Large structured induced subgraphs with close homomorphism statistics

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Introduction

Motivation: counting statistics and limits

$$\begin{tikzcd} F_1 \arrow[dr] & \\ F_2 \arrow[r] & G \\ F_3 \arrow[ur] & \end{tikzcd}$$

- Dense graphs:

$$t(F, G) = \frac{\hom(F, G)}{|G|^{|F|}}$$

- Very sparse graphs:

$$\mathrm{dens}(F, G) = \frac{\hom(F, G)}{|G|}$$

Motivation: counting statistics and limits

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- Dense graphs:

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- Very sparse graphs:

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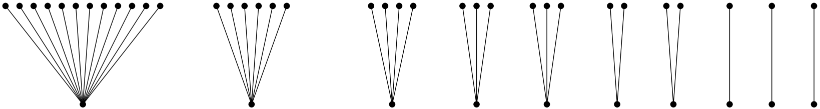
Problem

Given a graph G. Does there exist $G[A]$ such that:

- For every small F, $\hom(F,G[A]) \approx \hom(F,G)$;
- $G[A]$ has a very regular structure?

What can we expect?



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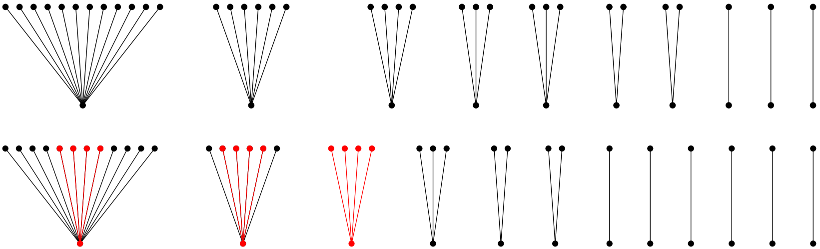

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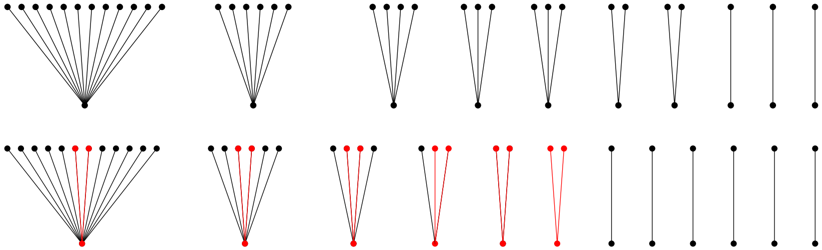
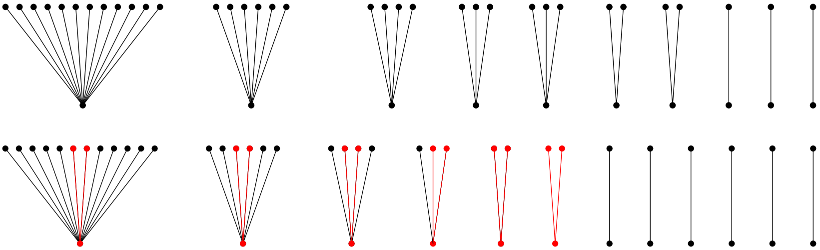


What can we expect?



For start forests, we cannot expect more than

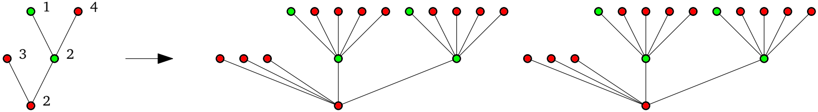
$$\hom(F, G[A]) \approx \frac{\hom(F, G)}{\log \hom(F, G)}$$

Small Colored Forests

Blowing Colored Forests

Y : Colored rooted forest

μ : Mapping $V(Y) \to \mathbb{N}$



(Y, μ) $Y * \mu$

Colored Rooted Forests with Bounded Height

Theorem (Nešetřil, POM — 2012)

