for all } j = 1,\ldots,k, \text{ with strict inequality for some } j \right\}.$$

(Recall how $\boldsymbol{x}$ and $\boldsymbol{y}$ are defined from $x$ and $y$ (Section 3).)

Note, first, that $EA$ may be empty, but that $EA$ is nonempty whenever $A$ is closed. Second, for $A \subseteq R$, $A$ must be a singleton and so $EA = A$. Finally, $EA$, when it exists, is generally not a singleton set—this follows from the partial nature of the Lorenz ordering.

Next, we define recursively the *Lorenz cores* of coalitions. The Lorenz core of a singleton coalition is $L(\{i\}) = \{v(i)\}$. Now suppose that the Lorenz cores for all coalitions of cardinality $k$ or less have been defined, where $1 < k < n$. The Lorenz core of a coalition of size $(k+1)$ is defined by

(2) $$L(S) = \{x \in \mathbb{R}^S | x \text{ is feasible for } S, \text{ and there is no } T \subset S \text{ and } y \in EL(T) \text{ such that } y > x(T)\}.$$

If $x \in S$, and there is $T \subset S$ and $y \in EL(T)$ such that $y > x(T)$, then we say that $y$ *Lorenz-blocks* ($L$-blocks) $x$. We shall also say in this case that $T$ $L$-blocks $x$.

A coalition $S$ will be called *viable* if $EL(S)$ is nonempty. An *egalitarian allocation exists* if the grand coalition is viable. $EL(N)$ will denote the set of *egalitarian allocations*.

REMARKS (i): A viable coalition, by our definition, has a credibility property. It is potentially capable of (Lorenz) blocking an allocation proposed for the grand coalition. As we have discussed in the Introduction, it is only permitted to block by using egalitarian allocations of its own. And coalitions that are not viable can simply be ignored, so far as the grand coalition is concerned.[15]

(ii) For any coalition $S$, the *core* of $S$ is defined by

$$(3) \qquad C(S) = \left\{ x \in \mathbb{R}^S | x \text{ is feasible for } S\text{, and there is no } T \subset S \text{ such that } v(T) > \sum_{i \in T} x_i \right\}.$$

Clearly, for each $S, C(S) \subseteq L(S)$. Thus our stringent requirement on blocking enlarges the set of "permissible" allocations.

(iii) $L(S)$ is therefore the set of all allocations which respect participation constraints for subcoalitions of $S$. From these, $S$ must choose the most egalitarian ones. In our definition, we accomplish this by applying the Lorenz map to the Lorenz core of $S$. This explains our definitions of viability and of the set of egalitarian allocations.

### 4.2. *Three Examples*

At this stage, a question that comes naturally to mind is: does an egalitarian allocation always exist? The answer is, not surprisingly, no. Our first example describes a three-person game where an egalitarian allocation does not exist.

Example 1 (A totally balanced game with no egalitarian allocation): Let $N = \{1,2,3\}$, $v(\{i\}) = 0$, $i \in N$, $v(\{1,2\}) = v(\{1,3\}) = v(N) = 1$, $v(\{2,3\}) = 0$. The core of this game is the single allocation $(1,0,0)$. The Lorenz core of the grand coalition is larger: it consists of all feasible allocations $(x_1, x_2, x_3)$ with $x_1 > \frac{1}{2}$. However, the reader can easily check that no egalitarian allocation exists.

Remarks (i): We have deliberately chosen a totally balanced game for the nonexistence example, to show that a full characterization of existence may be difficult to obtain using the standard concepts. However, existence *is* guaranteed within the class of all *convex* games, as we show below in Section 5. For a general treatment using a computational algorithm, see Dutta and Ray (1987).

(ii) By changing the notion of $L$-blocking to require that *every* individual in the blocking coalition be made strictly better off, a new set of "strong egalitarian allocations" is obtained. Such a set is nonempty for all superadditive games, though the properties are markedly different from the solution concept consid-

[15]This last item may appear to be an overly strong restriction. Coalitions may not be viable in the sense that we have defined, but this lack of viability may be due to the fact that its Lorenz core is not closed, so that a Lorenz-maximal element does not exist. In this case, ignoring such a coalition may not be reasonable. However, it can be shown that even if we treat such coalitions as viable and permit them to block with "limit points" of their egalitarian exercise, no difference is made to the analysis. Of course, if its lack of viability is due to the fact that its Lorenz core is *empty*, then there is no problem. We are grateful to a referee for raising this point.

ered here. A separate treatment of strong egalitarian allocations can be found in Dutta and Ray (1988).

That the existence of an egalitarian allocation is not tightly linked to the nonemptiness of the core is further illustrated by our next example. Here, an egalitarian allocation exists but the core is empty.

EXAMPLE 2 (A game where $C(N) = \varnothing$, but $EL(N) \neq \varnothing$): Let $N = \{1,2,3\}$, $v(\{1\}) = 0$, $v(\{2\}) = v(\{3\}) = 1$; $v(\{12\}) = v(\{13\}) = 1.4$, and $v(\{23\}) = v(N) = 2.2$. In this example, the reader can easily check that the core $C(N)$ is empty. But $EL(N) = \{(0, 1.1, 1.1)\}$. Observe that this allocation is blocked (in the usual core sense) by both coalitions $\{1,2\}$ and $\{1,3\}$. However, these coalitions cannot $L$-block the allocation (they can use only the vector $(0.4, 1)$ to $L$-block).

