Algorithmic game theory – 3rd tutorial exercises*

26th October 2023

1 Matrix games

Matrix game is normal form game of 2 players.

Matrix game is *not degenerated* if each player has at most k best responses to any other strategy with domain of quantity of k. *Zero-sum* game of 2 players is a matrix game where $u_1(a) = -u_2(a)$ for any action a. Matrix game $G = (\{1,2\}, A, u)$ with actions $A_1 = \{1,\dots,m\}$ and $A_2 = \{1,\dots,n\}$ have payoff matrices M and N, where $(M)_{i,j} = u_1(i,j)$ a $(N)_{i,j} = u_2(i,j)$ for all $i \in A_1$ and $j \in A_2$. In the lecture the following algorithm for computing the Nash equilibria of not degenerated games was introduced.

Algorithm 1.1: SUPPORT ENUMERATION(G)

Input : Not degenerated matrix game G.
Output : All NE of the game G.
for all $k \in \min\{m,n\}$ and pair of domains (I,J) of quantity k

$$\begin{cases} \text{solve the system of equalities } (N^\top)_j x = v, (M)_i y = u, \sum_{i\in I} x_i = 1, \\ \sum_{j\in J} y_j = 1 \text{ for each } i \in I, j \in J \\ \text{when } x, y \geq \mathbf{0} \text{ and } u = \max\{(M)_i y \colon i \in A_1\}, v = \max\{(N^\top)_j x \colon j \in A_2\}, \\ \text{then give } (x,y) \text{ as a NE} \end{cases}$$

Exercise 1. *Use Support enumeration algorithm from the lecture to find Nash equilibrium in the Game of chicken.*

	Turn (1)	Go straight (2)
Turn (1)	(0,0)	(-1,1)
Go straight (2)	(1,-1)	(-10,-10)

Table 1: Game of chicken.

Exercise 2. *Is the following game degenerated? Find all the NE in this game. What is specific for NE in this example?*

	Cooperate (1)	Detonate (2)
Cooperate (1)	(0,0)	(0,1)
Detonate (2)	(1,0)	(0,0)

Table 2: Game of soul of Gotham

Exercise 3. *Decide which of these payoff matrices correspond to degenerated games.*

(a) $M = \begin{pmatrix} 0 & 4 & 1 \\ 2 & 2 & 4 \\ 3 & 2 & 2 \end{pmatrix}$ a $N = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 1 \end{pmatrix}$.

*Tutorial information http://kam.mff.cuni.cz/~ryzak/

(b) $M = \begin{pmatrix} 0 & 4 & 1 \\ 2 & 2 & 4 \\ 3 & 2 & 2 \end{pmatrix}$ *a* $N = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{pmatrix}$.

Exercise 4. *Prove that the following linear programs from the proof of Minimax theorem are dual to each other.*

(a) *For matrix* $M \in \mathbb{R}^{m \times n}$,

	Program P	Program D
Variable	$y_1, \ldots, y_n$	x_0
Objective function	$\min x^\top M y$	$\max x_0$
Constraints	$\sum_{j=1}^n y_j = 1$,	$\mathbf{1}x_0 \leq M^\top x$.
	$y_1, \ldots, y_n \geq 0$.	

(b) *For matrix* $M \in \mathbb{R}^{m \times n}$,

	Program P'	Program D'
Variable	$y_0, y_1, \ldots, y_n$	$x_0, x_1, \ldots, x_m$
Objective	$\min y_0$	$\max x_0$
Constraints	$\mathbf{1}y_0 - My \geq \mathbf{0}$,	$\mathbf{1}x_0 - M^\top x \leq \mathbf{0}$,
	$\sum_{j=1}^n y_j = 1$,	$\sum_{i=1}^m x_i = 1$,
	$y_1, \ldots, y_n \geq 0$.	$x_1, \ldots, x_m \geq 0$.

You can use recipe table.

Exercise 5. *Prove that if* (s_1, s_2) *and* (s_1', s_2') *are mixed NE of a zero-sum game then also the strategy profiles* (s_1, s_2') *and* (s_1', s_2) *are mixed NE of the game.*

	Primal	Dual
Variables	$\mathbf{x} = (x_1, \ldots, x_m)$	$\mathbf{y} = (y_1, \ldots, y_n)$
Matrices	$A \in \mathbb{R}^{n \times m}$	$A^\top \in \mathbb{R}^{m \times n}$
Right hand side	$\mathbf{b} \in \mathbb{R}^n$	$\mathbf{c} \in \mathbb{R}^m$
Objective function	$\max \mathbf{c}^\top \mathbf{x}$	$\min \mathbf{b}^\top \mathbf{y}$
Constraints	i-th constraint has $\leq$	$y_i \geq 0$
	$\geq$	$y_i \leq 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \geq 0$	j-th constraint has $\geq$
	$x_j \leq 0$	$\leq$
	$x_j \in \mathbb{R}$	$=$