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COALITION FORMATION AND BARGAINING PROTOCOLS: A REVIEW OF THE LITERATURE

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Abstract. This paper offers a review of the vast literature regarding bargaining and coalition formation. This topic has been generally described as the attempt to provide strategic foundations to cooperative solution concepts. It can therefore be seen as the linking ring between the non-cooperative and the cooperative game-theoretic approach to coalition formation. Its central role in economic theory and its relatively long history that goes back to the Nash program have fostered a large academic production, including surveys. Nonetheless, this paper will focus on an aspect that is often neglected in the dedicated surveys: the specificities of the bargaining protocols leading to different outcomes. Although generally downgraded to the rank of details, the differences in bargaining protocols, even when minor, can cause significant changes in fundamental aspects such as the possibility to reach full cooperation, the distribution of final pay-offs and the time taken to reach an agreement. Focused on externalities-free games, therefore on bargaining protocols sustaining solution concepts for cooperative games in characteristic function form, the paper aims at providing a brief but exhaustive review of the topic that could result in a very useful tool for any researcher approaching the subject of coalitional bargaining.

Keywords. Bargaining; Coalition formation; Cooperative solution concepts; Nash program; Protocols

1. Introduction

In his survey, Serrano (2004) offers an enlightening similitude between the efforts to provide micro-foundations to macro-economic and the Nash program (Nash, 1953), whose aim is to 'bridge the gap between the two counterparts of game theory (cooperative and non-cooperative)'. This comparison is sufficient to shed light on the importance of the topic. If taken alone, each side of the coin, the cooperative and the non-cooperative approach, has its own weaknesses as stressed by Gul (1989). In particular, the cooperative approach has been criticized for lacking of strategic foundations, whereas the non-cooperative one has been judged to be heavily dependent on the choice of the extensive form of the game and on the equilibrium concept adopted, choices that are far from being commonly agreed. Another problematic aspect of the non-cooperative approach is the multiplicity of equilibria that it might lead to (Gul, 1989). Their combination, therefore, can be a useful operation to overcome the flaws of each side.

In order to understand the way in which one can strengthen the other, it is necessary to fully comprehend what the two approaches are and their differences. In particular, as Serrano (2004) underlines, the idea of cooperative game theory as a mere normative approach detached from strategic consideration that pursues cooperation and equity through desirable axioms is basically wrong. According to the same author, the proper definition of cooperative game theory is that 'of a theory in which coalitions and the set of pay-offs

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feasible for each coalition are the primitives'. A coalitional game is therefore a game in normal form where the set of pay-offs is given by the value that each coalition has attached dependently on each coalition structure (games in partition function form) or independently from it (games in characteristic function form). Cooperative game theory, therefore, operates a simplification omitting the extensive form of a game in favour of its normal, strategic form (Serrano, 2004). What is lost in richness, however, is gained in sharpness since it restricts the attention to the fundamental aspects of a strategic situation.

On the other side, non-cooperative game theory models explicitly the game in its extensive form. Behavioural assumptions underpinning players' moves, masked in the coalitional approach, must now be stated. The bargaining process between individuals becomes the focal point. This sheds light on the mechanism apt to lead to a certain pay-off distribution predicted by a cooperative solution concept. It could then be argued that one could start directly from the extensive form of a game. But in a multi-player environment, where players dispose of a large sets of feasible actions, this can lead to an impossible effort of comprehension without any guideline (Winter, 2002). Starting with the normal form of the game, either in partition or characteristic function form, offers then such a guideline. It comes with no surprise therefore that some of the most popular solution concepts in cooperative game theory, such as the Nash bargaining solution (Nash, 1950), the Shapley value (SV) (Shapley, 1953), the nucleolus (Schmeidler, 1969) and the constrained egalitarian solution (Thomson, 1983; Dutta and Ray, 1989; Dutta, 1990; Dutta and Ray, 1991), have all preceded their non-cooperative foundation.

If the cooperative approach was, and still is, a valuable simplification of strategic interactions among individuals, once several solution concepts, and the axioms upon which they are based, have been proposed, its role has lost momentum in favour of the non-cooperative side (Compte and Jehiel, 2010). In the last decade and a half, there has been a shrink of new solution concepts for games in characteristic function form,[1] whereas the same cannot be said for their extensive form counterpart. A few examples may serve to confirm this: Yan (2003), Montero (2006), Compte and Jehiel (2010) and Okada (2011). A different argument applies to games in partition function form. Although known since the work of Thrall and Lucas (1963), the higher complexity of the situation they serve to depict has considerably slowed the emergence of suitable solution concepts and this seems to be a still active research agenda. Furthermore, it is interesting to note that some of the solution concepts emerged for games in partition faction form are extensions or modifications of concepts originally envisaged for games in characteristic function form. See, for example, the works of Kóczy (2007), McQuillin (2009) and Bloch and Van den Nouweland (2014).

In light of what said till now, this paper aims at reviewing the literature on bargaining and coalition formation. The focus will be on the non-cooperative side, although the link between bargaining protocols and the cooperative solutions that they support will always be pointed. Furthermore, it will adopt a narrow perspective, carefully examining how specific variations in bargaining protocols lead to different conclusion about efficiency and timing of coalition formation and, obviously, on pay-offs distribution. Given the abundance of the literature on this topic, some works will be necessarily omitted and we apologize for this with their authors. Moreover, the analysis will be almost exclusively limited to bargaining games whose normal form counterpart is represented by a game in transferable utility (TU) characteristic function form. This excludes the interesting case of externalities between coalitions. The reason to apply this restriction is primarily due to the mentioned extent of the topic, under the obvious consideration that scope comes at the expense of depth.

It could be argued that the excellent works of Serrano (2004), Bandyopadhyay and Chatterjee (2006) and Ray (2007) already cover this topic. This is true, but several years have passed from their publication and, in the meanwhile, the literature has done significant steps forward. Furthermore, if the work of Serrano (2004) points at the relation of cooperative solution concepts with bargaining games, the present one, as mentioned, will give prevalence to the relation between outcomes of bargaining and protocols' specificities. Moreover, two players bargaining, largely covered by Serrano (2004) will be omitted in favour of games with at least three players. Compared to Bandyopadhyay and Chatterjee (2006), this paper focuses on 'pure' coalitional bargaining, whereas the former has a large section dedicated to legislative bargaining. Initiated by the seminal paper of Baron and Ferejohn (1989), although this strand of the literature is closely connected to coalitional bargaining, it nonetheless has significant points of departure, among which the most important is its focus on games with an empty core.[2]

Section 2 briefly summarizes the idea of a characteristic function form game, then it sketches the fundamental elements of a game in extensive form and, starting from these lasts, it depicts an ideal coalitional bargaining model. The core of this paper, Section 3, examines a sample of coalitional bargaining protocols present in the dedicated literature. Models will be divided according to the cooperative solution concepts they support, starting with models supporting the nucleolus, proceeding with the ones supporting the SV and concluding with models supporting a family of egalitarian solutions and the core. The final section is devoted to conclusions.

2. Structure, Equilibrium Concept and Other Common Features of Coalitional Bargaining Models

Before describing the structure, the selection of the equilibrium concept and other common elements of coalitional bargaining games, it seems opportune to describe their simplified normal form: the characteristic function form. Introduced by the seminal work of Von Neuman and Morgenstern (1947), a game in characteristic function form, G, is constituted by a 2-tuple (N, v) whose elements are the finite set of players, N, and a real-valued function, v, that assigns to each non-empty element of the power set of N a real value[3] (Osborne and Rubinstein, 1994). If we indicate with S a coalition, a generic element of $\mathscr{P}(N)$[4], $v(S)$ is therefore the total pay-off that is available for division among the members of S. When no restrictions are posed on the possibility to divide $v(S)$ among coalition's members, the game is said to be a TU game.

For a TU game (N, v) and a coalition of players S with value $v(S)$, define $X(S)$ as the set of all feasible divisions of the worth of S among its members: $X(S) = \{\mathbf{x} \in \mathbb{R}^{|S|} : \sum_{i \in S} x_i \leq v(S)\}$. The set $X(S)$ is closed, convex and comprehensive. If we define as Γ the set of all games $G(N, v)$ in characteristic function form, a cooperative solution concept, Φ, is a mapping from an element of Γ to a set, $\Phi(N, v) \subset \mathbb{R}^{|N|}$, of feasible vectors, called pay-off profiles, such that, for each element $\mathbf{x} \in \Phi(N, v)$, there exists a coalition structure – $S_1, S_2, ..., S_k, \bigcap_{i=1}^{k} S_i = \emptyset, \bigcup_{i=1}^{k} S_i = N$ – for which $\mathbf{x}(S_i) \in X(S_i)$ for all $i = 1, 2, ..., k$ (Serrano, 2004). A pay-off profile is therefore a vector of values that assigns to each of the n players – since now on n will be used as $|N|$ – a pay-off under the feasibility constraint represented by the amount of pay-off available for distribution given by $v(S)$. If this is a constraint common to all solution concepts, the difference in axioms which they are based upon generates the peculiar image of each Φ. It has to be noted that solution concepts can be classified into set valued, if the cardinality of $\Phi(N, v)$ can be greater than one, or single valued, when $\Phi(N, v)$ is necessarily a singleton. Finally, note that an element of $\Phi(N, v)$, let us name it $\boldsymbol{\phi}$, is called an imputation if it satisfies the following properties: $\sum_{i=1}^{n} \phi_i = v(N)$, $\phi_i \geq v(\{i\})$, $i = 1, 2, ..., n$.

Coalitional games are generally divided into classes according to some properties hold by the characteristic function. Since these properties influence the extensive form of the game as well, it is opportune to provide their formal definition:

Definition 2.1 (Essential game). A TU game (N, v) is said to be essential if

$$v(N) > \sum_{i \in N} v(\{i\})$$

Definition 2.2 (Cohesivness). A TU coalitional game (N, v) is said to be cohesive if

$$v(N) \geq \sum_{i=1}^{k} v(S_i), \quad \forall S_i \in \mathscr{P}(N) : \bigcap_{i=1}^{k} S_i = \emptyset, \bigcup_{i=1}^{k} S_i = N$$

Definition 2.3 (Super-additivity). A TU coalitional game (N, v) is said to be super-additive if

$$v(S \cup T) \geq v(S) + v(T), \quad \forall S, T \in \mathscr{P}(N) : S \cap T = \emptyset$$

Definition 2.4 (Convexity). A TU coalitional game (N, v) is said to be convex if

$$v(S \cup T) \geq v(S) + v(T) - v(S \cap T), \quad \forall S, T \in \mathscr{P}$$

Definition 2.5 (Normalized game). Given an essential TU coalitional game (N, v), its normalized form, (N, v') is obtained by the following two steps procedure:

$$1)\, v(S)^0 = v(S) - \sum_{i \in S} v(\{i\}),\ \forall S \in \mathscr{P}$$

$$2)\, v(S)' = \frac{v(S)^0}{v(N)^0},\ \forall S \in \mathscr{P}$$

Clearly, in a normalized game, $v(\{i\})' = 0,\ \forall i \in N$ and $v(N)' = 1$.

