Algorithmic game theory – Tutorial 3*

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1 ε-Nash and correlated equilibria

Let $G = (P, A, u)$ be a normal-form game of n players and let $\varepsilon > 0$. A strategy profile $s = (s_1, \ldots, s_n)$ is an *ε-Nash equilibrium* if, for every player $i \in P$ and for every strategy $s'_i \in S_i$, we have $u_i(s_i; s_{-i}) \geq u_i(s'_i; s_{-i}) - \varepsilon$.

Let p be a probability distribution on A, that is, $p(a) \geq 0$ for every $a \in A$ and $\sum_{a \in A} p(a) = 1$. The distribution p is a *correlated equilibrium* in G if

$$\sum_{a_{-i} \in A_{-i}} u_i(a_i; a_{-i}) p(a_i; a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} u_i(a'_i; a_{-i}) p(a_i; a_{-i})$$

for every player $i \in P$ and all pure strategies $a_i, a'_i \in A_i$.

Exercise 1. *Show that, in every normal-form game $G = (P, A, u)$, every convex combination of correlated equilibria is a correlated equilibrium.*

Proof. Assume p and p' are two correlated equilibria of G. We show that $p'' = tp + (1-t)p'$ is a correlated equilibrium as well. First, observe that p'' is a probability distribution. The rest follows immediately, as the fact that p and p' are correlated equilibria gives

$$\sum_{a_{-i} \in A_{-i}} u_i(a_i; a_{-i}) p''(a_i; a_{-i}) = \sum_{a_{-i} \in A_{-i}} u_i(a_i; a_{-i})(t \cdot p(a_i; a_{-i}) + (1-t) \cdot p'(a_i; a_{-i})) \geq$$
$$\sum_{a_{-i} \in A_{-i}} u_i(a'_i; a_{-i})(t \cdot p(a_i; a_{-i}) + (1-t) \cdot p'(a_i; a_{-i})) = \sum_{a_{-i} \in A_{-i}} u_i(a'_i; a_{-i}) p''(a_i; a_{-i})$$

for every player $i \in P$ and all pure strategies $a_i, a'_i \in A_i$. □

Exercise 2. *Let $G = (P = \{1, 2\}, A, u)$ be a normal-form game of two players with $A_1 = \{U, D\}$ and $A_2 = \{L, R\}$ with payoff function u depicted in Table 1.*

	L	R
U	(1,1)	(0,0)
D	$(1 + \frac{\varepsilon}{2}, 1)$	(500,500)

Table 1: A game from Exercise 2.

Show that there is an ε-Nash equilibrium s of G such that $u_i(s') > 10u_i(s)$ for every $i \in P$ and every Nash equilibrium s' of G. In other words, there might be games where some ε-Nash equilibria are far away from any Nash equilibrium.

Solution. First, we determine all Nash equilibria of G. For $i \in \{1, 2\}$, let s_i by mixed strategy for player i that chooses each action a from A_i with probability p_a and let $s = (s_1, s_2)$ be the corresponding mixed-strategy profile. The expected payoff of player 1 is

$$\begin{aligned} u_1(s) &= 1p_U p_L + 0p_U p_R + (1 + \varepsilon/2)p_D p_L + 500p_D p_R \\ &= p_U p_L + (1 + \varepsilon/2)(1 - p_U)p_L + 500(1 - p_U)(1 - p_L) \\ &= (500 - \varepsilon/2)p_U p_L - (499 - \varepsilon/2)p_L - 500p_U + 500 \\ &= p_U((500 - \varepsilon/2)p_L - 500) - (499 - \varepsilon/2)p_L + 500 \end{aligned}$$

*Information about the course can be found at http://kam.mff.cuni.cz/˜ryzak/

and, similarly, for player 2,

$$\begin{aligned} u_2(s) &= 1p_Up_L + 0p_Up_R + 1p_Dp_L + 500p_Dp_R \\ &= p_Up_L + (1-p_U)p_L + 500(1-p_U)(1-p_L) \\ &= 500p_Up_L - 499p_L - 500p_U + 500 \\ &= p_L(500p_U - 499) - 500p_U + 500 \end{aligned}$$

For fixed p_L, the function $u_1(s)$ is decreasing in p_U and setting $p_U = 0$ is a best response for player 1, achieving expected payoff $500 - (499 - \varepsilon/2)p_L - \frac{\varepsilon}{2}$. Setting $p_L = 0$ is a best response for player 2 if $500p_U \leq 499$. Altogether, we get that there is a unique Nash equilibrium $s' = (D, R)$ (with probability vector $(p_U, p_L) = (0, 0)$). Note that $u_1(s') = 500 = u_2(s')$. Of course, this is also ε-Nash equilibrium.