Let Y be a colored rooted forest of height h and let $p \in \mathbb{N}$. There exists $\widehat{Y}$ and a blowing μ of $\widehat{Y}$ such that $|\widehat{Y}| < F(h,p)$ and $\forall F,\ |F| \leq p \implies \exists \mu_F$ with

$$F * \mu_F \hookrightarrow \widehat{Y} * \mu \hookrightarrow Y$$

and

$$\hom(F, \widehat{Y} * \mu) \geq \prod_{v \in V(F)} \mu_F(v) \geq \frac{\epsilon(p,h) \hom(F,Y)}{(1 + \log \hom(F,Y))^{p^h}}$$

Sketch of the Proof

Lemma (Approximating sum of products)

Let $a_{i,j} \in \mathbb{N}$ where $1 \leq j \leq p$. Assume

$$N = \sum_i \prod_{j=1}^{p} a_{i,j}.$$

Then there exist $c_1, \ldots, c_p \in \mathbb{N}$ such that

$$\sum_i \prod_{j=1}^{p} \begin{Bmatrix} c_j, & \text{if } a_{i,j} \geq c_j \\ 0, & \text{otherwise} \end{Bmatrix} \geq \frac{N}{(1+\log N)^p}.$$

Sketch of the Proof

Lemma (Common Refinement)

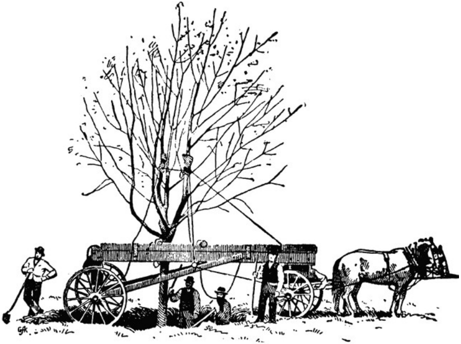
Assume

$$\begin{tikzcd} F_1 * \mu_1 \arrow[drr, hook] & & \\ F_2 * \mu_2 \arrow[rr, hook] & & Y \end{tikzcd}$$

Then $\exists F, \mu : \quad \mu_1' \approx \mu_1/2|F_2|, \quad \mu_2' \approx \mu_2/2|F_1|,$ such that $|F| \leq f(|F_1|, |F_2|)$ and

$$\begin{tikzcd} F_1 * \mu_1' \arrow[dr, hook] & & \\ F_2 * \mu_2' \arrow[r, hook] & F * \mu \arrow[r, hook] & Y \end{tikzcd}$$

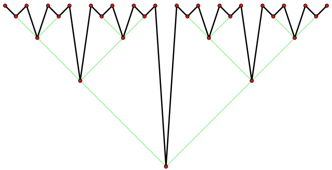
Moving to Broader Classes



Low tree-depth decompositions

Tree-depth td(G)

The *tree-depth* of G is the minimum height of a rooted forest F such that $G \subseteq \mathrm{Clos}(F)$.



Chromatic numbers $\chi_p(G)$

$\chi_p(G)$ is the minimum of colors such that any subset I of $\leq p$ colors induce a subgraph G_I so that $\mathrm{td}(G_I) \leq |I|$.

Low tree-depth decomposition

Lemma

Let G be a graph, let $p \in \mathbb{N}$, and let $q = p2^{p^2}$.
Then there exists $G[A]$ such that

- $\mathrm{td}(G[A]) \leq q$;
- for every F with $|F| \leq p$ it holds

$$(\#F \subseteq G[A]) \geq \frac{1}{\chi_q(G)^p}(\#F \subseteq G).$$

Low tree-depth decomposition

Lemma

Let G be a graph, let $p \in \mathbb{N}$, and let $q = p2^{p^2}$.
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Color Encoding

Graphs with td $\leq t$ $\approx$ 2^t-colored forests with height $\leq t$.

Low tree-depth decompositions

Let $\mathscr{C}$ be an infinite class of graphs.

Theorem (Nešetřil, POM — 2006)

$$\forall p,\ \chi_p(G) = O(1) \text{ in } \mathscr{C} \quad \Longleftrightarrow \quad \mathscr{C} \text{ has bounded expansion.}$$

Low tree-depth decompositions

Let $\mathscr{C}$ be an infinite class of graphs.

