The hypergraph 2-colouring threshold

Amin Coja-Oghlan

Goethe University Frankfurt

Random Constraint Satisfaction Problems

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- *Constraints* $C_1, \ldots, C_m$ chosen independently, uniformly at random.

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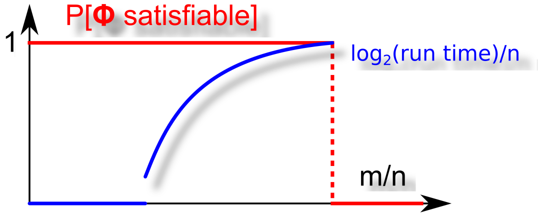
Kirkpatrick, Selman (experimental) [Science 1994]

There occurs a sharp satisfiability phase transition.



Random Hypergraph 2-colouring

Random Hypergraphs

- $V = \{v_1, \ldots, v_n\}$: vertices.
- $\mathcal{H} =$ random *k-uniform hypergraph* with m edges.
- Let $r = m/n$ be *fixed* while $n \to \infty$.

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Hypergraph 2-colouring

- Is there $\sigma : V \to \{\bullet, \bullet\}$ s.t. no edge is *monochromatic*.
- NP-hard in the *worst case*.

Prior work

Rigorous work

- Existence of *non-uniform* thresholds *[Friedgut 1999]*
- Second moment method *[Achlioptas, Moore'02]*
  - $r < 2^{k-1}\ln 2 - \frac{\ln 2}{2} - \frac{1}{2} + o_k(1) \Rightarrow \mathcal{H}$ is 2-col w.h.p.
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Non-rigorous stuff

- Cavity method: "Survey propagation" *[Mézard, Parisi, Zecchina 2002]*
- The condensation transition *[KMRSZ 2007]*
- *Universal* picture (random k-SAT, graph colouring, ...)

The statistical mechanics perspective

- Phase transitions in glasses hypothesized by *Kauzmann* (1948).
- Mean-field models of disorered systems (such as glasses).
- This work: first *proof* of condensation in "diluted mean-field model".

Random CSPs

This work: random hypergraph 2-colouring

- Pinning down the *threshold* in a problem with condensation.
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Known thresholds

- Random 2-SAT *[Chvátal, Reed'92; Goerdt'92]*
- Random 1-in-k-SAT *[Achlioptas, Chtcherba, Istrate, Moore'01]*
- Random k-XORSAT *[Dubois, Mandler'02]*
- Uniquely extendible problems *[Connamacher, Molloy'04]*
- Random k-SAT with $k > \log_2 n$ *[Frieze, Wormald'05]*

The partition function

- Let $\beta > 0$ be a parameter *("inverse temperature")*.
- For $\sigma : V \to \{\bullet, \bullet\}$ let

$$w(\sigma) = \#\text{monochromatic edges in } \mathcal{H} \text{ under } \sigma.$$

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- Goal: to find

$$(\beta, r) \mapsto \lim_{n \to \infty} \frac{1}{n} \mathrm{E}\left[\ln Z_\beta\right].$$

- *Bayati, Gamarnik, Tetali 2010:* the limit exists for any $0 < \beta < \infty$.

The partition function

Zero temperature

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Conjecture

The limit $\lim_{n\to\infty} \frac{1}{n} \mathrm{E}\left[\ln\left(1 \vee Z_\infty\right)\right]$ exists for any $r > 0$.

This implies the *"sharp threshold conjecture"*.

The partition function

Phase transitions

- a point (β, r) where the limit is non-analytic.
- a density r where the *zero temperature* limit is non-analytic.

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Key questions

- Do *one or more* phase transitions exist?
- *Zero temperature*: the 2-colouring threshold r_{col}, plus ...?

Results

Theorem [ACO, Zdeborová 2012]

1. The *zero temperature* limit is non-analytic at

$$r_{cond} = 2^{k-1}\ln 2 - \ln 2 + o_k(1) \quad \textit{and} \quad r_{col} > r_{cond}.$$

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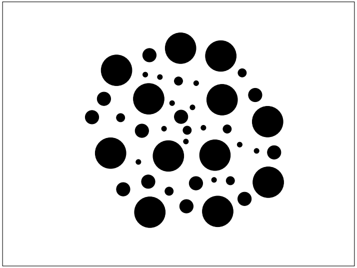
- *Zero temperature:* (at least) two phase transitions.
- *Low temperature:* at least one.

The solution space

- Let $\mathcal{S}(\mathcal{H}) = \{\text{all 2-colourings of } \mathcal{H}\} \subset \{0,1\}^n$.

