Algorithmic game theory – Tutorial 1*

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1 Linear programming boot camp

Many problems in practice as well as purely combinatorial problems can be formulated as an instance of linear programming (LP). To solve such instance, we can apply known methods and resolve it efficiently. Each instance of LP can be transformed into the following *canonical form* which is given by a matrix $A \in \mathbb{R}^{n\times m}$ and vectors $\mathbf{b} \in \mathbb{R}^n$ a $\mathbf{c} \in \mathbb{R}^m$:

$$\begin{aligned} \max \mathbf{c}^\top \mathbf{x} \\ \text{for } \mathbf{x} \in \mathbb{R}^m \\ \text{under conditions } A\mathbf{x} \leq \mathbf{b}. \end{aligned}$$

Exercise 1. *A bakery sells bread, buns, baguettes, and doughnuts.*

- *To bake a single piece of bread, we need half a kilo of flour, 10 eggs, and 50 g of salt.*
- *To bake a bun, we need 150 g of lour, 2 eggs, and 10 g of salt.*
- *For a baguette, we need 230 g of flour, 7 eggs, and 15 g of salt.*
- *For a doughnut, we need 100 g of flour and 1 egg.*

The bakery has 5 kilos of flour, 125 eggs, and half a kilo of salt. The prices are 20 crowns for bread, 2 crowns for a bun, 10 crowns for a baguette, and7 crowns for a doughnut.

The bakery's goal is to maximize the revenue. How many pieces of pastry does it need to produce to maximize the profit? Formulate the corresponding LP.

Exercise 2. *Show how to:*

1. *Transform maximizing LP to a minimizing LP and vice versa.*
2. *Transform LP with variables $\mathbf{x} \geq \mathbf{0}$ into LP with variables $\mathbf{x}' \in \mathbb{R}^m$ and vice versa.*
3. *Transform LP with conditions given by inequalities into LP with conditions given by equalities and vice versa.*

If we require the variables to take integer values, then we can formulate many NP-hard problems using LP. Without such conditions, the task of solving an LP is solvable in polynomial time. In practice, we use the *simplex method* to solve LP, which is efficient in practice, but can take exponential time on certain degenerate instances.

Exercise 3. *Formulate* The knapsack problem *using integer linear programming. That is, for n items where the ith one has weight v_i and cost c_i and a knapsack with capacity V, we want to fit some of the items into the knapsack so that the total value of items inside is maximized while we do not exceed the capacity..*

*Information about the course can be found at http://kam.mff.cuni.cz/~balko/

2 The duality

Consider the following linear program P with m variables and with n conditions:

$$\max \mathbf{c}^\top \mathbf{x} \text{ under conditions } A\mathbf{x} \le \mathbf{b} \text{ a } \mathbf{x} \ge \mathbf{0}. \tag{P}$$

We call P the *primal linear program* (or simply *primal*). Its *dual linear program* (or simply *dual*) is the following linear program D with n variables and m conditions:

$$\min \mathbf{b}^\top \mathbf{y} \text{ under conditions } A^\top \mathbf{y} \ge \mathbf{c} \text{ a } \mathbf{y} \ge \mathbf{0}. \tag{D}$$

Exaplnation: to solve P, we are trying to find a linear combination of the n conditions $A\mathbf{x} \le \mathbf{b}$ with some coefficients $y_1, \dots, y_n \ge 0$ such that the resulting inequality has the jth coefficient at least c_j for each $j \in \{1, \dots, m\}$ while the right hand side is as small as possible.

Exercise 4. *Construct the dual D for the following primal linear program P:*

$$\begin{aligned} \max 6x_1 + 4x_2 + 2x_3 \\ 5x_1 + 2x_2 + x_3 &\le 5 \\ x_1 + x_2 &\le 2 \\ x_2 + x_3 &\le 2 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

The following theorem is probably the most important result about linear programs.

Theorem 1 (Strong duality theorem)**.** *For linear programs P and D, one of the following four possibilities occurs:*

(a) None of the programs P and D has a feasible solution.

(b) The program P is unbounded and D does not have a feasible solution.

(c) The program D is unbounded and P does not have a feasible solution.

(d) The programs P and D both have a feasible solution. Then, they also have optimal solutions $\mathbf{x}^*$ and $\mathbf{y}^*$ and we have $\mathbf{c}^\top \mathbf{x}^* = \mathbf{b}^\top \mathbf{y}^*$.

Exercise 5. *Construct a dual D for the following primal linear program P:*

$$\begin{aligned} \max x_1 - 2x_2 + 3x_4 \\ x_2 - 6x_3 + x_4 &\le 4 \\ -x_1 + 3x_2 - 3x_3 &= 0 \\ 6x_1 - 2x_2 + 2x_3 - 4x_4 &\ge 5 \\ x_2 &\le 0 \\ x_4 &\ge 0 \end{aligned}$$

Dualization Recipe

	Primal linear program	Dual linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Matrix	A	A^T
Right-hand side	$\mathbf{b}$	$\mathbf{c}$
Objective function	$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{b}^T\mathbf{y}$
Constraints	ith constraint has $\leq$	$y_i \geq 0$
	$\geq$	$y_i \leq 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \geq 0$	jth constraint has $\geq$
	$x_j \leq 0$	$\leq$
	$x_j \in \mathbb{R}$	$=$

Obrázek 1: Recipe table