Definition 2.6 (Zero-normalized game). Given an essential TU coalitional game (N, v), its zero-normalized form, (N, v'), is obtained by applying only step 1 in Definition 2.5.

Now that the basic features of a cooperative game have been briefly summarized, it is possible to describe the extensive form of the bargaining game.

2.1 *The Extensive Form of a Coalitional Bargaining Game*

In his presentation of the concept of trembling hand perfect equilibrium, Selten (1975) offers a very clear and concise description of an extensive form game with perfect recall. Given that perfect recall, introduced by Kuhn (2016), is a standard assumption in coalitional bargaining games, – what can be considered as the most famous model of bargaining, the Rubinstein model (Rubinstein, 1982), although not coalitional, falls into this category – the description of Selten can be taken as our guideline.

An extensive game is fully described by a 6-tuple:

$$G = (H, P, F, C, \alpha, \pi)$$

where the set H represents the game tree, P is the set of players' partitions, F the information partition, C the choice partition, α is a function that assigns probabilities over the elements of C and π is a pay-off function that associates n real values (pay-offs), where n is the number of the players in G, to each final node of H. Selten (1975) assumes that H represents a finite tree, whereas coalitional bargaining games are generally infinite.

Selten (1975) describes H as the game tree, therefore as a collection of vertices and edges connecting them. The tree has an origin. The set of all edges except the final nodes is indicated with K, whereas Z stays for the set of endpoints. The set P, having cardinality $n + 1$, collects subsets of H – $P = \{P_0, P_1, ..., P_n\}$ – each of which, in turn, collects the vertices where player i – $i = 1, 2, ..., n$ – is entitled to make a move. The set P_0 is dedicated to the random mechanism operating in the game. Each P_i, including P_0, can be further subdivided into subsets, named information sets. Let us call each of them U. Informally speaking, if we draw the game tree, the elements of U are the vertices where player i is entitled to move that lay on the same horizontal line. Therefore, for each vertex k belonging to U, player i will have the same set of moves at her disposal. Moreover, the game-play can intersect U at most once. The set Υ_i collects all the sets U belonging to player i – all the subsets of P_i – and set Υ groups together all Υ_i. Note that $\Upsilon_0 = P_0$. The last set characterizing the game G is C, which collects all sets C_U. Sets C_U, in turn, list all the possible moves c, also called actions or choices, that are available at vertex k belonging to a certain information set U. The last two elements of G are functions. The first, α, is the probability distribution of the actions belonging to the random mechanism of the game. Its argument is therefore c. The second, instead, π, is the pay-off function that assigns a specific value for each player of the game at each ending vertex $z \in Z : \pi(z) = (\pi_1(z), \pi_2(z), ..., \pi_n(z))$ (Selten, 1975).

If the one just provided is the general description of a game in extensive form, it is instructive to try to characterize a coalitional bargaining game in its archetypal form and to relate the two descriptions. Given our objective of finding an archetype, the following coalitional bargaining game might have more or less elements compared to specific games present in the literature. Furthermore, the bargaining protocol depicted can be ascribed to the family of Rubinstein-type models. In general, authors use one of two types of ways for describing a coalitional bargaining game. The first way is very parsimonious and consider the game tree and the pay-off function to fully characterize the game. All the other elements, such as players' available choices, the probabilistic function of the random mechanism and the strategy space are considered to be elements of the same game tree. The game therefore can be described by a 2-tuple: $B = (H, \pi)$; see, for example, Kim and Jeon (2009). A second approach, instead, considers the game tree as the result of the combination of other basic elements and therefore lists them in the tuple describing the game; see, for example, Nguyen (2015). We will actually adopt this second, more explicit, approach.

A coalitional bargaining game is generally described by a 6-tuple:

$$B = (N, \mathscr{P}, v, \Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta})$$

The first three elements, N, $\mathscr{P}$ and v, are the same as the components of a game in coalitional form: N is the set of players, $\mathscr{P}$ is the set of non-empty coalitions and v is the function that associates a value to each element of $\mathscr{P}$. Although we have defined $\mathscr{P}$ as the power set of N, it must be noted that the set of coalitions can actually be a subset of $\mathscr{P}(N)$. Nonetheless, we will continue to use this symbol in order to underlying that, potentially, every subset of N can be a feasible coalition in a bargaining game. Although some works consider explicitly the way in which the value of a coalition is created, for example, Gul (1989) – with the characteristic function having as argument parameter values specific for each player, this is not generally the case.[5] Furthermore, the extensive form of a coalitional game generally abstracts from the way in which the worth of a coalition is produced. The other three elements, Σ, $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, are specific of the bargaining game. Σ represents the whole strategic space. Since, usually, a coalitional bargaining game is sequential and infinite, therefore it has a temporal dimension with $t = 1, 2, ..., \infty$, we have $\Sigma = \times_{t=1}^{\infty} \sigma_t$, where $\sigma_t = \times_{i=1}^{n} \sigma_{it}$. If we add σ_{0t} has the set of actions, at time t, of the random mechanism, Σ corresponds to the set C described in Selten (1975).

Let us see, concretely, which are the actions at disposal of the players, starting from the random mechanism. This initiates the game by selecting the first proposer, one of the n players, according to a given probability distribution. This is why the function α has been changed into vector $\boldsymbol{\alpha}$, where $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\sum_{i=1}^{n} \alpha_i = 1$. In the *rejector-proposes* protocol (e.g. Chatterjee *et al.*, 1993; Kim and Jeon, 2009), where the first rejector of an offer becomes the next proposer, the random mechanism operates only at the origin of the game tree by selecting an order of players that will persist for the whole game (*fixed proposers order*), whereas in the *random proposer* protocol (e.g. Okada, 1996; Compte and Jehiel, 2010; Okada, 2011), every node after a rejection of an offer is dedicated to the random mechanism. It worth to note that some variants of the coalitional bargaining model have the random mechanism operating at other levels. In Gul (1989) and Nguyen (2015), for example, also the formation of a certain coalition is treated as a random event, whereas in Hart and Mas-Colell (1996), the proposer suffers from a positive probability to be excluded from the game if her proposal is refused. Our archetypal presentation, however, focuses on the simple case in which the random mechanism operates only for the selection of a proposer.

At each time period t, there are two different information sets to which the choice of a player can belong. The available choices, therefore, vary according to which of the two states the player is in. If she has been selected as a proposer by the random mechanism, her choice can be described by a 2-tuple: $c_p = (\boldsymbol{x}, S)$ with $\boldsymbol{x} \in \mathbb{R}^{|S|}$ and $\sum_{i \in S} x_i \leq v(S)$. A proposer can select a coalition and propose a certain, feasible, division of its worth among the members. A proposer can always choose to pass her turn, but this is generally equivalent to propose a division that will be certainly refused. Furthermore, a proposer can be granted the possibility to choose the order of the respondents, but since this element does not generally influence the game in any way (see, for example, Chatterjee *et al.*, 1993; Okada, 1996; Compte and Jehiel, 2010), this is not a real strategic choice. If a player belongs to the coalition selected by the proposer, she will then be a respondent. In this case, her action space is described by a dichotomous choice: $c_r = \{accept,\ reject\}$.

The last element of the 6-tuple describing B is the vector of discount factors $\boldsymbol{\delta}$ – $\boldsymbol{\delta} \in \mathbb{R}^n$ and $\delta_i \in [0, 1]$, $\forall i \in N$ – which is strongly interrelated with the role of time t in the bargaining game. In the first time period – $t = 1$ – all the three steps previously described are present in the following order: random mechanism, proposal, and response. If all responders accept the offer received, the game is already at its terminal node z, provided that the model allows for the formation of a single coalition.[6] The pay-off function can be described as follows:

$$\pi_i(z) = \begin{cases} \delta_i^{t-1} x_i, & \forall i \in S,\ S \in c_p \\ v(\{i\}), & \forall i \in N \setminus S,\ S \in c_p \end{cases}$$

A refusal from one of the responders gives rise to an identical sub-game tree in the random proposal protocol or to an equal sub-game tree but without the random mechanism move in the rejector-proposes case. The game has therefore a clear recursive structure and the only change from one period to another happens through the action of discounting.

A final remark is related to players' preferences that follow in all respects the assumptions of Rubinstein (1982).[7] Being a game of complete information, every player is assumed to know her own and other players' preferences and all the other elements of the game such as the discount factors or the worth of each coalition. Furthermore, perfect recall implies that the information set is a singleton. Now that the structure of an archetypal coalitional bargaining model has been described, it should be easier to identify the relation between outcomes and protocol variants. Before, however, we will briefly discuss the selection of the equilibrium concept adopted.

2.2 *Stationarity and Sub-Game Perfection*

Once having a set of bargaining rules (a protocol) as the archetypal type described in the previous section, and supposing to have defined a characteristic function that assigns a value to each possible coalition, the result of the strategic interaction of players in the bargaining process is still heavily dependent on the equilibrium concept that is adopted. Nash equilibrium is clearly unsatisfactory since, even in two-player bargaining, it admits every solution that is efficient and that guarantees each player to obtain at least her disagreement pay-off (Rubinstein, 1982). Therefore, if we consider the Rubinstein model as a balanced two-player coalitional bargaining model where the outside options are $d_i = v(\{i\})$, for $i = 1, 2$, and there is a common discount factor equal to 1 ($\delta_1 = \delta_2 \to 1$), every point in the set X is a Nash equilibrium, with $X = \{\boldsymbol{x} \in \mathbb{R}^2 : x_i \geq v(\{i\}), \text{for } i = 1, 2 \wedge \sum_{i=1}^{2} x_i = v(\{1, 2\})\}$.

Sub-game perfection, introduced by Selten (1973), has been proved to reduce the set of equilibrium points of the Rubinstein model to a singleton (Rubinstein, 1982). However, in a 'proper' coalitional bargaining model, where the adjective proper means $|N| \geq 3$, it is not of much help to sharpen feasible equilibria if it is not coupled with stationarity. In fact, as shown in proposition 0 of Chatterjee *et al.* (1993), if a coalitional bargaining game has a super-additive characteristic function, there always exists a sufficiently high discount factor $\delta^* \in [0, 1]$ – assumed to be common to all players – such that each allocation $\boldsymbol{x}$ that is individually rational – $x_i \geq v(\{i\})\ \forall i \in N$ – is a sub-game perfect equilibrium (SPE).