EXAMPLE 3 (A game where $C(N) \neq \varnothing$, $EL(N) \neq \varnothing$, but $EL(N) \cap C(N) = \varnothing$): Let $N = \{1,2,3,4\}$, $v(\{i\}) = 0$ for all $i$, $v(N) = 2$, $v(\{2,3\}) = 1.05$, $v(\{3,4\}) = 1.9$, and for all other $S$, $v(S)$ is the minimal superadditive function compatible with these values. Here, $EL(N)$ is the singleton set consisting of (0.05, 0.05, 0.95, 0.95), and this is *not* in the core ($\{23\}$ can block). Moreover, the core is nonempty; e.g., $(0, 0.1, 0.95, 0.95)$ is in the core.

REMARK: Example 2 underlines the observation that the existence of an egalitarian allocation is unrelated to the nonemptiness of the core. Example 3 shows that even if the core is nonempty and an egalitarian allocation exists, the two sets of allocations may be disjoint.

Observe that in Examples 2 and 3, there is only *one* egalitarian allocation. Despite the partial nature of the Lorenz ordering, it turns out that this is a general feature of the problem which holds regardless of the game under consideration. This is what we turn to next.

### 4.3. *Uniqueness of the Egalitarian Allocation*

THEOREM 1: *There is at most one egalitarian allocation.*

PROOF: We will show that $EL(S)$ can have at most one point, for any coalition $S$. Clearly, the statement is true if $|S| \leqslant 2$. Suppose, then, that the statement is true for all coalitions of cardinality $k$ or less, $k < n$. We will show that it is true for all coalitions of cardinality $(k+1)$.

Suppose, on the contrary, that there exists a coalition $S$ of size $(k+1)$ and two distinct $y$, $y' \in EL(S)$. W.l.o.g., number indices so that $y_{i+1} \leqslant y_i$. Let $i$ be the smallest integer such that $y_i \neq y_i'$. Either (i) $y_i < y_i'$ or (ii) $y_i > y_i'$. Let us take these in turn. The treatment is not completely symmetric.

*Case (i):* Suppose that $y_i < y_i'$. It is clear, then, that $P(i, y') \equiv \{j \in S | y_j' < y_i'\} \neq \varnothing$. Define $M(i) = \{T \subset S | i \in T, T \text{ is viable}\}$. Choose any $T \in M(i)$, and let $EL(T) = \{y^T\}$ (the induction hypothesis guarantees uniqueness of $EL(T)$ for such viable $T$). Since $y' \in EL(S)$ and is consequently not $L$-blocked by $T$, one of the following statements must be true:

(4a) $y_i' > y_i^T$,

(4b) $y_j' > y_j^T$ for some $j \in T$, $j \neq i$,

(4c) $y_j' = y_j^T$ for all $j \in T$.

First, we claim that if (4c) is true, then $T \cap P(i, y') \neq \varnothing$. Suppose not. Then $y_j' \geqslant y_i'$ for all $j \in T$. But then observe that for $j \in T$ and $1 \leqslant j \leqslant i-1$, we have $y_j^T = y_j' = y_j$, $y_i^T = y_i' > y_i$, and for $j \in T$ with $j > i$, $y_j^T = y_j' \geqslant y_i' > y_i \geqslant y_j$. But then $T$ $L$-blocks the allocation $y$, contradicting the hypothesis that $y \in EL(S)$. So the claim is established.

Now let $M'(i)$ be the subset of coalitions of $M(i)$ for which (4a) is true. $M'(i) \neq \varnothing$, because $y_i' > y_i \geqslant v(\{i\})$, and so $\{i\} \in M'(i)$. Choose $\delta > 0$ so that $\delta < \min_{T \in M'(i)}(y_i' - y_i^T)$. Construct a feasible allocation for $S$, $y''$, in the following way: $y_j'' = y_j'$ for all $j$ not in $P(i, y') \cup \{i\}$, $y_i'' = y_i' - \delta$, and $y_j'' = y_j' + \delta/z$ for $j \in P(i, y')$, where $z = |P(i, y')|$. Clearly, $y''$ Lorenz dominates $y'$. So we have arrived at a contradiction if we can show that no $T \subset S$ can $L$-block $y''$.

First suppose that $T$ is not in $M(i)$. Then clearly $T$ cannot block $y''$ (since $y' \in EL(S)$ by assumption). By our choice of $\delta$, it is also clear that if $T \in M'(i)$, $T$ cannot $L$-block $y''$. Finally, if $T \in M(i)$, and satisfies (4b) or (4c), then we use the claim above and the definition of $y''$ to argue that there exists $j \in T$ with $y_j'' > y_j^T$. So in this case, too, $T$ cannot $L$-block $y''$. But then all possibilities are exhausted, and $y'' \in L(S)$. This is a contradiction, so Case (i) cannot hold.

*Case (ii):* Suppose $y_i > y_i'$. We can then assume, w.l.o.g., that $P(i, y) = \{j \in S | y_j < y_i\}$ is nonempty. For, if this is not true, then $y_j = y_i$ for all $j > i$. But then, since $y$ and $y'$ are both feasible, there must exist some $j > i$ such that $y_j' > y_i'$. Simply renumber the indices by switching $i$ and $j$ and keeping all else unchanged—we are then in Case (i) which yields a contradiction as before.

Again, for any $T \in M(i)$, one of the following will be true.

(5a) $y_i > y_i^T$,

(5b) $y_j > y_j^T$ for some $j \in T$, $j \neq i$,

(5c) $y_j = y_j^T$ for all $j \in T$.