Theorem (Nešetřil, POM — 2006)

$\forall p,\ \chi_p(G) = O(1)$ in $\mathscr{C}$ $\iff$ $\mathscr{C}$ has bounded expansion.

Theorem (Nešetřil, POM — 2010)

$\forall p,\ \log \chi_p(G) = o(\log |G|)$ in $\mathscr{C}$ $\iff$ $\mathscr{C}$ is nowhere dense.

Bounded Expansion Classes

Definition

A class $\mathscr{C}$ has bounded expansion if there exists $f : \mathbb{N} \to \mathbb{N}$ such that for integer k it holds:
No $G \in \mathscr{C}$ contains a $\leq k$-subdivision of a graph with average degree $> f(k)$.

Examples

Bounded degree, planar, classes excluding a topological minor, etc.

Nowhere Dense Classes

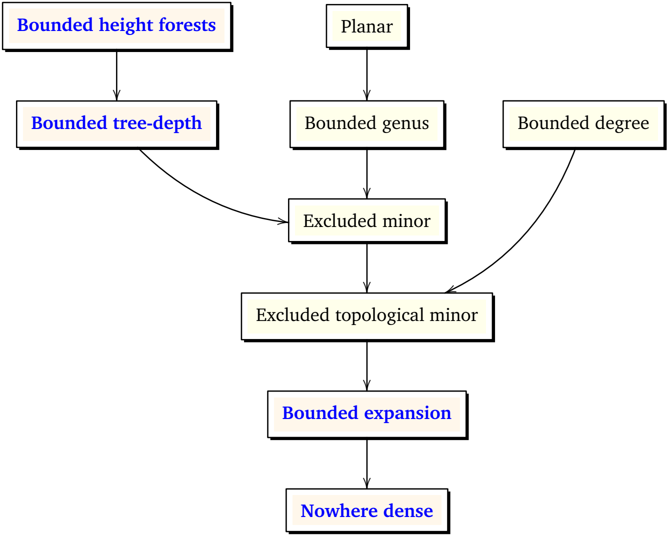
Definition

A class $\mathscr{C}$ is nowhere dense if there exists $f : \mathbb{N} \to \mathbb{N}$ such that for integer k it holds:
No $G \in \mathscr{C}$ contains a $\leq k$-subdivision of $K_{f(k)}$.

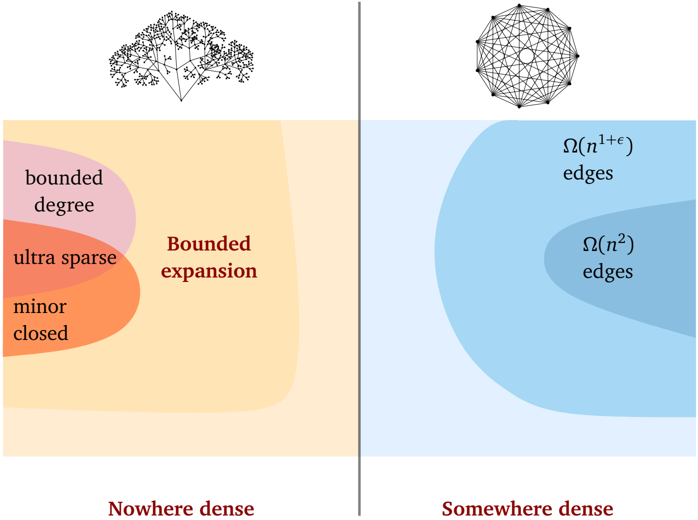
Examples

Classes with bounded expansion, classes such that $\Delta(G) < \mathrm{girth}(G)$, etc.

Class Taxonomy



Class Taxonomy



Consequence

Theorem (Nešetřil, POM — 2012)

Let $\mathscr{C}$ be a class with bounded expansion and let $p \in \mathbb{N}$. Then every graph $G \in \mathscr{C}$ has an induced subgraph $G[A]$ such that:

- $\mathrm{td}(G[A]) \le p2^{p^2}$,
- $G[A]$ is a "blowing" of a graph of order $\le f_{\mathscr{C}}(p)$;
- for every graph F of order at most p

$$\hom(F, G[A]) \ge \frac{\hom(F, G)}{\operatorname{polylog} \hom(F, G)}.$$

Consequence

Theorem (Nešetřil, POM — 2012)