Kim and Jeon (2009) provide a concise but exhaustive definition of stationary strategy:

A stationary strategy is a mapping from the player's position in the bargaining process to the choice set available to him for every player.

In other words, this implies that strategies are not dependent on the history of the game, but only on the position in the bargaining process that a player covers. While Chatterjee *et al.* (1993) do not find a compelling reason to restrict the attention to a stationary SPE (SSPE) if not the fact that it is analytically tractable and that it enables to reach uniqueness, Kim and Jeon (2009) justify its adoption by the implicit stationarity of preferences and by the intrinsic recursiveness of the game.

This section has presented the main common elements of a coalitional bargaining game through the description of an archetypal sequential offer game. The next section, the core of this paper, will present several variants of this model and their relation with some of the most well-known cooperative solution concepts. We will start presenting models that departs the most from the basic protocol just sketched to come back to it at the end of the section.

3. Bargaining Protocols' Variants and Results

3.1 *The Nucleolus and the Bankruptcy Problem*

Introduced by Schmeidler (1969), the nucleolus does not have a straightforward definition and some steps are required in order to achieve it. First of all, it is necessary to define the notion of excess.

Definition 3.1 (Excess). Given an allocation $\boldsymbol{x}$, with $\boldsymbol{x}$ being a vector in $\mathbb{R}^n$, the excess $e(S, \boldsymbol{x})$ of coalition S given $\boldsymbol{x}$ is defined as

$$e(S, \boldsymbol{x}) = v(S) - \sum_{i \in S} x_i$$

Given a coalitional game (N, v) and a feasible allocation $\boldsymbol{x}$ such that $\sum_{i \in N} x_i = v(N)$, define $\boldsymbol{e}(\boldsymbol{x})$ as the vector of the excesses for all the non-empty coalitions of (N, v) other than N. Given a TU coalitional game (N, v), the nucleolus (Nu) is then given by

$$Nu = \left\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} = Lmin_{\boldsymbol{x}} \boldsymbol{e}(\boldsymbol{x})\right\}$$

where *Lmin* is here defined as the lexicographical minimum operator. The nucleolus is always a singleton and satisfies symmetry, covariance and consistency, where the last is the axiom that peculiarly characterizes this solution concept (Serrano, 1993).

3.1.1 *A Three-Players Model with Non-Contingent Offers*

In pointing the equivalence between the nucleolus and the solution proposed in the Talmud for the contested garment problem, Aumann and Maschler (1985) describe an implementation process leading to the nucleolus in solving any bankruptcy case. This process, however, is not formally described as a non-cooperative bargaining model. Serrano (1993) filled the gap some years later, presenting a bargaining protocol for a three players, super-additive and normalized coalitional game.

In the first step, the random mechanism selects an order of proposers (fixed-order proposer model) that will remain unvaried for the whole game. Given that, the first player in the queue will make a proposal to the other players that will then reply simultaneously. Note that, in this last step, perfect recall is violated. Another peculiar characteristic is the fact that offers are non-contingent, meaning that, if one of the two responders accepts the offer and the other does not, the accepter obtains what has been offered to her. In general, coalitional bargaining games display contingent offers where unanimity is required in order for the offer to produce an effect. Another example of non-contingent offers can be found in Chaturvedi

(2016). If the case of an offer followed by one acceptance and one refusal materializes, the game goes back to the random mechanism that selects, with equal probability, one between the former proposer and the refuser to be the new proposer. The respondent, in this case, has the option to 'buy' the resources of the accepter so that her outside option will now be $\max\{v(i, j) - x_j, 0\}$ where j is the index of the former accepter and i the one of the actual responder. In case the offer is rejected by both responders in the first turn, the second player in the queue (as determined by the initial random move) will become the new proposer. Obviously, if the offer is accepted by both responders, the game ends.

The model has another important peculiarity. Discounting is substituted by a fixed cost of bargaining c, which, however, applies only to the proposer whose offer has been refused by at least one player. Every new proposal, therefore, coincides with an increase of one unit of time. The game is infinite and a perpetual disagreement leads to a pay-off for each player equal to $-\infty$.

The game is solved using SSPE in pure strategy. Serrano (1993) assumes a particular relation in the worth of each coalition: $0 \leq v(2, 3) \leq v(1, 3) \leq v(1, 2) \leq 1$, where $v(\{i, j\})$ has been substituted by $v(i, j)$ for ease of notation. This implies that, defining m_i as the sum of the marginal contributions of player i, we have $m_1 \geq m_2 \geq m_3$. Define as $v_{x\{i,k\}}(i)$ the outside option of player i if j has accepted an offer x_j. Serrano (1993) shows that the solution of the bargaining problem can be translated into a system of equations:

$$
\begin{aligned}
x_i &= 1 - x_j - x_k && \text{(Proposer)}\\
x_j &= \frac{1}{2}\left[1 - x_k - v_{x\{i,j\}}(i) + v_{x\{i,j\}}(j)\right] && \text{(Responder)}\\
x_k &= \frac{1}{2}\left[1 - x_j - v_{x\{i,k\}}(i) + v_{x\{i,k\}}(k)\right] && \text{(Responder)}
\end{aligned}
$$

Proposition 1 of Serrano (1993) states the main results. The system of equation has a unique solution $\boldsymbol{x}^* = \{x_1^*, x_2^*, x_3^*\}$, the nucleolus, provided that the order of proponents mirrors the decreasing order of their sum of marginal contributions (order $= \{i, j, k\}$ if $m_i > m_j > m_k$). If the first proposer is changed, keeping the remaining order identical, the proponent will always obtain the equilibrium value x_i^*, but there is a continuum of feasible solutions, containing $\boldsymbol{x}^*$, for the pay-off of the remaining players. In each case, the agreement is reached in the first round of negotiations. For other orders of proposers, without relation with m, and bargaining costs sufficiently low, there are multiple SSPE in period 2 only.

3.1.2 *A Generalization to n-Players for Bankruptcy Problems*

In the model just described, the nucleolus is the unique SSPE only if a precise order of proponents is adopted. Further, this result is valid only in case of three players (Serrano, 1993). A generalization to n players is offered in Serrano (1995). This alternative bargaining protocol relies on a particular coalitional setting: a bankruptcy problem or a surplus sharing problem. It is instructive to look at the peculiar characteristic function of these games. In a bankruptcy problem, there are n claimants, each of which claims a sum $d_i \geq 0$, with $i = 1, 2, ..., n$. The available sum to be distributed is equal to W, with $W \leq \sum_{i=1}^{n} d_i$. Defining $d(T) = \sum_{i \in T} d_i$, for T being a set in $\mathscr{P}$, the worth of a coalition $S \in \mathscr{P}$ is given by: $v(S) = \max\{0, W - d(N \setminus S)\}$. Since $W \leq d(N)$, the bankruptcy game can be described as a loss sharing problem. The other typology of game examined by Serrano (1995), the surplus sharing game, can be seen as the dual of the bankruptcy problem (Aumann and Maschler, 1985). Here, W is the value of a joint project undertaken by the n players, each of which contributed with an amount $d_i \geq 0$ to its realization. By assumption, $W \geq \sum_{i=1}^{n} d_i$. The characteristic function is defined exactly as in the bankruptcy case, and therefore it is sub-additive. Consequently, the surplus sharing game is not balanced.

The bargaining game is a perfect recall, finite horizon game solved in SPE since its finite nature renders stationarity superfluous. The first random move selects the proposer and the sequence of responders.

Non-contingency of offers is still valid, but what changes is the bargaining mechanism after eventual refusals. In fact, each rejector will enter into a bilateral sub-game with the proposer consisting into an initial random draw, with equal probability, to select who among the two players will act as a dictator. The non-selected player will get half of his claim d, plus what the dictator decides to leave her. Serrano (1995) shows that, given a pair of players $\{i, j\}$ entered into such a sub-game, the expected pay-off for player i is equal to $\max\{0, W - d_j\} + \frac{1}{2}[W - \max\{0, W - d_i\} - \max\{0, W - d_j\}]$. Defining as y the formula for the expected pay-off just shown, after a proposal $\boldsymbol{x}$ is made by player i, the final pay-offs vector $\boldsymbol{\pi}$ will be:

$$\begin{aligned} \pi_j &= x_j && \text{for each accepter } j \\ \pi_k &= y^{k-1} && \text{for each rejector } k \\ \pi_i &= W - \pi(N \setminus \{i\}) && \text{for proposer } i \end{aligned}$$

where y^{k-1} is defined recursively by substituting W, in the formula for the expected pay-off, with $W - \sum_{j \in A} x_j - \sum_{k \in R_k^{k-1}} x_k$, where A is the set of accepters and R_k^{k-1} is the set of rejectors preceding rejector k. Finally, it must be noted that the bargaining game does not have either a discount factor and bargaining costs.

As in Serrano (1993), the game has a unique SPE, coincident with the nucleolus, only if the order of players produced by the initial random move is such that it respects the magnitude of the claims/contributions of players. For example, given the order $1, 2, \cdots, n$, it must hold that $d_1 > d_2 > \cdots > d_n$. Apart from being based on peculiar characteristic form games, such as the bankruptcy game, the implementation of the nucleolus is heavily dependent from the selected order of proposers.

3.2 *Three Roads to the Shapley Value*

Being one of the most popular cooperative solution concepts, the SV (Shapley, 1953) has been extensively studied in the domain of non-cooperative bargaining processes. Three families of models seem to have emerged, leading, under specific conditions, to this cooperative solution. Remembering that the SV is deeply rooted into marginalism, in fact, it grants to each player her marginal contribution to all coalitions of a game (N, v), weighted by the probability that this contribution takes place, the formulas to compute it are:

$$\phi_i(N, v) = \sum_{S \subseteq N} \frac{(n - |S|)!(|S| - 1)!}{n!}[v(S) - v(S \setminus \{i\})] \quad \forall\, i \in N$$

$$\phi_i(N, v) = \frac{1}{n}(v(N) - v(N \setminus \{i\})) \sum_{i \neq j} \phi_i(N \setminus \{j\}, v) \qquad \forall\, i \in N$$

The second equation, presented in Maschler and Owen (1989), will turn out to be useful to understand the mechanism behind one of the family of non-cooperative approaches leading to the SV. Finally, let us remember that this solution concept satisfies efficiency, symmetry, additivity, strong monotonicity and null-player (dummy player).