Define $M''(i) = \{T \in M(i) | y_i > y^T\}$. $M''(i) \neq \varnothing$ because $y_i > y_i' \geqslant v(\{i\})$, and so $\{i\} \in M''(i)$. Again, choose $\delta > 0$ so that $\delta < \min_{T \in M''(i)}(y_i - y_i^T)$. Construct a feasible allocation for $S$, $y^*$, in the following way: $y_j^* = y_j$ for all $j$ not in $P(i, y) \cup \{i\}$, $y_i^* = y_i - \delta$, and $y_j^* = y_j + \delta/w$ for $j \in P(i, y)$, where $w = |P(i, y)|$.

Clearly, $y^*$ Lorenz dominates $y$. So if $y^* \in L(S)$, we have a contradiction and the theorem is proved. We invite the reader to check that, given $y \in L(S)$, $y^*$ cannot be blocked by *any* viable coalition $T$ satisfying any one of the following conditions: (a) $T$ is not in $M(i)$, (b) $T$ satisfies (5a), (c) $T$ satisfies (5b), and (d) $T$ satisfies (5c) and $T \cap P(i, y) \neq \varnothing$. So if $T$ is to $L$-block $y^*$, it must be that $T \in M(i)$, $T$ satisfies (5c), and $T \cap P(i, y) = \varnothing$.

In such a case, $T$ cannot be a subset of $\{1, \dots, i\}$. For if it were, then note that $y_j^T = y_j \geqslant y_j'$ for all $j \in T$, with strict inequality holding for $j = 1$. But then $T$ would $L$-block $y'$, a contradiction.

So $T$ is not a subset of $\{1, \dots, i\}$ and $T \cap P(i, y) = \varnothing$. Therefore there exists $t > i$ such that $y_i = y_{i+1} = \cdots = y_t$ and with $y_{t+1} < y_t$ if $t < k+1$. But then it cannot be the case that $y_j \geqslant y_j'$ for all $j \in T$, with strict inequality for $j = i$. But then $T$ $L$-blocks $y'$, a contradiction.

So there exists $j \in T$, with $i + 1 \leqslant j \leqslant t$, such that $y_j' > y_j$. Now simply permute $i$ and $j$ and keep all other indices unchanged. It is easy to check that we are then in Case (i), which, as we already know, yields a contradiction.

Therefore no $T \subset S$ can $L$-block $y^*$, which contradicts $y \in EL(S)$. This completes the proof of the theorem.

Theorem 1 is a strong result. Consider the grand coalition $N$. As we have already observed, the Lorenz core $L(N)$ is at least as large as the core of $N$ (and is often strictly larger). And we do know that the core may be "large"—recall, for example, that the cores of convex games are complete (Shapley (1971)). Moreover, the Lorenz core will, in general, *not* contain the equal division allocation.[16] Yet the Lorenz map applied to the Lorenz core can yield *at most one* allocation, despite the partial nature of the ordering involved.[17]

## 5. CONVEX GAMES

In this section, we show that for the class of convex games $EL(N)$ is always nonempty, and establish a close connection between $EL(N)$ and the core. In addition, we also describe a simple algorithm to locate the (unique) egalitarian allocation.

Recall that a game is *convex* if for all coalitions $S, T$, we have

$$(6) \qquad v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T).$$

We start by describing the algorithm for locating the egalitarian allocation in a convex game. This will also establish existence. Define first, for any characteristic function $v'$ and any coalition $S$, $e(S, v') = v'(S)/|S|$, so that $e(S, v')$ is the *average worth of S under* $v'$. Define $v_1 = v$.

[16] Of course, it is a trivial observation that if the equal division is in the Lorenz core, it *must* be the egalitarian allocation.

[17] We should emphasize that there *cannot* even be two egalitarian allocations which are identical to each other in terms of the Lorenz curve that they generate, but with the individuals permuted.

STEP 1: Define by $S_1$ the unique coalition such that (i) $e(S_1, v_1) \geqslant e(S, v_1)$ for all coalitions $S$; (ii) $|S_1| > |S|$ for all $S \neq S_1$ such that $e(S, v_1) = e(S_1, v_1)$; so that $S_1$ is *the* largest coalition having the highest average worth. The reader can verify (using convexity) that such an $S_1$ exists. Define

(7) $\quad x_i^* = e(S_1, v_1) \quad \text{for all } i \in S_1.$

STEP $k$: Suppose that $S_1, \ldots, S_{k-1}$ have been defined recursively and $S_1 \cup \cdots \cup S_{k-1} \neq N$. Define a new game with player set $N \backslash \{S_1 \cup \cdots \cup S_{k-1}\}$. For all subcoalitions $S$ of this new player set, define $v_k(S) = v_{k-1}(S_{k-1} \cup S) - v_{k-1}(S_{k-1})$. The reader can check that this new game is also convex. Just as in Step 1, define $S_k$ to be the largest coalition with the highest average worth in this game. Define

(8) $\quad x_i^* = e(S_k, v_k) \quad \text{for all } i \in S_k.$

Clearly, in $m$ of these steps ($m \leqslant n$), there will be a partition of $N$ into sets $S_1, \ldots, S_m$. Let $x^*$ be the allocation defined by the equations of the form (7) and (8). *We are going to show that $x^*$ is the egalitarian allocation of $N$.* It is important to keep in mind that $x^*$, as constructed, satisfies the following conditions:

(9) $\quad x_j^* = x_k^* \qquad \text{for all } j, k \in S_t \text{ and } t = 1, \ldots, m,$

(10) $$\sum_{k=1}^{t} \sum_{j \in S_k} x_j^* = v(S_1 \cup \cdots \cup S_t) \qquad (t = 1, \ldots, m),$$

(11) $\quad x_i^* > x_j^* \quad \text{if } i \in S_k,\ j \in S_t, \text{ and } k < t.$

Our main result of this section is the following theorem.