Now, we show that the strategy profile $s = (U, L)$ is another ε-Nash equilibrium. We are then finished, as $u_1(s) = 1u_2(s)$. We need to show that for every player $i \in P$ and for every strategy $s'_i \in S_i$, we have $u_i(s_i; s_{-i}) \geq u_i(s'_i; s_{-i}) - \varepsilon$. That is,

$$u_1(U; L) \geq u_1(s'_1; L) - \varepsilon \quad \text{and} \quad u_2(L; U) \geq u_2(s'_2; U) - \varepsilon.$$

Let s' be a strategy where player 1 selects U with probability p and D otherwise and player 2 selects L with probability q and R otherwise. Then

$$u_1(s'_1; L) = p \cdot 1 + (1-p)(1 + \varepsilon/2) = 1 + \frac{\varepsilon}{2} - \frac{\varepsilon}{2}p$$

and

$$u_2(s'_2; U) = q \cdot 1 + (1-q) \cdot 0 = q.$$

Thus the inequalities are satisfied, as

$$1 = u_1(U; L) \geq u_1(s'_1; L) - \varepsilon = 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}p \quad \text{and} \quad 1 = u_2(L; U) \geq u_2(s'_2; U) - \varepsilon = q - \varepsilon,$$

and s is an ε-Nash equilibrium. Note that the first inequality is not satisfied for $p < 1$ without subtracting ε, thus (U, L) is not a Nash equilibrium. □

Exercise 3. *Let $G = (P = \{1, 2\}, A, u)$ be a normal-form game of two players with $A_1 = \{U, D\}$ and $A_2 = \{L, R\}$ with payoff function u depicted in Table 2.*

	L	R
U	(6,6)	(2,7)
D	(7,2)	(0,0)

Table 2: A game from Exercise 3.

(a) *Compute all Nash equilibria of G and draw the convex hull of Nash equilibrium payoffs.*

(b) *Is there any correlated equilibrium of G (for some ditribution p) that yields payoffs outside this convex hull?*

Solution. (*a*) For $i \in \{1, 2\}$, let s_i by mixed strategy for player i that chooses each action a from A_i with probability p_a and let $s = (s_1, s_2)$ be the corresponding mixed-strategy profile. The expected payoff of player 1 is

$$\begin{aligned} u_1(s) &= 6p_Up_L + 2p_Up_R + 7p_Dp_L + 0p_Dp_R \\ &= 6p_Up_L + 2p_U(1-p_L) + 7(1-p_U)p_L \\ &= 2p_U + 7p_L - 3p_Up_L = p_U(2 - 3p_L) + 7p_L \end{aligned}$$

and, similarly, for player 2,

$$u_2(s) = p_L(2 - 3p_U) + 7p_U.$$

For fixed $p_L < 2/3$, the function $u_1(s)$ is increasing in p_U and thus setting $p_U = 1$ is a best response for player 1 with expected payoff $2 + 4p_L$. If $p_L > 2/3$ is fixed, the function $u_1(s)$ is decreasing in p_U and setting $p_U = 0$ is a best response for player 1, achieving expected payoff $7p_L$. If $P_L = 2/3$, then an arbitrary p_U gives expected payoff $7p_L = 14/3$. Analogously, setting $p_L = 1$ is a best response for player 2 if $p_U < 2/3$, $p_L = 0$ if $p_U > 2/3$ and arbitrary p_l if $p_U = 2/3$. Altogether, we get the Nash equilibria with probability vectors $(1, 0)$, $(0, 1)$, and $(2/3, 2/3)$ with expected payoffs $(2, 7)$, $(7, 2)$, and $(14/3, 14/3)$, respectively. The convex hull of Nash equilibrium payoffs is depicted in Figure 1.

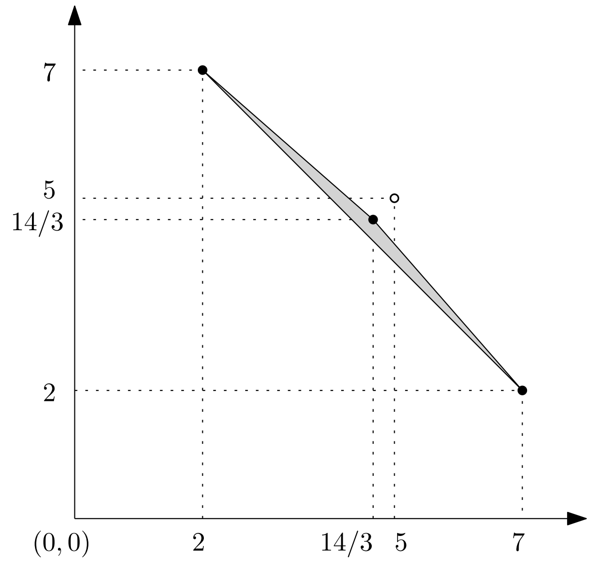

Figure 1: A convex hull of Nash equilibrium payoffs from Exercise 3, depicted by the grey polygon. The correlated equilibrium payoff achieving value $(5, 5)$ is denoted by empty circle. Note that it lies outside the convex hull of Nash equilibrium payoffs.