3.2.1 *A Bargaining Model of Bilateral Random Meetings*

The first, following a chronological order, typology of non-cooperative game described is the one presented in Gul (1989), which could be entitled ‘bilateral random meetings’. The coalitional side presupposes that there are n players, each endowed with a valuable resource M_i for $i = 1, 2, \cdots, n$. Their combination form a bundle that, in turn, generates utility according to a specific function (the characteristic function): $v(S) = v(\sum_{i \in S} M_i)$. The bargaining game, on the other side, is a perfect recall, infinite horizon model, with a peculiar aspect. At each time period t, a single random meeting between two players, say $\{i, j\}$, happens with probability $\frac{2}{n_t}(n_t - 1)$, where n_t is the number of players still in the game at time t. With equal probability, one of the two is then appointed to make a take-it-or-leave offer in terms of utility. If accepted, the responder exits the game with such offer and sells her resource to the proposer, otherwise the meeting breaks. The next turn, till there are players active in the game, opens with a new bilateral random meeting. It must be noted that pay-offs are expressed in terms of utility streams; therefore, the utility of a player, say i, associated with this game is equal to $\sum_{t=0}^{\infty}[(1-\delta)v(M_t) - r_t]\delta^t$, where δ is the common discount factor, M_t represents the bundle owned by i at period t and r_t the payment made, still at period t, to buy a player's resource.[8]

The main result of Gul (1989), as stated in theorem 1, asserts that, for the common discount factor approaching unity, the SSPE unique equilibrium of the bargaining process tends to the SV. Furthermore, the same process is efficient, meaning that at each random meeting, the proposer will offer exactly the expected continuation pay-off to the responder and this last will accept. Initially, Gul (1989) supposed that strict super-additivity of the characteristic function was a sufficient condition to obtain this result, namely efficiency, but an example of Hart and Levy (1999) disproved it. Although the convergence towards the SV is retained under super-additivity, efficiency in the bargaining process requires the more stringent condition of strict convexity.

3.2.2 *A Model Rooted into Simple Demand Commitment Games*

The second type of non-cooperative bargaining game to be considered follows into the category of simple demand commitment games (SDCGs). Firstly envisaged by Bennett and Van Damme (1991) and Selten (1992), Winter (1994) has shown that a particular SDCG protocol leads to the SV in expected terms if the underlining coalitional game is strictly convex. Dasgupta and Chiu (1998) have subsequently simplified the protocol of Winter (1994) obtaining a more general result.[9] We will therefore describe this last protocol.

The bargaining process of Dasgupta and Chiu (1998) can be described as a finite horizon, perfect recall game with no discounting or bargaining costs. It has an underlining coalitional game (N, v) assumed to be strictly convex. In the first step, a random move selects, with equal probability, one of the possible permutations of the set N, a 'fixed' order of players. Let us call Q the list of players generated by this random move. At each next step, a single player, say i, moves. Her choice set includes two possibilities: asking an amount $d_i \in \mathbb{R}$ and passing the turn to the next player or choosing a set of players among her predecessors and forming a coalition with them. This second option entails the obligation for i to pay the chosen players their demands and causes the game to stop, with the remaining players receiving their stand-alone pay-off.[10] Suppose the ordinal position of i in Q is equal to k and define set Q_{k^i} as the truncation of Q at position k. Therefore, a player j is in Q_{x^i} if her ordinal position in Q is lower than k. If player i decides to leave the game, she will then choose a set of players S, with $S \in \mathscr{P}(Q_{k^i})$ such that this choice maximizes her utility. This implies the following equation: $U_i = \max_{S \in \mathscr{P}(Q_{k^i})}(v(S \cup \{i\}) - \sum_{j \in S} d_j)$. The game has, at most, $|N| + 1$ steps.

If we define $Q^n_{k^i}$ as the set of players following i in Q, having, therefore, $Q_{k^i} \cup \{i\} \cup Q^n_{k^i} = N$ and $Q_{k^i} \cap \{i\} \cap Q^n_{k^i} = \emptyset$, theorem 2 of Dasgupta and Chiu (1998) shows that there is an SPE strategy for which, given a random permutation Q of N, player i gets $v(\{i\} \cup Q^n_{k^i}) - v(Q^n_{k^i})$, $\forall i \in Q_{n-1}$ or $v(\{i\})$ for $i \in Q_n$, where Q_{n-1} is the set of all players in Q but the last and Q_n is the singleton set of the last player in the queue. By considering all the possible permutations of N, it is clear that the expected pay-off of a player is her SV of (N, v). Dasgupta and Chiu (1998) derive four results. The first, theorem 1, states that for a strictly convex game, there is only one SPE strategy for each player; therefore, this result is the unique SPE of the game. In convex games, instead, such a strategy exists (theorem 2), but it is not necessarily unique. Furthermore, they show (theorem 3) that this result can be obtained for each type of game by 'convexifying' it, meaning, by adding a monotonically increasing reward in the size of the formed coalition and then applying an opportune system of taxation to players' realizations to ensure budget balance. Finally, in Corollary 6, they show that, for three players (and only three) non-convex games with non-empty core, their bargaining protocol always supports an SPE equilibrium leading to an allocation inside the core.

3.2.3 *A Model with Risk of Partial Breakdown of Negotiations*

The two types of games just described departed significantly from the general model presented in the previous section. We turn now the attention towards a bargaining game whose rules are more in line with a standard Rubinstein-type model. The present game can be shortly described as a sequential bargaining game with risk of exclusion, or partial breakdown. It has been firstly proposed by Hart and Mas-Colell (1992), refined by Krishna and Serrano (1995) and finally extended to the NTU case in Hart and Mas-Colell (1996). It is a finite horizon, perfect recall game with no discounting or fixed costs of bargaining although, as we will see, an additional parameter will partially simulate the effect of discounting.

Furthermore, the present bargaining model is based on a standard game in coalitional form (N, v) that is assumed to be 0-normalized and where v is monotonic:[11] $\forall S \in \mathscr{P} \setminus N$ and $i \notin S$, we have $v(S \cup \{i\}) \geq v(S) + v(\{i\})$. The game opens with the random mechanism selecting, with equal probability, one of the n players to be the proposer (random proposer model). This will make a feasible proposal ($\boldsymbol{x} \in \mathbb{R}^{|S|}$ and $\sum_{i \in S} x_i = v(S)$), to a coalition $S \in \mathscr{P}$. If unanimously accepted (it is therefore a contingent offer) the game ends with the selected players receiving their offer and the remaining getting their stand-alone pay-off $v(\{i\})$. The departure from a standard Rubinstein model with random proposer comes when a refusal takes place since the games, before reverting to the random move that selects a proposer, has another, intermediate, chance move: with probability $\alpha \in [0, 1)$, the actual proposer can remain in the game, but with probability $(1 - \alpha)$, she is excluded, receiving her stand-alone pay-off and the game proceeds without her. This is what renders the game, potentially infinite, a finite horizon one. Furthermore, it actually acts as a form of discounting (Hart and Mas-Colell, 1996).

On the side of results, Hart and Mas-Colell (1992) solve the model applying SSPE, whereas Krishna and Serrano (1995) show that, under particular conditions, SPE is sufficient to support the uniqueness of the findings of their predecessors. Theorem 2 of Hart and Mas-Colell (1996) states that, for $0 \leq \alpha < 1$, there is a unique SSPE equilibrium in the game and the associated pay-off vector is equal to the SV of the underlying cooperative game (N, v) and that, in the limit of α tending to one, this equilibrium vector will be proposed (and unanimously accepted), regardless the identity of the selected player, in the first step. Therefore, for $\alpha \to 1$, the bargaining process is efficient. It is interesting to note that efficiency is obtained the more the model gets closed to a pure Rubistein-type model with random proposer and vanishing discounting.

As anticipated, Krishna and Serrano (1995), in theorem 3.1, extended the validity of the first part of the mentioned theorem 2 by showing that, for $\alpha < \frac{1}{(n-1)}$, the uniqueness of the SV as equilibrium pay-off holds under sub-game perfection without requiring stationarity. Instead, for values of α above a minimum variable threshold value, say α^*, they prove that there is effectively a set of SPE equilibria. They point, however, that it is a strict subset of the set of individually rational vectors. This is important since it shows that, under this bargaining protocol, what asserted by the mentioned proposition 0 of Chatterjee *et al.* (1993) does not hold. For symmetric games, where the adjective symmetric implies that for S and $T \in \mathscr{P}$ with $|S| = |T|$, it must hold that $v(S) = v(T)$, they managed to precisely characterize the set of SPE equilibrium points, let us call it E, and the range of α values supporting it: $E = \{\boldsymbol{x} : \sum_{i=1}^{n} x_i \leq v(N) \wedge \forall i, x_i \geq \frac{v(S)}{n}\}$ if $\frac{1}{|S|} \leq \alpha \leq \frac{1}{(|S|-1)}$. For $\alpha < \frac{1}{(n-1)}$, E is still a singleton having the SV of (N, v) as its sole vector element.

Although this paper does not deal with NTU games, it is worth to note a result of Hart and Mas-Colell (1996), namely that when this assumption holds for the underlying coalitional game, the limits of the

SSPE equilibria pay-off vectors for $\alpha \to 1$ are not the popular solution concepts for this class of games, namely the NTU Shapley and Harsanyi solutions (Harsanyi, 1958, 1963), but the consistent values of Maschler and Owen (1989).

3.2.4 *Partial Breakdown Model's Variants: The Bidding Approach*

In the last section of their paper, Hart and Mas-Colell (1996) revert to the TU assumption and offer a summary description of variants of their model with related results. Basically, these variations affect two elements of the model: the identity of the player to be excluded in case of rejection and the probability of exclusion. In particular, it is relaxed the assumption that it is only the proposer the one to be potentially forced out and, furthermore, it is introduced the possibility of players having heterogeneous probabilities of dropping out. Interacting these variations generates a multiplicity of outcomes, in some of which the SV looses its centrality as equilibrium result. Further variants, have emerged subsequently. Before analysing this literature strand, we will consider another modification of the Hart and Mas-Colell (1996) model since this has also been used as the baseline for the previously described variations. This is the model of Pérez-Castrillo and Wettstein (2001).

Excluding the initial random move, the model runs almost identically as the described version of Krishna and Serrano (1995), with the same assumptions holding for the underlying cooperative game (N, v), apart from the fact that Pérez-Castrillo and Wettstein (2001) consider only the limiting case of α being equal to zero. A rejection, therefore, excludes automatically the proposer. The real innovation results in the substitution of the initial random selection of the proposer with a bidding stage to select it. At this step, each player proposes contemporaneously (perfect recall is then dropped) a vector of offers to each of the other players: $\boldsymbol{b}_i \in \mathbb{R}^{n-1}, \forall i \in N$ where each element is $b_i^j \in \mathbb{R}, \forall j \in N \setminus \{i\}$. If we define vector $\boldsymbol{\beta}_i = \sum_{i \neq j} b_i^j - \sum_{j \neq i} b_j^i$, and the scalar $r = \arg\max_i \boldsymbol{\beta}_i$, we see that r individuates the index of the player that has made the highest bid and that therefore will be appointed as a proposer. In case of equal maximizers, a random draw will select among them. Player $i = r$ will then have to pay immediately the bidden amount and make an offer to all other players that will answer sequentially. The game is then identical as the one described before, with $\alpha = 0$, except that each refusal is followed by a new bidding stage rather than a random move.