THEOREM 2: *In a convex game, $x^*$ as constructed by the algorithm above is the unique egalitarian allocation. Moreover, $x^*$ is in the core.*

PROOF: The proof of this theorem is an immediate corollary of two steps.

STEP 1: In a convex game, $x^* \in C(N)$.

PROOF: Suppose that the procedure outlined above terminates in $m$ steps and the induced partition of $N$ is $(S_1, \ldots, S_m)$. Pick any $t < m$, and an arbitrary coalition $S$ in $S_1 \cup \cdots \cup S_t$. We shall first prove that the following statement is true:

(12) $$\text{For all } T \subseteq S_{t+1}, \quad v(S) \leqslant \sum_{j \in S} x_j^* \quad \text{implies} \quad v(S \cup T) \leqslant \sum_{j \in S \cup T} x_j^*.$$

Suppose that for some $T \subseteq S_{t+1}$, (12) is not true. Then while $S$ has the property that $v(S) \leqslant \sum_{j \in S} x_j^*$,

(13) $$v(S \cup T) > \sum_{j \in S \cup T} x_j^*.$$

Now, since the game is convex,

$$(14)\qquad v(S\cup T)+v(S_1\cup\cdots\cup S_t)\leqslant v(S_1\cup\cdots\cup S_t\cup T)+v(S).$$

Moreover,

$$(15)\qquad \sum_{j\in S\cup T}x_j^*=\sum_{j\in S}x_j^*+\sum_{j\in T}x_j^*\geqslant v(S)+\sum_{j\in T}x_j^*.$$

Combining (13), (14), and (15), we have

$$(16)\qquad v_{t+1}(T)=v(S_1\cup\cdots\cup S_t\cup T)-v(S_1\cup\cdots\cup S_t)>\sum_{j\in T}x_j^*$$

$$=|T|e(S_{t+1},v_{t+1})$$

but (16) contradicts the definition of $S_{t+1}$ as the coalition having the highest average worth in $v_{t+1}$. Therefore, (12) is true.

It is also clear that if $S\subseteq S_1$, then $\sum_{j\in S}x_j^*=|S|e(S_1,v_1)\geqslant v(S)$. Now use (12) and an obvious induction argument to show that for all $S\subset N$, $\sum_{j\in S}x_j^*\geqslant v(S)$. Therefore $x^*\in C(N)$.

The next step is of some independent interest as it shows that a condition weaker than the convexity of the game is sufficient to guarantee the nonemptiness of $EL(N)$.

STEP 2: If $x^*\in L(N)$, then $x^*=EL(N)$.

PROOF: Again let the partition of $S$ induced by the algorithm be denoted by $(S_1,\ldots,S_m)$. Suppose that $x^*\in L(N)$. That $\{x^*(S_1)\}=EL(S_1)$ is obvious enough. We prove the lemma by induction on the $S_t$'s. In particular, we shall prove the following statement: for all $t=1,\ldots,m-1$, if $\{x^*(S_1\cup\cdots\cup S_t)\}=EL(S_1\cup\cdots\cup S_t)$, then $\{x^*(S_1\cup\cdots\cup S_{t+1})\}=EL(S_1\cup\cdots\cup S_{t+1})$.

Suppose this is not true for some $t$. If $x^*(S_1\cup\cdots\cup S_{t+1})\notin L(S_1\cup\cdots\cup S_{t+1})$, then $x^*\notin L(N)$, a contradiction. Therefore $x^*(S_1\cup\cdots\cup S_{t+1})$ is Lorenz dominated by some $y\in L(S_1\cup\cdots\cup S_{t+1})$. Then there exists $j$ such that (i) $y_j<x_j^*$, and (ii) for all $k\in T=\{p\in S_1\cup\cdots\cup S_{t+1}|x_p^*\geqslant x_j^*\}$, we have $y_k\leqslant x_k^*$.

By the construction of $x^*$, $T=S_1\cup\cdots\cup S_q$ for some $q\leqslant t$. By hypothesis $\{x^*(T)\}=EL(T)$. But then $T$ $L$-blocks $y$, and so $y\notin L(S_1\cup\cdots\cup S_{t+1})$, a contradiction. This establishes the step.

Now combine Steps 1 and 2, remembering that $C(N)\subseteq L(N)$. This completes the proof of the theorem.

In particular, Theorem 2 tells us that the algorithm works in locating the unique egalitarian allocation, which always exists in the convex case. In our next result, we show that not only is $x^*$ the unique element of $EL(N)$, but it also Lorenz dominates every other core allocation:

THEOREM 3: *In a convex game, $x^*$ Lorenz dominates every allocation in the core $C(N)$.*

PROOF: Let the partition of $S$ induced by the algorithm be denoted by $(S_1, \ldots, S_m)$. Suppose that $x \in C(N)$, $x \neq x^*$. Two cases are possible.

