Using algebra in geometry (updated and corrected version)

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Abstract

In this thesis, we develop a technique that combines algebra, algebraic topology and combinatorial arguments and provides non-embeddability results. The novelty of our approach is to examine non-embeddability arguments from a homological point of view. This turns out to be a surprisingly effective idea, as homological analogues of embeddings appear to be much richer and easier to build than their homotopic counterparts. We illustrate the strength of this approach by proving two interesting theorems.

The first one states that it is impossible to embed k-dimensional skeleton of $\left(b\binom{2k+2}{k} + k + 3\right)$-dimensional simplex into a $2k$-dimensional manifold with kth $\mathbb{Z}_2$-Betti number at most b. So far as we now this is the first finite upper bound for Kühnel's conjecture of non-embeddability of simplices into manifolds.

The second one is a very general topological Helly type theorem for sets in $\mathbb{R}^d$: There exists a function $h(b,d)$ such that the following holds. If $\mathcal{F}$ is a finite family of sets in $\mathbb{R}^d$ such that $\tilde{\beta}_i\left(\bigcap \mathcal{G}; \mathbb{Z}_2\right) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and every $0 \leq i \leq \lceil d/2 \rceil - 1$, then $\mathcal{F}$ has Helly number at most $h(b,d)$. The symbol $\tilde{\beta}_i\left(\bigcap \mathcal{G}; \mathbb{Z}_2\right)$ stands for the ith reduced Betti number of $\mathcal{G}$. If we are only interested whether the Helly numbers are bounded or not, this theorem subsumes a broad class of Helly types theorems for sets in $\mathbb{R}^d$. The bounds it provides are very suboptimal, though.

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Chapter 1

Introduction

In this thesis we introduce a general framework which combines algebra, algebraic topology and combinatorial arguments to yield non-embeddability results. The novelty of our approach is to examine non-embeddability arguments from a homological point of view. This turns out to be a surprisingly effective idea, as homological analogues of embeddings appear to be much richer and easier to build than their homotopic counterparts. So far, we have two main applications of the developed methods: an upper bound for Kühnel's conjecture [Küh94, Conjecture B] of non-embeddability of skeleta of simplices into manifolds (Theorem 1.1) and a very general topological Helly type theorem for sets in $\mathbb{R}^d$ (Theorem 1.8).

The thesis is based on the following papers:

1. X. Goaoc, I. Mabillard, P. Paták, Z. Patáková, M. Tancer, U. Wagner: *On Generalized Heawood Inequalities for Manifolds: a Van Kampen–Flores Type Nonembeddability Result*, conference version in Proceedings of Symposium of Computational Geometry, 2015
   Here we show that if $n \geq 2b_k\binom{2k+2}{k} + 2k + 5$, then the k-dimensional skeleton of n-dimensional simplex does not embed into any $2k$-dimensional manifold with kth $\mathbb{Z}_2$-Betti number at most b_k (Theorem 1.1). This generalizes van Kampen-Flores theorem [vK32, Flo33], although with a slightly suboptimal bound, and constitutes the first finite upper bound for Kühnel's conjecture [Küh94, Conjecture B], so far as we know. Moreover, our bound is roughly only kth power of the conjectured value.

2. X. Goaoc, P. Paták, Z. Patáková, M. Tancer, U. Wagner: *Bounding Helly numbers via Betti numbers*, conference version in Proceedings of Symposium of Computational Geometry, 2015
   Using induction, we obtain a very general topological Helly type theorem (Theorem 1.8): There exists a function $h(b, d)$ such that the following holds. If $\mathcal{F}$ is a finite family of sets in $\mathbb{R}^d$ such that the reduced Betti numbers satisfy $\tilde{\beta}_i\left(\bigcap \mathcal{G}; \mathbb{Z}_2\right) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and every $0 \leq i \leq \lceil d/2 \rceil - 1$, then $\mathcal{F}$ has Helly number at most $h(b, d)$. If we are only interested whether the Helly numbers are bounded or not, this theorem subsumes a broad class of Helly type theorems for sets in $\mathbb{R}^d$.