Pérez-Castrillo and Wettstein (2001) adopt the second formula used to define the SV to prove, in theorem 1, that the bidding mechanism just described always implement the same SV in SPE for 0-normalized, monotonic TU games. In particular, all the bids will be identical and the final pay-off to each player is given by:

$$\pi_i = v(N) - v(N \setminus \{i\}) - \sum_{j \in N \setminus \{i\}} b_i^j \qquad i = \text{proposer}$$

$$\pi_j = \phi_j(N \setminus \{j\}) + b_i^j, \ \forall j \in N \setminus \{i\} \quad j = \text{accepter}$$

According to Pérez-Castrillo and Wettstein (2001), the bidding mechanism offers a conceptual advantage compared to the other models implementing the SV since this last is not implemented in expectation but in the first round of negotiations regardless of the identity of the selected proposer. They finally show that the model can be duly modified to support the weighted SV (see Kalai and Samet, 1987) in SPE.

3.2.5 *Partial Breakdown Model: Other Variants*

Before considering the various modifications of the random proposer\bidding stage models with partial breakdown, it is necessary to shortly introduce two new solution concepts. The first, called solidarity value (SO) and introduced by Nowak and Radzik (1994), is a variant of the SV that substitutes the marginal contribution of a player to a coalition with the average marginal contribution brought by all players to the same coalition, whereas the other, called in Hart and Mas-Colell (1996) the equal split (ES),[12] is the simple equal division of the value of a coalition among its members. The solidarity value assigns to each player i the following pay-off:

$$\tau_i(N, v) = \sum_{S \ni i} \frac{(n - |S|)!(|S| - 1)!}{n!} A^v(S)$$

with

$$A^v(S) = \frac{1}{|S|} \sum_{j \in S} [v(S) - v(S \setminus j)]$$

Therefore, $A^v(S)$ is the average marginal contribution of all players to coalition S. The ES attributes to each player i the following pay-off:

$$\gamma_i(N, v) = \frac{v(N)}{n}$$

Note that this solution does not take into account the values of any sub-coalitions, but just $v(N)$. Both solutions share with the SV the axioms of efficiency, symmetry and additivity and both discard the null-player axiom. While the first substitutes it with the axiom of A-null player (see Nowak and Radzik, 1994), the latter does it with the property of nullifying player (see van den Brink, 2007).[13]

Hart and Mas-Colell (1996) show that, holding fixed the equal chance of being a proposer, when this role is not affected by the probability of dropping out, whereas the responders, with equal probability, are, the ES solution is obtained in SSPE. Instead, when the possibility of being excluded affects all players and, particularly, the proposer drops out with probability $(1 - \alpha)\theta$, whereas the responders, all equally, suffer the risk of exclusion with probability $\frac{(1-\alpha)(1-\theta)}{(s-1)}$, with s being the number of players still active and holding $\theta \in [0, 1]$ the SSPE will converge towards a 'compromise' between the SV and the ES dependent on the value of θ: $\theta\text{SV} + (1 - \theta)ES$. Calvo (2008) considers a similar case, where a refusal gives rise to an equal probability of exclusion for each of the players still active in the game. Therefore, either the proposer and each of the responders may drop out after a rejection with probability $(1 - \alpha)\frac{1}{s}$, where s is again defined as the number of players still active. For a TU game (N, v), this new setting generates a unique SSPE that is the solidarity value. Finally, if only the proposers face the risk of exclusion, but players have attached different probabilities both of being excluded and of being selected as proposers, the SSPE result is the weighted SV, where each player weight is given by $w_i = (\pi_i(1 - \alpha_i))_{i \in N}$, with π_i being the personal probability of being proposer and, analogously, $(1 - \alpha_i)$ the one of dropping out.

There is a last general modification, and related variants, of this model to be considered: the introduction of discounting. Several authors have investigated the possibility to add, besides the risk of partial breakdown of negotiations, a discounting of players' pay-off value each time a new round of bargaining takes place. Once again, before looking at this family of models, it is opportune to introduce the cooperative solutions they support. The first, introduced by Joosten (1996) and named egalitarian SV (ESV) by van den Brink *et al.* (2013), has already been encountered in the previous paragraph and it is a simple convex combination of the SV and the ES. It will therefore grant to each player the following pay-off:

$$\xi_i(N, v) = \theta\psi_i + (1 - \theta)\gamma_i, \quad \theta \in [0, 1]$$

The second, still proposed by Joosten (1996) and Driessen and Radzik (2002) and named δ-discounted SV (DSV) in van den Brink and Funaki (2010), awards a player with:

$$\mu_i(N, v) = \sum_{S \subseteq N} \frac{(n - |S|)!(|S| - 1)!}{n!} [\delta^{n-|S|} v(S) - \delta^{n-|S \setminus \{i\}|} v(S \setminus \{i\})], \ \delta \in [0, 1]$$

When $\delta = 0$, this solution collapses into ES, whereas when $\delta = 1$, it gives the SV. Both the solutions satisfy efficiency, symmetry and additivity, while their distinguishing features are weak monotonicity for the ESV and δ-reducing player property for the DSV (van den Brink and Funaki, 2010).

van den Brink *et al.* (2013) show that the ESV can be implemented in SSPE with a modification of the model of Pérez-Castrillo and Wettstein (2001). At first stage, their variant runs identical unless there is a rejection. At this point, the two models depart. In van den Brink *et al.* (2013), $\alpha \in [0, 1]$ is the probability that the proposer is eliminated and $(1 - \alpha)$ is instead the possibility of a total breakdown, meaning that the bargaining game ceases with all players getting zero. In case the game reaches the second stage, instead, the possibility of total breakdown is removed and a rejection will cause the proposer to be excluded with probability equal to one; therefore, the model reverts to Pérez-Castrillo and Wettstein (2001). Although no proper discount is present here, the possibility of a total collapse can be interpreted in this way according to Dagan and Serrano (1998). In van den Brink and Funaki (2010), instead, it can be found a support for the DSV by simply adding a common discount factor δ for all players, while the model remains equal in all other aspects to the one of Pérez-Castrillo and Wettstein (2001). Calvo and Gutiérrez-López (2016) show that the simple introduction of the discount factor into the model of Hart and Mas-Colell (1996) leads to the DSV in SSPE. In particular, if the introduced discounting factor is represented by ρ, the δ in the formula of the DSV given before will take the following value: $\delta = \frac{\rho(1-\alpha)}{1-\alpha\rho}$, where α and $(1 - \alpha)$ are defined as previously. The extension holds also for the NTU case. Finally, Kawamori (2016) shows that the same result holds even without the restriction that a proposal must be done to all active players, assumption present in the model of Hart and Mas-Colell (1996) and its variants. The proposer is now free to choose the coalition in addition to the proposal to make to its members. Note that the remaining players are free to negotiate in the next round if the coalition S called by the proposer forms, assuming $S \subset N$ and $|N \setminus S| > 1$. The result obtained by introducing a discount factor is the same as the one of Calvo and Gutiérrez-López (2016), but the greater flexibility of this model is paid by a restriction, stated in theorem 2 of Kawamori (2016), in terms of the values of the coalitions for the result to hold.

3.3 *Egalitarianism and the Standard Bargaining Protocol*

After having presented an archetypal model of coalitional bargaining, we have examined a series of protocols supporting different solution concepts, namely the nucleolus, the SV, the equal division and various combinations of these last two. Several of the models described were actually departing considerably from the given benchmark. In this last section, instead, we will revert to a family of models that is very close to our starting protocol and that resembles more faithfully the two-person bargaining model of Rubinstein. As done till here, before considering the non-cooperative side, we will briefly introduce the cooperative solution concepts related to this family of bargaining protocols.

We will start by providing a formal definition of what is probably the most influential set solution concept in cooperative game theory: the core (Gillies, 1959).

$$C(N, v) = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \sum_{i \in S} x_i \geq v(S), \forall S \in \mathscr{P} \setminus N \wedge \sum_{i \in N} = v(N) \right\}$$

Shapley and Shubik (1966) proposed a variant of the core, the strong ϵ-core, which is obtained by subtracting a constant ϵ to all coalition's characteristic values apart from the one of the grand coalition:

$$C_\epsilon(N, v) = \left\{ \boldsymbol{x} \in \mathbb{R}^n : \sum_{i \in S} x_i \geq v(S) - \epsilon, \forall S \in \mathscr{P} \setminus N \wedge \sum_{i \in N} = v(N) \right\}$$

Maschler *et al.* (1979) defined the intersection of all possible strong ϵ-cores as the least core: $C_l(N, v) = \cap_{\epsilon\in\mathbb{R}} C_\epsilon(N, v)$. Another interpretation describes the least core as the non-empty strong ϵ-core for which the value of ϵ is minimum (note that ϵ can be negative).