CASE (i): $\Sigma_{k=1}^{t}\Sigma_{j \in S_k} x_j = v(S_1 \cup \cdots \cup S_t)$ for all $t = 1, \ldots, m$. Then there must exist $S_t$ and $i$, $j \in S_t$ such that $x_j \neq x_i$. For all such $S_t$, a sequence of "rich-to-poor" transfers can be made to convert $x$ to $x^*$. But then $x^*$ clearly Lorenz dominates $x$.

CASE (ii): $\Sigma_{k=1}^{t}\Sigma_{j \in S_k} x_j > v(S_1 \cup \cdots \cup S_t)$ for some $t = 1, \ldots, m$. In this case, it is easily seen that there must exist $t, q$, with $t < q$, such that (a) $\Sigma_{j \in S_t} x_j^* < \Sigma_{j \in S_t} x_j$ and (b) $\Sigma_{j \in S_q} x_j^* > \Sigma_{j \in S_q} x_j$. Then, an appropriate sequence of "rich-to-poor" transfers can be made to convert $x$ into $x'$, where for all $i = 1, \ldots, m$, $\Sigma_{j \in S_i} x_j^* = \Sigma_{j \in S_i} x_j'$ (see equation (11) to confirm that this can be done). Now, use another sequence of "rich-to-poor" transfers to equalize individual allocations *within* each $S_i$ (just as in Case (i)). Therefore we have arrived from $x$ to $x^*$ by a sequence of such transfers. It follows that $x^*$ Lorenz dominates $x$, and the theorem is proved.

Theorem 3 tells us, in effect, that if we were to choose the alternative route to the problem, which is simply to select Lorenz-undominated allocations from within the core, we would get the same solution in the class of convex games. We reiterate that the Lorenz ordering is a *partial* one, and the sharpness of the present result should be viewed in this light.

Is Theorem 3 true in general? That is, does the egalitarian allocation *always* Lorenz dominate other core allocations, whether or not the game is convex? The answer is in the negative, and is illustrated by our next example.

EXAMPLE 4 (A game where the egalitarian allocation does not Lorenz dominate all core allocations): $N = \{1, 2, 3, 4\}$ and $v$ is described in the table below:

| $S$ | $v(S)$ | $S$ | $v(S)$ |
|---|---|---|---|
| {1} | 1 | {24} | 6 |
| {2} | 1.5 | {34} | 7 |
| {3} | 2.5 | {123} | 5.5 |
| {4} | 4 | {124} | 7 |
| {12} | 3 | {134} | 8 |
| {13} | 4 | {234} | 8.5 |
| {14} | 5 | {1234} | 10 |
| {23} | 4 | | |

Here, the set of Lorenz undominated outcomes in $C(N)$ is $\{x | x = \theta(1,2,3,4) + (1-\theta)\,(1\tfrac{1}{2}, 1\tfrac{1}{2}, 2\tfrac{1}{2}, 4\tfrac{1}{2})$ for $\theta \in [0,1]\}$. Moreover, the egalitarian allocation exists and routine calculation shows it to be equal to $(1,2,3,4)$. Of course, this game is not convex.

Theorem 4 also suggests a different line of generalization, this time within the class of convex games. Does $x^*$ Lorenz dominate every other point in the *Lorenz core* $C(N)$, not only in the core, when the game is convex? This is also not true in general, and our final example is designed to show this.

EXAMPLE 5 (A convex game where $x^*$ does not Lorenz dominate every other allocation in $L(N)$): Let $N = \{1,2,3\}$; $v(1) = 4$, $v(2) = 6$, $v(3) = 8$, $v(12) = 11$, $v(23) = 15$, $v(13) = 12$; and $v(N) = 21$. This game is convex. It is easy to check that $x^* = (6,7,8)$. Moreover, the allocation $x = (6.25,\ 6.5,\ 8.25) \in L(N)$, and clearly $x$ is not Lorenz dominated by $x^*$. (Of course, $x$ cannot be a core allocation (by Theorem 3), and by Theorem 1, $x$ must be Lorenz dominated by some other point in the Lorenz core $L(N)$.)

## 6. APPLICATIONS

In this section, we describe the allocations recommended by the egalitarian solution in two economic problems, and compare them with the allocations prescribed by the Shapley value. As a first application, we have chosen the problem of how to share the cost of a public good. Our second application is concerned with distributing the surplus of a labor-managed firm.

### 6.1. *Sharing the Cost of a Public Good*

Consider an economy with two goods, a public good $y$ and a private good ("money"), $z$. The $n$ individuals in the economy each have an initial endowment of the private good, the $i$th individual endowment being denoted by $\bar{z}_i$. The public good is produced by means of the private good, the cost function being

$$z = c(y), \quad c' > 0, \quad c'' > 0. \tag{17}$$

Individuals have identical utility functions defined over combinations of public and private good:[18]

$$V(y, z) = u(v) + z, \quad u' > 0, \quad u'' \leqslant 0. \tag{18}$$

Since utility functions are identical, the optimal provision of the public good for any coalition depends only on the *size* of the coalition. Let $y_s$ denote the optimal level of the public good for a coalition of size $s$. How should the cost $c(y_n)$ be shared by the individuals in the economy? One way of allocating costs is to convert this problem to a game in characteristic function form, and apply various solution concepts. This is what we do below.

[18] The assumption of identical utility functions is in no way essential for the application of the egalitarian solution. However, there are restrictions involved (See Section 7.1).

