Algorithmic game theory – tutorial*

9th October 2023

1 Lemke–Howson algorithm

Polyhedron of best responses of the player 1 in a matrix game $G = (\{1,2\}, A, u)$ with payoff matrices M and N is the polyhedron

$$\overline{P} = \{(x, v) \in \mathbb{R}^m \times \mathbb{R} \colon x \geq \mathbf{0}, \mathbf{1}^\top x = 1, N^\top x \leq \mathbf{1}v\}.$$

Player 2 has the following polyhedron

$$\overline{Q} = \{(y, u) \in \mathbb{R}^n \times \mathbb{R} \colon y \geq \mathbf{0}, \mathbf{1}^\top y = 1, My \leq \mathbf{1}u\}.$$

Point (x, v) of a polyhedron $\overline{P}$ has a *label* $i \in A_1 \cup A_2$, if either $i \in A_1$ and $x_i = 0$ or $i \in A_2$ and $(N^\top)_i x = v$. A point (y, u) of the polyhedron $\overline{Q}$ has a label $i \in A_1 \cup A_2$, if either $i \in A_1$ and $(M)_i y = u$ or $i \in A_2$ and $y_i = 0$.

For nonnegative matrices M and $N^\top$, which do not have a null column is a *normalized polytope of best responses* of player 1 the following polytope

$$P = \{x \in \mathbb{R}^m \colon x \geq \mathbf{0}, N^\top x \leq \mathbf{1}\}.$$

Player 2 has the following polytope

$$Q = \{y \in \mathbb{R}^n \colon y \geq \mathbf{0}, My \leq \mathbf{1}\}.$$

Labels in P a Q are defined analogically as in $\overline{P}$ a $\overline{Q}$.

Nash equilibria of not degenerated game correspond to pairs of vertices from $P \times Q \setminus \{(\mathbf{0}, \mathbf{0})\}$, which have all the labels.

Exercise 1. *Draw a polyhedron of best responses and normalized polytope of best responses in the game of chicken. Then in the given polyhedra find all pairs of vertices corresponding to Nash equilibria.*

	Turn (3)	Go straight (4)
Turn (1)	$(10, 10)$	$(9, 11)$
Go straight (2)	$(11, 9)$	$(0, 0)$

Table 1: Game of chicken.

Exercise 2. *Use Lemke–Howson algorith and compute all the Nash equilibria in the following game of two players:*

$$M = \begin{pmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{pmatrix} \quad a \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{pmatrix}.$$

Begin with the label 2.

A configuration graph has vertices composed of pairs of (x, y) vertices from $P \times Q$, which are *k-almost fully labeled*, in other words each label $A_1 \cup A_2 \setminus \{k\}$ is a label either of point x or point y. Vertices (x, y) and (x', y') are an edge, if either $x = x'$ and yy' is edge in Q or if xx' is an edge P and $y = y'$.

Exercise 3. *Prove that Lemke–Howson algorithm can not end in vertices of a form* $(x, \mathbf{0})$ *or* $(\mathbf{0}, y)$ *in the configuration graph.*

*Info about tutorial http://kam.mff.cuni.cz/~ryzak/