In the previous section, we have encountered a solution concept strongly pervaded by the idea of egalitarianism: the ES. We have seen that this solution disregards all coalitions with cardinality different from one and $|N|$. Dutta and Ray (1989) have tried to reconcile egalitarianism with the strategic dimension of a coalitional game. The core, which satisfies personal and group rationality, together with efficiency, seems to be a perfect candidate to meet the strategic requirements; egalitarianism can then come into play by selecting the Lorenz maximal set of points inside the core. This solution has been proposed in Hougaard *et al.* (2001) and, given the partial nature of the Lorenz ordering, it is a set-valued solution. Before giving its mathematical description, it is necessary to define the concept of Lorenz domination that is strictly related to the set of Lorenz maximal points since these lasts are that allocations that are not Lorenz dominated by any other. Given a vector of scalars $\boldsymbol{a} \in \mathbb{R}^n$, define an equally dimensional vector $\boldsymbol{a}^l$, whose elements a^l_j are given by the mapping $f_j(\boldsymbol{a}) = \min(\sum_{i=1}^{j} a_i), \forall j = 1, 2, ..., n$. Now, given two vectors of scalars, $\boldsymbol{a}$ and $\boldsymbol{b} \in \mathbb{R}^n$, we say that $\boldsymbol{a} >^L \boldsymbol{b}$, where the symbol $>^L$ stays for Lorenz dominates, if $\boldsymbol{a}^l > \boldsymbol{b}^l$, meaning that $a^l_j \geq b^l_j, \forall j = 1, 2, ..., n$, with strict inequality holding for at least one j. Therefore, the core-constrained Lorenz maximal imputations set of Hougaard *et al.* (2001) can be defined as

$$C_L(N, v) = \left\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in C(N, v) \wedge \nexists \boldsymbol{y} \in C(N, v) : \boldsymbol{y} >^L \boldsymbol{x}\right\}$$

Dutta and Ray (1989), however, were not satisfied with this formulation. Their idea was to have a solution concept that accounts for the desire of equity, as to say egalitarianism, and for selfish stimuli, where the first element is meant to be a normative principle. Using the core conditions to satisfy the selfish side and then applying egalitarianism is inappropriate, according to them, since the same egalitarian principle should be at the base of the division of the value of every coalition. Consequently, they developed the idea of the Lorenz cores (C^L).[14] They are defined recursively, starting from the bottom of the cooperative pyramid, meaning that first are considered coalitions of size equal to one, then equal to two, proceeding till $|N|$. At each step, it is computed the set of Lorenz un-dominated allocations for each coalition, call it $EL(S)$, which is not necessarily a singleton. Suppose now that the Lorenz core has been defined for all coalitions with cardinality lower than $|S|$. The Lorenz core of S is then defined as

$$C^L(S) = \left\{\boldsymbol{x} \in \mathbb{R}^{|S|} : \sum_{i\in S} x_i = v(S) \wedge (\nexists T \subset S \wedge \nexists \boldsymbol{y} \in EL(T)) : \boldsymbol{y} > \boldsymbol{x}(T)\right\}$$

where $\boldsymbol{x}(T)$ is the restriction of vector $\mathbf{x}$ that includes only that allocations whose index is an element of T. Once defined the Lorenz core, the set of Lorenz maximal imputations is defined, muta mutandis, in an identical way as before:

$$C^L_L(N, v) = \left\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in C^L(N, v) \wedge \nexists \boldsymbol{y} \in C^L(N, v) : \boldsymbol{y} >^L \boldsymbol{x}\right\}$$

In theorem 1, Dutta and Ray (1989) state the uniqueness of their egalitarian allocation. Therefore, $C^L_L(N, v)$ is a singleton and it can be seen as a single-valued solution.[15] It must be noted that the Lorenz core is a superset of the core, but this does not imply that the non-emptiness of the core guarantees the existence of the egalitarian solution. However, for convex games, the authors have proved its existence and they have further shown that its location is inside the core. For this class of games, it must be noted that the Lorenz maximal point over the Lorenz core and over the 'standard' core coincide (Hougaard *et al.*, 2001).

3.3.1 *Fixed-Order Proposers Protocol*

As already mentioned, a bargaining game with fixed order of proposers is one in which the initial random move does not only select the current proposer among the n players, but lists them in a specific order, according to which the proposer is determined,[16] which will be maintained along the whole game. Generally, these models consider a specific order as given and evaluate the outcomes of the game for different possible orders. A first model in this strand has been proposed in Selten (1988): a zero-normalized, rejector-proposes model without discounting where only one coalition can form. Solved in SSPE, the model had to be coupled with axiomatic properties to refine the large number of equilibria due to the absence of discounting (Bandyopadhyay and Chatterjee, 2006). Due to this partial non-cooperative nature, we will skip a further analysis of this model in favour of the one proposed in Chatterjee *et al.* (1993).

The protocol of Chatterjee *et al.* (1993) is very similar to the one of Selten (1988) apart from two major changes: it has discounting and it allows the formation of more than one coalition. This last aspect implies that, if a coalition S forms after a proposal has been unanimously (by the members of S) accepted and $|S| < |N|$, the remaining players, the members of $N \backslash S$, continue the bargaining process over the set of coalitions $\mathscr{P}(N \setminus S)$, with the order of proposers being unchanged. The addition of discounting leaded Chatterjee *et al.* (1993) to drop the zero normalization assumption since strategic equivalence is not preserved in this case (Bandyopadhyay and Chatterjee, 2006). As said, the remaining assumptions underpinning the bargaining protocol[17] are identical to Selten (1988), namely it is applied the rejector-proposes rule. The solution concept adopted, SSPE, is the same as well.

Before looking at the results obtained by Chatterjee *et al.* (1993), it is instructive to consider the building blocks used to reach them, starting with the condition (lemma 1) for a stationary equilibrium response for each player i at player set $S \in \mathscr{P}$. Remembering that, after the formation of a coalition, say P, the bargaining game continues among the remaining players $N \backslash P$, the player set S simply identifies the set of players still active in the game after a given history: $S = N \backslash P$, with P being the union of all coalitions already exited from the game at time t. Therefore, in a stationary equilibrium response vector $\boldsymbol{x}(S, \delta) \in \mathbb{R}^{|S|}$, each element must respect the following inequality:

$$\boldsymbol{x}_i(S, \delta) \geq \delta \max_{i \in T \subseteq S} [v(T) - \sum_{j \in T \setminus \{i\}} \boldsymbol{x}_j(S, \delta)], \quad \forall i \in S$$

For i being a proposer, and willing to make an accepted proposal, the expression must be an equality. By solving the simultaneous set of equations for each $i \in S$, it is obtained a no-delay stationary equilibrium vector, meaning a vector for which, after every history and for any proposer, this last will make an acceptable proposal. In proposition 1, Chatterjee *et al.* (1993) prove that for every $S \in \mathscr{P}$ and $\delta \in [0, 1]$, such vector exists and it is unique. It must be remembered, however, that a no-delay equilibrium vector does not imply by any mean that full cooperation is obtained without delay. It is a much weaker condition, stating that at every proposal, it will correspond the formation of a coalition, but not necessarily the coalition with all players. Furthermore, it is dependent from the selected order of proposers.

Proposition 3 states the very demanding condition for which the grand coalition is formed with no delay for any possible order of players and for $\delta \to 1$: the game (N, v) must be dominated by its grand coalition. Domination implies that $\frac{v(S)}{|S|} \leq \frac{v(N)}{n}, \forall S \in \mathscr{P} \setminus N$. In this case, the equilibrium pay-off vector will coincide with the previously seen ES. Note also that, for such game, this vector is the only element of both $C_L(N, v)$ and $C_L^L(N, v)$. The stringency of this condition is made very clear in proposition 6 that, conversely, states the requirements for having inefficiency for every order of proposers. Consider a strictly super-additive[18] game and define $\boldsymbol{x}(N, \delta)$ as the equilibrium response vector resulting from solving the simultaneous system of equations described before for $S = N$. According to proposition 6, then, there always exists a lower bound of the discount factor, call it $\underline{\delta}$, such that for $\delta \in (\underline{\delta}, 1)$, and for every proposers order, inefficiency will arise if $\sum_{i \in N} x_i(N, \delta) > v(N)$. This last condition can materialize for super-additive games with non-empty core. Its absence, instead, implies that there will be some order of proposers leading to an efficient (without delay) bargaining outcome for δ tending to one. In proposition 5, Chatterjee *et al.* (1993) show that, in such case, the resulting equilibrium allocation is in the core. Strictly convex games follow into this last case where an efficient, full cooperative outcome is dependent upon the selection of the order of proposers. In particular, given a strictly convex game and for a discount factor $\delta \in (\underline{\delta}, 1)$ with $\underline{\delta}$ being again a lower bound, there is an order of proposers that assures a no-delay efficient equilibrium. Moreover, for such an order, the resulting allocation will converge to the Lorenz maximal allocation inside the core.[19]

Resuming briefly what seen in the previous paragraph, it is possible to state that non-emptiness of the core is a necessary but not sufficient condition for having full cooperation. Besides it, $\sum_{i \in N} x_i(N, \delta) < v(N)$ is actually a necessary condition for reaching an efficient outcome, in the limit of δ tending to one, for at least some proposers' order. Holding $\delta \to 1$, if efficiency arises, the equilibrium allocation is in the core and, for strictly convex games, this coincides with the Lorenz maximal allocation.

3.3.2 *Random Proposer Protocol*

Besides the structure of the underlying coalitional game and the value of the common discounting factor, the model of Chatterjee *et al.* (1993) is strongly influenced by the order of proposers. Both delay and inefficiency can or cannot arise for the same game (N, v) and δ depending on the selected order. Okada (1996) tries to obviate to this problem adopting the random proposer mechanism. Therefore, either at the beginning of the bargaining process and after a rejection, a random move selects a proposer among the remaining players in the game with equal probability. The other assumptions are identical as in Chatterjee *et al.* (1993), apart that Okada (1996) considers the underlying coalitional game (N, v) as being essential, super-additive and zero-normalized. The equilibrium concept, SSPE in pure strategies, is also common to both papers.

By slightly changing the notation in Okada (1996) for a better comparison with Chatterjee *et al.* (1993), it is possible to see that the maximization problem defining an equilibrium proposal is actually very similar:

$$x_i(S, \delta) \geq \delta \max_{i \in T \subseteq S} \left[v(T) - \sum_{j \in T \setminus \{i\}} x_j \right], \quad \forall i \in S$$
$$\text{s.t.}$$
$$x_j \geq v_j^S, \quad \forall j \in T \subseteq S$$

where S is defined again as the set of players still active in the game. The difference therefore is represented by the condition $x_j \geq v_j^S$, where v_j^S is defined as the expectation pay-off for player j of the continuation of the game with players set S. In equilibrium, both inequalities $x_i(S, \delta) \geq \delta \max(\cdot)$ and $x_j \geq v_j^S$ must hold with equality. This implies that $\sum_{j \in T \setminus \{i\}} x_j$ can be substituted with $\sum_{j \in T \setminus \{i\}} v_j^S$. It remains to see what this expectation is by using the expression provided in Okada (1996):

$$v_i^S = \frac{1}{|S|} \left\{ \left[v(T_i) - \delta \sum_{j \in T \setminus \{i\}} v_j^S \right] + \delta \sum_{i \in Q_k | k} v_i^S + \delta \sum_{i \notin P_m | m} v_i^{S \setminus P_m} \right\}$$

The expected pay-off of the continuation of game (S, v) is composed by three additive elements multiplied by $\frac{1}{|S|}$, which reflects the equal probability of the random move. The first element, inside the square brackets, is the pay-off that player i will obtain if called to be the proposer and assuming that she will make an equilibrium proposal to coalition $T \subseteq S$. The second term, instead, displays the pay-off i gets by receiving an acceptable proposal in the next round, given that the next proposer, k, will be one that will select a coalition to which i belongs. Finally, the last term describes the continuation pay-off of i beyond the next round, provided that the selected player m will be one proposing to a coalition that excludes i.