Clearly, for any coalition $S$ with $|S| = s$,

$$(19) \qquad v(S) = su(v_s) - c(y_s) + \sum_{i \in S} \bar{z}_i.$$

It therefore follows that $v$ is convex. Hence, from Theorem 2, there exists a unique egalitarian solution to this cost-sharing problem. Moreover, from Shapley (1971), the Shapley value belongs to the core. Theorem 3 then tells us that the egalitarian solution Lorenz dominates the Shapley value.

Under the assumptions made here, the Shapley value recommends *equal absolute taxation*, that is, no matter what may be the distribution of initial endowments, each individual contributes the same amount of private good. This is obviously an *extremely regressive form of taxation* since no attempt is made to make those with a higher endowment of $z$ pay more taxes. Note that the Shapley value has been recommended as an "equitable" solution for this game (see Champsaur (1975)).

Let $x^E = (x_1^E, x_2^E, \dots, x_n^E)$ be the allocation recommended by the egalitarian solution. Hence,

$$(20) \qquad x_i^E = u(y_n) + z_i^E$$

so that the implied tax structure $\{t_i^E\}$ is

$$(21) \qquad t_i^E = \bar{z}_i - z_i^E.$$

The egalitarian tax structure is less regressive than the Shapley value. This is clear from Proposition 1 below, a proof of which is available in Dutta and Ray (1987).

PROPOSITION 1: *If* $\bar{z}_i > \bar{z}_j$, *then* $t_i^E > t_j^E$.

Proposition 1 does not say whether $\{t_i^E\}$ is always a *progressive* tax structure, that is, one in which the average tax rate is an increasing function of individual "income." Indeed, in general it is *not* true that $t_i^E/\bar{z}_i$ (or $t_i^E/v(\{i\})$) is an increasing function of $\bar{z}_i$ (or $v(\{i\})$).

EXAMPLE 6: Let $n = 3$, $u(y) = 2\sqrt{y}$, $c(y) = y$, $\bar{z}_1 = a + 7$, $\bar{z}_2 = 5$, $\bar{z}_3 = 2$. Then $t_1^E = 5$, $t_2^E = 3$, $t_3^E = 1$. Here $t_3^E/\bar{z}_3$ is highest, although individual 3 is poorest. Also, $t_1^E = 5$ irrespective of the value of $a$. So, for high enough values of $a$, person 1's average tax rate $t_1^E/v(\{1\})$ can be made smaller than that of individual 2 or 3. However, it follows from Theorem 3, that any attempt to get a more progressive tax structure will result in an allocation which is not in the core of $v$.

### 6.2. *The Egalitarian Allocation in a Labor-Managed Firm*

Consider a firm of $n$ individuals. Output is produced by the combined effort of the individuals. Individual $i$ is capable of producing $\alpha_i$ units of output, $i \in N$. For any coalition $S$, the total output is $\sum_{i \in S} \alpha_i$.

To set up a firm requires a fixed cost of $c > 0$. Each coalition can choose whether or not to set up a firm. Therefore the *worth* of a coalition $S$ is given by

$$v(S) \equiv \max\left\{ \sum_{i \in S} \alpha_i - c, 0 \right\}. \tag{22}$$

This describes the game. The reader can check that this game is convex. To keep matters nontrivial, we assume that there exists a coalition $S$ with $v(S) > 0$.

Let us number the individuals so that $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$.

PROPOSITION 2: *The egalitarian allocation $x^*$ of the firm is given by*

$$x_i^* = \frac{1}{k}\left[ \sum_{s=1}^{k} \alpha_s - c \right] > 0 \quad \text{for} \quad 1 \leqslant i \leqslant k, \tag{23}$$

$$x_i^* = \alpha_i \quad \text{for } i > k, \tag{24}$$

*where $k$ is either $n$, or the smallest integer such that*

$$\frac{1}{k}\left[ \sum_{s=1}^{k} \alpha_s - c \right] > \alpha_{k+1}. \tag{25}$$

For a proof of this result, see Dutta and Ray (1987).

In general, then, the egalitarian allocation prescribes the following rule: A subset of the $k$th "richest" individuals is identified (where $k$ is chosen to maximize the *average* product, *net* of set-up costs for that subset). Individuals outside this set receive their "marginal product." Individuals within this set share the remaining surplus *equally*, and pay for the entire set-up cost. They therefore receive (on average) *less* than their marginal product. The egalitarian allocation here corresponds to the *levelling tax* (Young (1984)).

By Theorem 3, *no* allocation can Lorenz dominate this allocation and yet be a core allocation. Indeed, by Theorem 1, a Lorenz-dominating allocation cannot be achieved even if subcoalitions are restricted to block only with *their* egalitarian allocations.

Compare the egalitarian allocation with the Shapley value. For simplicity consider the case where $\min_i \alpha_i > c$. In this case, one can easily prove the following proposition:

PROPOSITION 3: *The Shapley value $x^S$ is characterized by*

$$x_i^S = \alpha_i - \frac{c}{n}, \quad i \in N. \tag{26}$$

That is, everyone contributes equally to the setting up of the firm and then receives his marginal product.[19] This, too, is a core allocation. However, as the

[19] In the case where $\min_i \alpha_i > c$, the Shapley value is always different from the egalitarian allocation. But sometimes they might coincide. Take for example, $N = \{1,2\}$ with $\alpha_i < c$, $i = 1, 2$ and $\alpha_1 + \alpha_2 > c$.

egalitarian allocation demonstrates, it is generally possible to come up with a more progressive allocation and still satisfy the no-deviation constraints.