(b) We consider the probability distribution p that selects each of the following three strategy profiles (U, R), (U, L), and (D, L) independently at random with probability $1/3$. First, we show that p is a correlated equilibrium. If player 1 is told to play D it must be that the outcome arising from the above probability distribution in the correlated equilibrium is L, since (D, R) does not receive a positive probability. In this setting, player 1's expected payoff from selecting D is 7, while that from unilaterally deviating towards U is only 6. Hence, player 1 does not have incentives to deviate from D. Similarly, if player 1 is told to play U, then he does not know whether the realization of the above probability distribution is outcome (U, L) or (U, R). His expected payoff from agreeing to select U is

$$\frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} \cdot 6 + \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} \cdot 2 = 4.$$

Note that the first ratio identifies the probability of outcome (U, L), conditional on U occurring. Similarly, the second term identifies the conditional probability of outcome (U, R), given that U occurs. If, instead, player 1 deviates to D, his expected utility becomes

$$\frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} \cdot 7 + \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}} \cdot 0 = \frac{7}{2} < 4.$$

Therefore, player 1 does not have strict incentives to deviate in this setting either. By symmetry, player 2 does not have incentives to deviate from the correlated equilibrium and thus we have a correlated equilibrium.

Then the expected payoff of player 1 at this correlated equilibrium s is

$$u_1(s) = \frac{1}{3}(1 \cdot 7 + 0 \cdot 0) + \frac{2}{3}\left(\frac{1}{2} \cdot 6 + \frac{1}{2} \cdot 2\right) = 5.$$

The first term is when player 1 plays D (with probability $1/3$) and player 2 plays L or R with probabilities 1 and 0, respectively. The second term is when player 1 plays U (with probability $1/3 + 1/3 = 2/3$) and player 2 plays L or R, each with probability $(1/3)/(1/3 + 1/3) = 1/2$. Analogously, the expected payoff of player 1 at this correlated equilibrium is $u_2(s) = 5$. As seen in Figure 1 this payoff lies outside the convex hull of Nash equilibrium payoffs of G. □

Exercise 4. *Spočítejte všechna korelovaná ekvilibria ve hře Věžňovo dilema.*

	T	S
T	(-2,-2)	(0,-3)
S	(-3,0)	(-1,-1)

Table 3: Game from example 4

Proof. We show that the only correlated equilibrium is the Nash equilibrium (T, T). We write out all the inequalities form the definition of correlated equilibrium p (omitting cases $a_i = a_i'$ that are trivially satisfied) and obtain

$$\begin{aligned}
\mathrm{u}_1(S,S)p(S,S) + u_1(S,T)p(S,T) &\geq u_1(T,S)p(S,S) + u_1(T,T)p(S,T)\\
u_1(T,S)p(T,S) + u_1(T,T)p(T,T) &\geq u_1(S,S)p(T,S) + u_1(S,T)p(T,T)\\
u_2(S,S)p(S,S) + u_2(T,S)p(T,S) &\geq u_2(S,T)p(S,S) + u_2(T,T)p(T,S)\\
u_2(S,T)p(S,T) + u_2(T,T)p(T,T) &\geq u_2(S,S)p(S,T) + u_2(T,S)p(T,T).
\end{aligned}$$

Plugging in the payoffs, we get

$$\begin{aligned}
-p(S,S) - 3p(S,T) &\geq -2p(S,T)\\
-2p(T,T) &\geq -p(T,S) - 3p(T,T)\\
-p(S,S) - 3p(T,S) &\geq -2p(T,S)\\
-2p(T,T) &\geq -p(S,T) - 3p(T,T).
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
0 &\geq p(S,T) + p(S,S)\\
p(T,T) + p(T,S) &\geq 0\\
0 &\geq p(T,S) + p(S,S)\\
p(T,T) + p(S,T) &\geq 0
\end{aligned}$$

from which we see $p(T,T) = 1$ and $p(T,S) = p(S,T) = p(S,S) = 0$. So there is unique CE (T, T). □