With regard to results, the main difference between the random proposer model compared to the fixed order one stays in the timing of agreement. We have seen that proposition 1 of Chatterjee *et al.* (1993) states that a no-delay equilibrium path exists and it is unique. But it should be stressed that the proposition implies that there is at least one order of proposers, possibly more, allowing for this path.[20] The random proposer model, instead, always features a no-delay equilibrium path (theorem 1 in Okada, 1996). This is mainly due to the fact that randomness eliminates the strategic opportunity that some players might enjoy given a certain order of proposers. Shifting the focus from timing to efficiency, instead, theorem 3 proofs that, in the limit of $\delta \to 1$, the equilibrium allocation for player set S entails the formation of coalition S itself only if $x_i(S, \delta) = v_i^S = \frac{v(S)}{|S|}, \forall i \in S$. This happens only if the vector $\boldsymbol{x} \in \mathbb{R}^{|S|} = [\frac{v(S)}{|S|}, ..., \frac{v(S)}{|S|}]$ is inside the core. This is obtained by considering that, in an SSPE equilibrium where coalition S forms, the last term in the expression for v_i^S, $\sum_{i \notin P_m|m} v_i^{S \setminus P_m}$, must necessarily be equal to zero for all $i \in S$ and by solving the resulting system of equations. At player set N, therefore, the ES, provided that this is in the core of (N, v), is the only SSPE equilibrium according to Okada's random proposer protocol.

3.3.3 *Extensions of the Random Proposer Protocol*

The random proposer protocol just examined has found two important extensions. One, due to Okada (2011), considers the possibility that players differ both in recognition probability and in time discounting, where the first is simply defined as the probability of being called as proposer by the random move. In Okada (1996), we have seen this was equal to $\frac{1}{|S|}$, with S being the set of active players in the game. This extension therefore takes into account asymmetries among players in these two crucial parameters, all other elements of the bargaining protocol being equal as in Okada (1996).

No delay in agreement persists even when considering the mentioned sources of asymmetries. The crucial change brought by their introduction is the allocation that takes place when a coalition is realized. In fact, for player set S, defined as before, and assuming that the SSPE bargaining outcome entails the formation of coalition S, the expected pay-off of a player i depends on a proportion between recognition probabilities and discount factors:

$$v_i^S = \frac{\frac{\alpha_i}{1-\delta_i}}{\sum_{j \in S} \frac{\alpha_j}{1-\delta_j}}$$

where α_i is the probability of player i of becoming a proposer. Okada (2011) further shows that the condition for coalition S to be implemented is that $\sum_{i \in T} v_i^S + \sum_{j \in S \setminus T} v_j^{S \setminus T}(1 - \delta_j) \geq v(S), \forall i \in S$. By substituting S with N, we then have a condition for, and the unique SSPE allocation in, the grand coalition of the underlying coalitional game (N, v). It is instructive to note that, for the grand coalition to take place, the SSPE allocation must lay on a non-empty ϵ-core, where ϵ is determined by the values of the discounting factors. In particular, when these are common and equal to $\delta \to 1 \Rightarrow \epsilon \to 0$, we revert to the 'standard core'. The last thing that worth to be mentioned is that, by dropping the asymmetry in discounting and considering the limit of δ tending to one while keeping the asymmetry in the recognition probabilities, the SSPE allocation in the grand coalition becomes totally dependent on the same recognition probabilities: $x_i(N, v) = \alpha_i v(N), \forall i \in N$.

The second extension of the random proposer protocol that will be considered is the model of Compte and Jehiel (2010). Their bargaining game reverts to the case of equal recognition probabilities and common discount factor, where the focus is placed on the limit of $\delta \to 1$. The peculiarities, instead, are twofold: first, only one coalition can form and players remaining outside the winning coalition get zero; second, the considered equilibrium, SSPE, is in mixed, rather than limited to pure strategies. Both these variants go in the direction of enlarging the feasibility of the grand coalition as a bargaining outcome. The first, in fact, increases the propensity of players to cooperate since their exclusion from a coalition that forms necessarily implies a zero pay-off. Recalling the equation defining the expected continuation value in Okada (1996), this translates into nullifying the third term in the curly bracket. Mixed strategies, on the other side, increase flexibility. As seen, in Okada (1996), full cooperation materializes, asymptotically, only if the ES allocation is inside the core. In Compte and Jehiel (2010), although the SSPE allocation is still unique, this becomes dependent on the value of the characteristic functions.

As anticipated, Compte and Jehiel (2010) is mainly devoted to investigate efficiency properties for δ tending to one. They then consider games with non-empty core. No delay holds under this new setting. Their main findings relate to the efficient SSPE allocation and to the conditions for obtaining it, which goes behind the non-emptiness of the core. In particular, when the grand coalition forms, the unique SSPE allocation is the vector, inside the core, which maximizes the Nash product, where the maximization is in terms of pay-off values rather than utilities:

$$\boldsymbol{x}(N, v) = \arg\max_{x_i \in C(N,v)} \prod_{i \in N} x_i$$

Clearly, this vector belongs to the set of Lorenz maximal imputations inside the core: $\boldsymbol{x}(N, v) \in C_L(N, v)$. Further, for convex games, it will coincide with the unique element of $C_L^L(N, v)$ and, if the ES solution is inside the core, they will also be coincident. The condition for this vector to be the no delay SSPE outcome of (N, v) is expressed in proposition 1, which reminds to Property P1, in Compte and Jehiel (2010). For defining Property P1, consider a balanced game (N, v) and define a new game (N, v, Δ) obtained by subtracting a positive constant Δ to each characteristic value of the original game: $(N, v, \Delta) = v(S) - \Delta, \forall S \in \mathscr{P}, v \in (N, v)$. Define then scalar $\mu^C(N, v)$ as the maximum value of the product of the coordinates of a point inside the core of a game: $\mu^C(N, v) = \max_{x \in C(N,v)} \prod_{i \in N} x_i$. A game for which Property P1 holds is a game for which there exists a scalar Δ_0 and an open interval $(0, \Delta_0)$ such that, for $\Delta \in (0, \Delta_0)$, $\mu^C(N, v, \Delta) < \mu^C(N, v)$ and $\mu^C(N, v, \Delta)$ is a decreasing function of Δ itself. Proposition 1 of Compte and Jehiel (2010) states that Property P1 is a necessary requisite of a balanced game (N, v) for having an asymptotically efficient SSPE outcome. The authors mention some conditions that a game has to possess for P1 to hold. Among them, it is worth to note that strict convexity is a sufficient one. By considering that the value of μ^C is given by both $v(N)$ and the degree of equality attainable in the bargaining game, we see that subtracting a positive constant to $v(N)$ has clearly a negative effect on μ^C, but if this is offset by the increase in equality granted by the enlargement of the core through the relaxation of its constraints, then full cooperation cannot take place.

3.3.4 *The Core and the Standard Bargaining Protocol*

Although we have seen several protocols leading, under certain conditions, to multiple equilibria, the prevalence, till now, has been for models sustaining single-valued solutions. Several authors, instead, focused on bargaining models explicitly supporting the core. It is instructive to consider these works since they further clarify which are the elements of the bargaining protocols that lead to certain outcomes. Four papers appear to be particularly interesting for our purposes: Moldovanu and Winter (1995), Evans (1997), Yan (2003) and Kim and Jeon (2009).[21] The reason of their interest is twofold: besides the fact that, for balanced coalitional games, their set of equilibria coincides with the core, they are also all based on bargaining protocols closely related with the presented archetypal model.

The first two listed papers share the absence of discounting. In particular, Moldovanu and Winter (1995) consider the same model of Selten (1988) and show that, if it is true that the set of equilibria is dependent on the order of proposers selected by the random move, the intersection of these sets for all the possible orders is exactly the core, provided that the underlying coalitional game is balanced. Therefore, the core is sustained as an ex ante expectation set of equilibria. The model adopted by Evans (1997) is, instead, very similar to Okada (1996), but coupled with the bidding selection mechanism á la Pérez-Castrillo and Wettstein (2001). There is, however, a significant difference between the assumptions shaping the two bidding stages, namely in Pérez-Castrillo and Wettstein (2001), the bid was a vector of payments directed at other players, whereas here, it constitutes a waste of resources. An efficient outcome therefore must be one in which all bids are equal to zero. In case of equal bids, there is still a random draw that selects the proposer with equal chances.

Under these settings and in the absence of discounting, Evans (1997, theorem 2.1) proved that the set of SSPE allocations in pure strategies for a balanced game coincides with its core. If mixed strategies are considered, however, inefficient allocations can arise as SSPE equilibria. The author further examines the behaviour of the model when discounting is introduced. No efficient outcome is obtained since players will make positive bids to gain the right of proposal. However, by introducing a minor artifice, namely that offers and bids must be made in discrete units, Evans (1997) is able to restore asymptotically the support of the core. A discrete offer is such if, given a scalar η, the elements of the proposed allocation vector are obtained by multiplying η for any arbitrary non-negative integer: $x_j \in \boldsymbol{x}(S) = k\eta, \forall j \in S, k \in \mathbb{N}, \eta \in \mathbb{R}_+$. In such case, for $\eta \to 0$ and $\delta \to 1$ and considering pure strategies, the set of SSPE allocations converges in Housdorff distance to the core (Evans, 1997, theorem 2.2).

Yan (2003) basically laid the basis for the results obtained in Okada (2011). Their bargaining protocols are almost identical, random-proposer with a variable vector $\boldsymbol{\alpha}$ of recognition probabilities, with $\boldsymbol{\alpha} \in \mathbb{R}^n$. Yan (2003), however, considers the case of a common discount factor, an essential, normalized and balanced underlying coalitional game (N, v) and adopts the same assumption of Compte and Jehiel (2010) of a single coalition being allowed to form. Remembering that normalization implies $v(N) = 1$, it shows that the ex ante unique SSPE allocation, say $\boldsymbol{x}(N, v) \in \mathbb{R}^n$, of game (N, v), given δ and α, is given by $\boldsymbol{x}(N, v) = \boldsymbol{\alpha}$. For $\delta \to 1$, also the ex post unique SSPE allocation will converge to this result. Recalling the result of Okada (2011) where the SSPE asymptotic allocation of a balanced game was $\boldsymbol{y}(N, v) = \boldsymbol{\alpha} v(N)$, the difference is just due to the assumption of normalization in Yan (2003). Being the results identical, this implies that the assumption of terminating the game after a coalition is formed is not crucial for its achievement, given that Okada (2011) does not make this assumption. The last important result worth to be mentioned is that the equality between the recognition probabilities and equilibrium pay-offs vectors falls apart if the same recognition vector is not, once translated into a pay-offs vector, an element of the core. In this case, inefficiency will arise.