## 7. SOME REMARKS ON THE CONCEPT OF AN EGALITARIAN ALLOCATION[20]

### 7.1. *Full Interpersonal Comparability and the Failure of Strategic Equivalence*

The egalitarian solution violates a "classical" requirement of game theory—it is *not invariant* to modifications of the game which are *strategically equivalent* (see Von Neumann and Morgenstern (1944), Luce and Raiffa (1957)). This invariance concept is so firmly rooted in game theory that it is worth discussing why our concept deviates from the norm, and outlining the significance of such a deviation.

Recall that two $n$-person games $v$ and $v'$ are *strategically equivalent* if there is a vector $(\delta_1, \dots, \delta_n)$ and a positive constant $c$ such that for all coalitions $S$,

$$v'(S) = cv(S) + \sum_{i \in S} \delta_i.$$

A solution concept $\Psi$ satisfies *covariance* (that is, the invariance requirement) if for all strategically equivalent games $v$ and $v'$,

$$\Psi(v') = c\Psi(v) + \sum_{i \in N} \delta_i.$$

The rationale for imposing the requirement of covariance is that two strategically equivalent games have the same strategic character. The use of characteristic functions to represent a transferable utility cooperative game implies that individual utility functions are *cardinal*. So, if $u_i$ is the "right" utility function for $i$, then so is $v_i$, where $v_i = \delta_i + cu_i$. Luce and Raiffa (1957) also argue that if each player $i$ is paid an amount $\delta_i$ prior to the play of the game, then no essential change has taken place in the structure of the game. Neither a change in the units of measurement of utility nor payments to individuals prior to the game alter the strategic considerations involved. Consequently, they should not have an effect on the rational selection of strategies nor on the outcomes of the game.

We are fully in sympathy with this view for positive solution concepts which attempt to predict the outcomes that will be reached by rational and selfish players attempting to maximize their own utilities. It bears repetition, however, that the egalitarian allocation is a hybrid concept,[21] in the sense that it incorporates both positive as well as normative aspects. The "positive" aspect of our solution concept is captured by the fact that "selfish" participation constraints for subcoalitions are respected. But there is a "normative" aspect too: the possible deviations themselves are constrained by the norms of the deviating

[20] This section is inspired by the useful comments of an anonymous referee.

[21] Although it is a hybrid solution concept, the egalitarian solution turns out to be the efficient *noncooperative* bargaining solution in the limit (as discount factors tend to unity) for convex games. On this, see Chatterjee, Dutta, Ray, and Sengupta (1987).

subcoalitions. These norms are derived from an explicit concern for equity. And it is this aspect of the concept that causes it to apply in nontrivially different ways across strategically equivalent games.

In particular, the social concern for equity necessitates inequality comparisons between different allocations in the attainable set. Suppose that $x$ and $x'$ are two allocations and $I$ is some measure of inequality. Let $\hat{\delta} \in \mathbb{R}^n$. Then it is quite possible that $I(x) > I(x')$ and $I(x + \hat{\delta}) < I(x' + \hat{\delta})$. In other words, inequality comparisons are *not* invariant to *all* kinds of transformations of the imputation space. It is for precisely this reason that the egalitarian solution fails to satisfy covariance.[22]

Indeed, apart from this "technical" reason, we feel strongly that egalitarian-type solutions *should not* be covariant. Consider, for instance, the public goods problem discussed in Section 6.1. Suppose that each individual has the same endowment of the private good, and that utility functions are identical. It is then only sensible that each individual pays the same tax. Shall we still prescribe the same tax allocation if one individual's initial wealth doubles, while every other endowment remains unchanged? "The rich should pay more" must be a minimal requirement underlying any egalitarian notion, particularly when everyone gets the same benefit from the public good. But this requirement *necessitates* a violation of covariance. Note, too, that this example is exactly analogous to Luce and Raiffa's example of players receiving monetary payments before the start of the game.

We now make some observations on the analytical framework required to make systematic egalitarian judgements. This issue has been discussed extensively in the social choice literature on interpersonal comparability of utilities (see, e.g., Sen (1970, 1987) and Roberts (1980)), so we shall be brief. Let an inequality measure be a function $I: U \to \mathbb{R}$, where $U$ is the set of available utility allocations. Of course, given the measurability assumptions on individual utilities, two seemingly different utility vectors may represent the same underlying reality. This problem can be met by imposing an invariance requirement on $I$, which takes the general form of specifying that for any two vectors $u$ and $u'$ in the same *comparability set* (see Sen (1987)), $I(u) = I(u')$.

Of course, since we are in the framework of transferable utility games, our framework assumes that individual utilities are cardinal. Systematic inequality comparisons then require the assumption of *cardinal full comparability*: for all $u, u'$ in $U$, if there exists $a \in \mathbb{R}$, $b \in \mathbb{R}_+$ such that $\forall i = 1, 2, \cdots n$, if $u'_i = a + bu_i$, then $I(u') = I(u)$.

Our framework implicitly assumes that the nature of individual utilities is such that they are cardinally fully comparable. This is, of course, stronger than the comparability assumption implicit in the requirement of covariance. As a consequence, the egalitarian solution only satisfies:

[22] However, the inequality measures which are consistent with the Lorenz ordering will rank $x$ and $x'$ in the same way as $cx$ and $cx'$ for any $c > 0$.

*Weak Covariance*: If for all coalitions $S$, $v'(S) = a + bv(S)$ where $a \in \mathbb{R}$, $b \in \mathbb{R}_+$, then $\Psi(v') = a + b\Psi(v)$.