3. P. Paták: *Colorful Algebraic Tverberg Type Theorem*, In preparation
   Tverberg's theorem states that given $(r-1)(d+1)+1$ points in $\mathbb{R}^d$, it is possible to split them into r parts $F_1, F_2, \ldots, F_r$ such that $\operatorname{conv} F_1 \cap \operatorname{conv} F_2 \cap \cdots \cap \operatorname{conv} F_r \neq \emptyset$. There is also a colorful version that places some additional constraints onto the resulting sets $F_1, \ldots, F_r$. So far the colorful version can only be proven if r is a prime number. Here we prove a variant of colorful Tverberg Theorem, where we replace convex combinations with affine ones. The result does hold for all fields and arbitrary non-negative integer values of r; and enables us to reduce the bound $n \geq 2b_k\binom{2k+2}{k} + 2k + 5$ in Theorem 1.1 to $n \geq 2b_k\binom{2k+2}{k} + 2k + 3$.

Before we describe main ideas of our method, some definitions are needed.[1].

Given a field $\mathbb{F}$ and a topological space X, there is a certain vector space $\widetilde{C}_*(X; \mathbb{F})$ assigned to X. It is called the augmented chain complex. Augmented chain complexes also exist for simplicial complexes. Any continuous map $f\colon X \to Y$ between two topological spaces X, Y induces a chain map (a special

[1] Proper definitions can be found in Section 3.1, here we only sketch the most important ones.

homomorphism) $f_\sharp\colon \widetilde{C}_*(X;\mathbb{F}) \to \widetilde{C}_*(Y;\mathbb{F})$ between the corresponding chain complexes. The induced chain map sends the empty chain, the generator of $\widetilde{C}_{-1}(X;\mathbb{F})$, onto the empty chain, generator of $\widetilde{C}_{-1}(Y;\mathbb{F})$. We call chain maps satisfying this condition *nontrivial*. Furthermore, if f is an embedding, the induced chain map maps chains with disjoint supports to chains with disjoint supports. We call nontrivial chain maps satisfying this condition *homological almost embeddings*.

Now the main idea of our method can be described as follows:

Suppose that L is a finite simplicial complex and $f\colon \Delta_n^{(k)} \to X$ is a continuous map of the k-dimensional skeleton of n-dimensional simplex, $\Delta_n^{(k)}$, into a topological space X. If $\mathbb{F}$ is finite and n is big enough (depending on X, L, $\mathbb{F}$ and k), we can, using Ramsey theory and the additive structure of the finite chain group $C_*\left(\Delta_n^{(k)};\mathbb{F}\right)$, find a homological almost embedding $\varphi\colon C_*(L;\mathbb{F}) \to C_*\left(\Delta_n^{(k)};\mathbb{F}\right)$ such that the composition $f_\sharp \circ \varphi$ is homologically trivial.

In the proof of Theorem 1.1, we combine this idea with a result by Volovikov [Vol96b] that every embedding f of $\Delta_{2k+2}^{(k)}$ into a $2k$-dimensional compact manifold M satisfies[2] $f_* \neq 0$.

In the proof of Theorem 1.8, assuming that the Helly number of a family $\mathcal{F}$ is unbounded, we inductively use the construction to obtain a homological almost embedding of $C_*\left(\Delta_{2k+2}^{(k)};\mathbb{Z}_2\right)$ into $C_*\left(\mathbb{R}^{2k};\mathbb{Z}_2\right)$, which contradicts our homological version of the Van Kampen-Flores theorem (Theorem 1.7).

1.1 Thesis outline

The thesis is divided into chapters as follows:

1. Chapter 1 relates our work to known results.

2. Chapter 2 contains a variant of colorful Tverberg theorem for affine combinations (Theorem 1.3). Although interesting on its own, the result is mainly used to find suitable combinations of non-trivial chain maps in Chapter 3. Since the proof is technical, we also provide an easier version (Lemma 2.8) for the readers who do not want to go through all the technical details and are willing to accept a slightly worse[3] bounds in Chapters 3 and 4.

3. In Chapter 3 we show that given a manifold M with bounded kth Betti number, a k-dimensional simplicial complex L and a continuous map f from a "sufficiently large" k-dimensional simplex K into M, f can be used to construct a map $g\colon L \to K$ satisfying $(f \circ g)_* = 0$ and some additional properties (Theorem 1.2). The chapter concentrates on the affine structure of non-trivial chain maps and shows that an affine combination $\varphi = \sum a_i h_i$ satisfying certain properties can be turned into a continuous map h such that $h_* = \varphi_*$. We combine this idea with Theorem 1.3 to obtain the desired result.

4. Chapter 4 provides the first striking application of our methods: Given a $2k$-dimensional manifold M, we prove non-embeddability of "sufficiently large" k-dimensional simplicial complexes into M (Theorem 1.1) and hence provide an upper bound for Kühnel's conjecture [Küh94, Conjecture B]. The proof is based on a combination of Theorem 1.2 with Volovikov's result that there is no embedding $f\colon \Delta_{2k+2}^{(k)} \to M$ into a compact $2k$-dimensional manifold M for which $f_* = 0$. We also provide several generalizations of Theorem 1.1.