The last paper to be considered is Kim and Jeon (2009). It is based on the rejector-proposes bargaining protocol of Chatterjee *et al.* (1993), with a fixed order of proposers determined by the initial random move. Despite both papers adopt an identical bargaining protocol, there is a crucial distinction in results due to the fact that in Kim and Jeon (2009) mixed strategies are allowed. In the limit of $\delta \to 1$, the possibility to adopt mixed strategies nullifies the prominent role that the selected order of proposers had in the paper of Chatterjee *et al.* (1993). No delay is always assured. Moreover, the authors show that the set of SSPE allocations, for $\delta = 1$, is equal to the set of solutions of the following minimization problem:

$$\begin{aligned} &\min_x \sum_{i \in N} x_i \\ &\text{s.t.} \\ &\sum_{i \in S} x_i \geq v(S), \quad \forall S \in \mathscr{P} \end{aligned}$$

For a balanced game (N, v), it is easily verified that minimum of $\sum_{i \in N} x_i$ is equal to $v(N)$. It then follows, from the constraints of the minimization problem, that the set of solutions coincides with $C(N, v)$ (Kim and Jeon, 2009, theorem 4). It must be further noted that, for a game with an empty core, the same minimization problem provides the set of cut-off values below which responders will refuse an offer. By using the words of Yan (2003), we can see better what these cut-off values are:

> In an SSPE players behave as if they used simple strategies, in which a player accepts a proposal if and only if she herself is offered at least a certain cut-off value, and a proposer always includes herself in the nominated coalition and offers the other coalition members their cut-off values.

As in the case of a balanced game, there can be multiple optimal vectors satisfying the given minimization problem. Define E as the set collecting them. Assume that a vector $\boldsymbol{x}^* \in E$ represents the cut-off values defining the SSPE strategies of the n players of game (N, v). If the game is balanced, the final vector of pay-offs, $\boldsymbol{\pi} \in \mathbb{R}^n$, for $\delta \to 1$, will converge to $\boldsymbol{x}^*$. However, for a game with an empty core, this is not the case. Since the equilibrium is reached without delay, this implies that the first players in the selected order, say i, will randomize the choice among the coalitions that guarantee her $\pi_i = v(S) - \sum_{j \in S \setminus \{i\}} \geq x_i^*$, holding $\delta \to 1$. Players in $N \setminus S$ are not guaranteed to obtain a pay-off equal to their cut-off value. According to Kim and Jeon (2009), once a coalition has formed, the bargaining game can continue according to the same rules and a new minimization problem, with players $k \in N \setminus S$ and coalitions $T \in \mathscr{P}(N \setminus S)$, applies. However, it is not very clear if allowing mixed strategies is enough to guarantee that no strategic delay is adopted by players.[22] It could be that only by adopting the assumption á la Compte and Jehiel (2010) of a single coalition could avoid strategic delay. Further investigation on this topic seems necessary since the authors appear to have glossed over the issue.

4 Conclusions

This paper has offered a panoramic view of the vast research production during the last three decades dedicated to bargaining over coalitions. It has started from the basis by describing the main elements of a characteristic function form cooperative model, intended as the normal form of a coalitional model, and then it has proceeded by analysing its non-cooperative counterpart, or else, the extensive form of a coalitional model. The Rubinstein bargaining game, due to its popularity and its ability to describe a general bargaining framework, is by far the most adopted base model even in a coalitional setting, although significant variants are not lacking. It has therefore been presented an archetypal way to adapt this model to the case where multiple coalitions, with distinct values to be distributed among their members in form of pay-offs, are allowed. The paper has then moved to consider the specific variants emerged in the literature. For clarity and convenience of exposition, these variants have been grouped according to the cooperative solution concepts they support.

Willing to underline some of the common features displayed by a widespread sample of the examined bargaining protocols, it is possible to start from the adopted equilibrium concept. SSPE is by far the most common given the recursive nature and the infinite temporal dimensionality of the bargaining problem they serve to depict. However, when particular characteristics render the model a finite horizon one, SPE is an option. The selection of the proposer, a crucial element in sequential bargaining, is evenly divided between the rejector-proposes assumption, generally coupled with the fixed order of proposers drawn at the very beginning of the game, and the random proposer device, according to which the random mechanism operates at the start of each new time period. The bidding stage approach is for sure a significant variant that has gained popularity in this field. Other two major divisions seen in this strand of the literature concern the nature of the allowed strategies, pure versus mixed, and the possibility of continuing the bargaining process after a coalition has formed, with some models allowing for it and others not.

Finally, we can state some concluding remarks regarding outcomes and, particularly, regarding the relation between protocols and cooperative solution concepts. Models sustaining the nucleolus are quite peculiar, both in the coalitional side, where the characteristic function is modelled in order to represent specific instances (such as a bankruptcy problem), and in the bargaining side, with a protocol that departs significantly from the standard Rubinstein typology. Furthermore, the implementation of the nucleolus rests upon a specific order of proposers taking place.

The SV is sustained by three families of models: bilateral random meetings, SDCG and partial breakdown. The first sustains the popular cooperative solution concept trough the device of random meetings, which has been accused to impose exogenously the relevance of each possible coalition and, consequently, of each player's marginal contribution. The second, instead, suffers from the drawback to lead to the SV only in expected terms and from the fact that the uniqueness of this result holds solely for a restricted class of games, namely strictly convex games. The third model type appears to be the more parsimonious in terms of assumptions. It is basically a Rubinstein bargaining model with random proposers where discounting is substituted by the risk of partial breakdown. Although it also leads to the SV in expected terms, through the substitution of the random selection of the proposer with the bidding stage, this drawback can be easily avoided. However, the sustained equilibrium varies according to the assumption regarding which player suffers the risk of exclusion after the rejection of a proposal. When it is only the proposer, then we have the SV, but the more this possibility becomes commonly and evenly shared among responders, the more the SSPE allocation shifts towards the equal division of the worth of the grand coalition.

Egalitarianism with regard to the distribution of final pay-offs appears to be a robust outcome of coalitional bargaining models. As seen in the last part of section three, the more a protocol resembles the original Rubinstein game, the more the previous statement holds true. This seems to come at the expense of efficiency, given that, even for super-additive and balanced games, the conditions to reach an SSPE imputation might be quite stringent, particularly when only pure strategies are allowed. The last fundamental aspect that must be underlined is therefore the role of recognition probabilities since they play a central role in determining efficient outcomes and in shaping pay-offs distribution.

A common feature of literature reviews is to suggest future directions for the research agenda. Given the apparent tension that has emerged between egalitarianism and efficiency, an interesting and promising theme of investigation seems to be the possibility to render endogenous the recognition probabilities in repeated bargaining games.

Notes

1. There are, however, remarkable exceptions. See, for example, González-Díaz and Sánchez-Rodríguez (2007) and Tijs *et al.* (2011).
2. Besides Bandyopadhyay and Chatterjee (2006), the reader interested in legislative bargaining should also consider the recent review of Binmore and Eguia (2017).
3. It could be argued that a collational game is properly defined by a 3-tuple, (N, c, v), with N and v defined as previously and where c is a map form N to a set of non-empty subsets of N itself (coalitions). Generally, however, the simplification (N, v) seems to prevail.
4. Along the paper, $\mathscr{P}$ will be used as a shorthand for $\mathscr{P}(N)$.
5. This approach is actually followed by cooperative games as well. See, for example, Owen (1975) and Chander and Tulkens (2006).
6. This assumption is by no means the prevalent one in the literature. Several models – for example, Chatterjee *et al.* (1993), Okada (1996) and Okada (2011) – allows for the continuation of the bargaining game with players set $N \setminus S$ after coalition S has formed. We have adopted it here merely for simplification purposes.
7. Rubinstein (1982) preferences' assumptions are fulfilled by two models: fixed bargaining costs and fixed time discounting. We have presented a bargaining model of the second kind that is by far the most common approach in the dedicated literature.

8. Therefore, the value for player i of holding in a single period t a generic bundle composed by the resources owned by a set S of players, with $S \ni i$, is equal to $(1-\delta)v(S)$. This holds since Gul (1989) assumes that $v(S)$ expresses a discounted value of utility.
9. While Winter (1994) result requires three equilibrium refinement concepts, namely SPE, sub-game consistency (Harsanyi *et al.*, 1988) and strategic equilibrium (Leininger, 1986), Dasgupta and Chiu (1998) only use the first.
10. Note that both non-selected players preceding i and the ones after i in Q will get their respective stand-alone pay-off.
11. Since this paper deals with TU games, it will follow the presentation of the model given by Krishna and Serrano (1995) rather than the one in Hart and Mas-Colell (1996). Although very similar, there are some minor differences. For example, in the latter, monotonicity is assumed, but the game is not 0-normalized.
12. van den Brink (2007) calls it the equal division solution.
13. van den Brink (2007) has introduced a variant, called equal surplus sharing, obtained by dropping the nullifying player property in favour of the axiom of invariance: $\gamma_i(N, v) = \frac{v(N)}{n} v(\{i\}) + \frac{v(N)-\sum_{j\in N} v(\{j\})}{n}$. In zero-normalized games they are obviously identical.
14. We will now provide a very short and summary description of this solution concept. The interested reader can refer to Dutta and Ray (1989) for a formal presentation and to Ray (2007) for a description of an algorithm to compute it.
15. This does not apply to the Lorenz maximal set defined over the strong Lorenz core, a further refinement presented in Dutta and Ray (1991) that substitutes the concept of Lorenz domination with the one of strong Lorenz domination.
16. It must be noted that in the models of Selten (1988) and Chatterjee *et al.* (1993), the selected order also governs the sequence of replies after a proposal. However, since a change in the order of replies is inconsequential for the model's outcome, we can restrict the attention to the sole order of proposers.
17. It must be noted that the term protocol used in this paper differs significantly in meaning from Chatterjee *et al.* (1993). Here, it defines the whole set of bargaining rules and it is quite close to a synonym of model, whereas in Chatterjee *et al.* (1993), it takes a more restrictive meaning being a synonym of the order of proposers.
18. Chatterjee *et al.* (1993) define efficiency as the condition according to which, for a game (N, v), there is not an equilibrium allocation such that every player can be made strictly better off. Strict super-additivity implies that efficiency can be obtained only when the grand coalition forms.
19. Recall that for convex games, $C_L(N, v) = C_L^L(N, v)$ and they are singletons.
20. In Okada (1996), it can be found an example of a four-player game with a given order that necessarily entails delay.
21. It must also be mentioned the work of Perry and Reny (1994). This model, however, is quite peculiar since it considers continuous rather than discrete time. Given this strong departure from the models mentioned in this paper, we prefer to omit its description. Another interesting paper is due to Serrano and Vohra (1997) that finds an implementation mechanism for the core in a market economy. Also, this model, however, departs considerably from the scope of the present review.
22. For an example of strategic delay with a fixed order of proposers, see the first example in Okada (1996).

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