We again stress that relaxation of the invariance requirement to weak covariance is the minimum price that has to be paid in order to be able to discuss the issues that we have raised here.

### 7.2. *Beyond Transferable Utility: an Alternative Approach*

We sketch here a general framework in which the present model can be easily embedded, using a characterization that dates back to Hardy, Littlewood, and Polya (1934). This yields some additional insight into the concept of an egalitarian allocation that may be of independent interest.

Consider a game in characteristic function form. Every coalition has attached to it a set $V(S) \in \mathbb{R}^S$ of feasible utility allocations. For the special case of a TU game, $V(S) = \{x \in \mathbb{R}^s | \sum_{i \in S} x_i \leqslant v(S)\}$. Each coalition possesses, moreover, a social welfare function $W_S$ defined on $V(S)$. Denote by $W$ the collection $\{W_S\}_{S \subseteq N}$.

We are interested in the welfare-maximal allocations for the grand coalition, subject to participation constraints. Of course, deviating coalitions are restricted in the nature of their deviations. They can only "block" with *their* welfare-maximal allocations, subject to further participation constraints.

Analogous to Lorenz cores, it is possible to define *attainable sets* $A(S)$ for each coalition $S$. A collection $\{A(S)\}$ is *attainable* if for each $S$: $A(S) = \{x \in V(S)|$ there is no $T \subset S$ and $x' \in A(T)$ such that (a) $x' \in \arg\max W_T(x)$ over $x \in A(T)$, and (b) $x' > x(T)\}$.

The *desired set of allocations* is the set $D(W) = \arg\max_{x \in A(N)} W_N(x)$. Note that in general, the desired set depends on $W$.

Specialize to TU games. The point of interest is that for a broad class of welfare functions, the desired set of allocations is *independent* of the particular form within that class, and *coincides with the egalitarian allocation*. To be precise, we have the following proposition:

PROPOSITION 4: *Let $V$ be a transferable utility game, and let $W_S$ be increasing and strictly Schur-concave for every coalition $S$. Then $D(W)$ is nonempty if and only if the egalitarian allocation exists, and contains at most one element—the egalitarian allocation itself.*

REMARK: A function $f: X \to \mathbb{R}$ is strictly Schur-concave if for all $x \in X$, $f(Qx) > f(x)$ for all bistochastic matrices $Q$, such that $Qx \neq x$. Notice that Proposition 4 does not make any explicit reference to any interpersonal comparisons of utility. However, Schur-concavity implies a preference for averaging, and it makes little sense to impose Schur-concavity on the welfare function, if individual utilities are not comparable.

We omit a proof of this proposition. As already mentioned, it hinges on the arguments of Hardy, Littlewood, and Polya (1934) and a statement of the relevant results for inequality measures may be found in Dasgupta, Sen, and Starrett (1973). We should also mention that the class of all strictly Schur-concave functions is quite broad, containing the class of all strictly concave functions.

This completes the embedding. Our analysis is therefore indicative of a general approach, which has as its distinctive feature the consistent use of norms, not only in the overall choice of society, but also in determining the constraints over which the choice must apply.

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## APPENDIX

In this appendix, we present a systematic coverage of all 3-person games. Our purpose here is to identify the conditions under which the egalitarian solution will exist. Since only routine checking is required, we have omitted proofs of the assertions made subsequently.

Let $v$ be any arbitrary superadditive characteristic function for a 3-person game. W.l.o.g. assume that $v(N)=1$. Recalling from Section 5 that $e(S, v)$ is the average worth of coalition $S$, let

(27) $$e(\{1,2\},v)=a, \quad e(\{1,3\},v)=b, \quad e(\{2,3\},v)=c.$$

CASE 1:

(28) $$\text{For all } S\subset\{1,2,3\}, \quad e(S,v)\leqslant \tfrac{1}{3}.$$

Then, the egalitarian solution exists, and is $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$. From now on, suppose (28) does not hold.

CASE 2:

(29) $$\forall i=1,2,3, \quad v(\{i\})\leqslant \min_{S\ni i} e(S,v).$$

W.l.o.g. assume that $a\geqslant b\geqslant c$. Then, the egalitarian solution exists iff $a>b$ and $EL(N)=\{a,a,1-2a\}$.

CASE 3: Suppose (29) does not hold for some $i$, say 1. Continue to assume that $a\geqslant b$, and that

(30) $$v(\{1\})=d>b.$$

Since $v$ is superadditive, the egalitarian solution exists for all 2-person coalitions. So, let

$$EL(\{1,2\})=(x_1,x_2),$$
$$EL(\{1,3\})=(d,y),$$
$$EL(\{2,3\})=(z_2,z_3).$$

Since $v(\{1\})=d$, clearly $x_1\geqslant d$.

CASE 3(i): Suppose $x_1 = d$. Then, $EL(N)$ exists iff one of the following conditions holds:

(i) $x_2 > z_2$ and $y \leqslant 1 - 2a$,

(ii) $x_2 = z_2$ and $\max(y, z_3) \leqslant 1 - 2a$,

(iii) $x_2 < z_2$ and $z_3 \geqslant y$,

(iv) $x_2 < z_2$, $y > z_3$ and $x_2 \leqslant 1 - 2b$.

CASE 3(ii): Suppose $x_1 > d$. Then, $EL(N)$ exists iff one of the following conditions holds:

(i) $x_2 > z_2$,

(ii) $x_2 = z_2$ and $z_3 \leqslant 1 - 2a$,

(iii) $x_2 < z_2$ and $z_3 \geqslant y$.

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