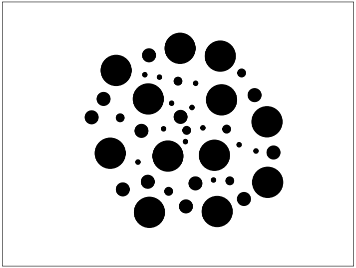


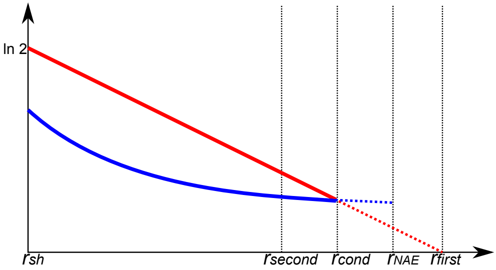
- For $r < 2^{k-1}\ln 2 - \ln 2 + o_k(1)$, the set shatters into tiny *clusters*.
- Each cluster is *exponentially small*.

A phase transition [ACO, Zdeborová 2012]

Let's plot the functions

$$r \mapsto \frac{1}{n}\mathrm{E}\left[\ln Z\right] \qquad \text{and} \qquad r \mapsto \frac{1}{n}\mathrm{E}\left[\ln\{\text{cluster size}\}\right].$$





The number of 2-colourings

Corollary [ACO, Zdeborová 2012]

- For $r \leq 2^{k-1} \ln 2 - \ln 2 + o_k(1)$ we have

$$\frac{1}{n}\mathrm{E}[\ln Z] \sim \frac{1}{n}\ln \mathrm{E}[Z] = \ln 2 + r \cdot \ln(1 - 2^{1-k}).$$

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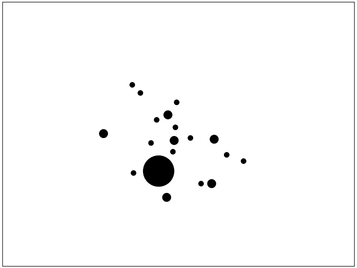
- By contrast, for $r > 2^{k-1} \ln 2 - \ln 2 + \varepsilon$ we have

$$\frac{1}{n} \mathrm{E}[\ln Z] < \frac{1}{n} \ln \mathrm{E}[Z] - \Omega(1).$$

The solution space

Stat mech hypothesis

- Let $\mathcal{S}(\mathcal{H}) = \{\text{all 2-colourings of } \mathcal{H}\}$.



- At $r = 2^{k-1}\ln 2 - \ln 2 + o_k(1)$, the set condenses.
- A *sub-exponential* number of clusters dominate.

The theshold for 2-colurability

Theorem [ACO, Panagiotou 2012]

We have $r_{col} = 2^{k-1} \ln 2 - \frac{\ln 2}{2} - \frac{1}{4} + o_k(1)$.

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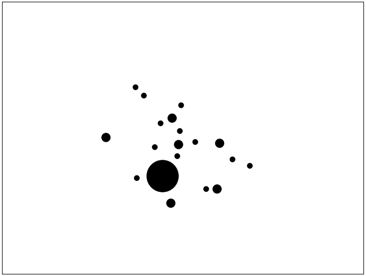
Density	What's happening?
$2^{k-1}\ln 2 - \frac{\ln 2}{2} - \frac{1}{2} + o_k(1)$	"vanilla" second moment *[AM'02]*
$2^{k-1}\ln 2 - \ln 2 + o_k(1)$	*phase transition* ("condensation")
$2^{k-1}\ln 2 - \frac{\ln 2}{2} - \frac{1}{4} + o_k(1)$	2-colouring *threshold*
$2^{k-1}\ln 2 - \frac{\ln 2}{2} + o_k(1)$	*first moment* upper bound

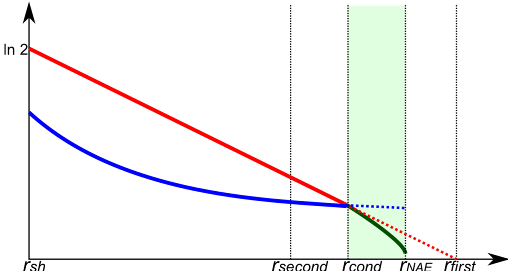
Into the condensation phase

Corollary [ACO, Panagiotou 2012]

Approximate expressions for. . .

- . . . the *partition function* $\frac{1}{n}\mathrm{E}[\ln Z]$,
- . . . the *number of clusters* (“complexity”).





Conclusion

- Main contricutions:
  - first *improvement* over the "vanilla" 2nd moment from [AM02],
  - first rigorous proof of a *condensation transition* in this kind of model,
  - pinned down the 2-*colouring threshold* up to $o_k(1)$.
- Open problems:
  - *exact* threshold for any k?
  - counting solutions in the condensation phase?