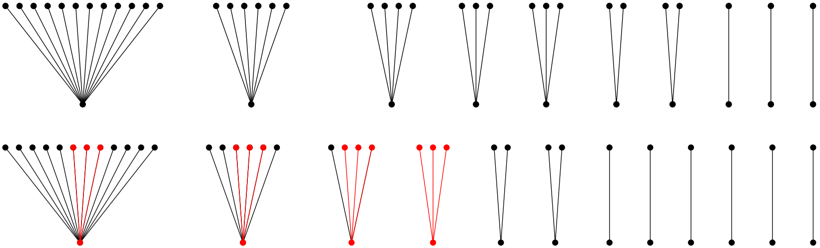
Let $\mathscr{C}$ be a nowhere dense class and let $p \in \mathbb{N}$. Then for every $\epsilon > 0$ there exists $N = N(\mathscr{C}, p, \epsilon)$ such that every graph $G \in \mathscr{C}$ of order at least N has an induced subgraph $G[A]$ with the properties:

- $\mathrm{td}(G[A]) \leq p2^{p^2}$,
- $G[A]$ is a "blowing" of a graph of order $\leq f_{\mathscr{C}}(p)$;
- for every graph F of order at most p

$$\log \hom(F, G[A]) \geq (1 - \epsilon) \log \hom(F, G).$$

Problems

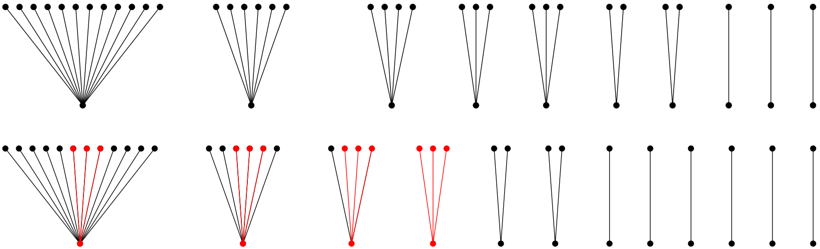
Worst Case vs Random Case



For star forest, we cannot expect more than

$$|G[A]| \approx \frac{|G|}{\log\ |G|}$$

Worst Case vs Random Case



But for a random star forest, we can achieve

$$|G[A]| \approx \frac{6(\log 2)^2}{\pi^2} |G| > \frac{|G|}{4}.$$

Random Case

Problem 1

Let $\mathscr{C}$ be a monotone class with bounded expansion and let $p \in \mathbb{N}$. Does there exists $\epsilon = \epsilon(\mathscr{C}, p)$ such that a random graph $G \in \mathscr{C}$ has (with high probability) an induced subgraph $G[A]$ such that:

- $\mathrm{td}(G[A]) \leq p2^{p^2}$,
- $G[A]$ is a "blowing" of a graph of order $\leq f_{\mathscr{C}}(p)$;
- for every graph F of order at most p

$$\hom(F, G[A]) \geq \epsilon \hom(F, G)?$$

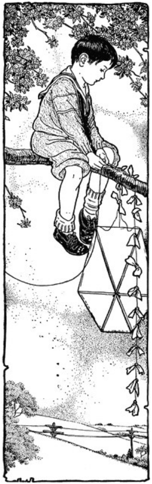
Homomorphism density

Problem 2

Let $\mathscr{C}$ be a monotone class with bounded expansion, let $p \in \mathbb{N}$, and let $\epsilon > 0$. Does there exists $N = N(\mathscr{C}, p, \epsilon)$ such that every graph $G \in \mathscr{C}$ of order $> N$ has an induced subgraph $G[A]$ such that:

- $\mathrm{td}(G[A]) \leq f_{\mathscr{C}}(p, \epsilon)$,
- $G[A]$ is a "blowing" of a graph of order $\leq g_{\mathscr{C}}(p, \epsilon)$;
- for every graph F of order at most p

$$\left| \frac{\hom(F, G[A])}{|A|^{\alpha(F)}} - \frac{\hom(F, G)}{|G|^{\alpha(F)}} \right| < \epsilon?$$



Thank you for your attention.