5. Chapter 5 shows the nonexistence of homological embeddings of $C_*\left(\Delta_{k+1}^{(k)};\mathbb{Z}_2\right)$ or $C_*\left(\Delta_{2k+2}^{(k)};\mathbb{Z}_2\right)$ into $C_*(\mathbb{R}^k;\mathbb{Z}_2)$ or $C_*\left(\mathbb{R}^{2k};\mathbb{Z}_2\right)$, respectively (Theorems 1.6 and 1.7). The proofs are based on the fact that the classical cohomological arguments easily translate into the setting of non-trivial chain maps.

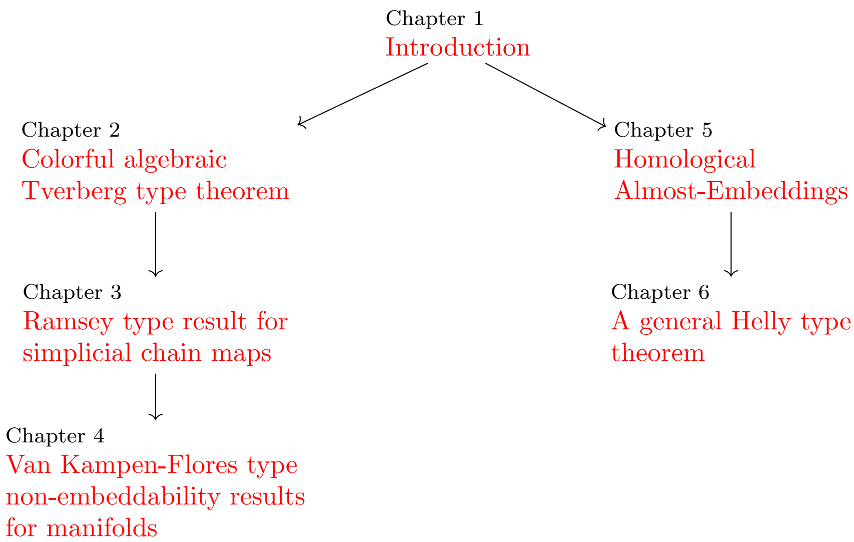
6. Chapter 6 shows another striking application of our approach, our very general Helly type theorem (Theorem 1.8): There exists a function $h(b,d)$ such that the following holds. If $\mathcal{F}$ is a finite family

[2] Volovikov's original result is stated in terms of cohomology but if $\mathbb{F}$ is a field, it implies that $f_* \neq 0$.

[3] We note that for the Kühnel's conjecture (Theorem 1.1), the bounds differ only by one. The gap becomes larger for other theorems.

of sets in $\mathbb{R}^d$ such that $\tilde{\beta}_i\left(\bigcap \mathcal{G}; \mathbb{Z}_2\right) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and every $0 \leq i \leq \lceil d/2 \rceil - 1$, then $\mathcal{F}$ has Helly number at most $h(b, d)$. The proof is obtained by contradiction. If the Helly number of $\mathcal{F}$ is sufficiently large, we use an inductive construction to build a homological almost embedding of $C_*\left(\Delta^{(k)}_{2k+2}; \mathbb{Z}_2\right)$ into $C_*\left(\mathbb{R}^{2k}; \mathbb{Z}_2\right)$. Existence of such an embedding contradicts Theorem 1.7 from Chapter 5, hence the Helly number of $\mathcal{F}$ has to be smaller.

The logical dependency of the chapters is as follows:



The remaining part of the introduction shows our results in the context of related work. It also depicts relations between various chapters of this thesis. In Section 1.2 we investigate non-embeddability results and provide a motivation for Theorem 1.1. In Section 1.3 we investigate Helly type theorems and show that if one only concentrates on the question "Are Helly numbers bounded, or not?", that many of the Helly type theorems are subsumed by our Theorem 1.8. Unfortunately, the bounds on Helly numbers it provides are enormous, so the main application of Theorem 1.8 is a quick identification of situations for which Helly numbers are bounded and for which a new Helly type theorems can be obtained by proving effective bounds.

1.2 Non-embeddability results

1.2.1 Non-embeddability

The fact that the complete graph K_5 does not embed in the plane has been generalized in two independent directions. On the one hand, the solution of the classical *Heawood problem* established that for surfaces other than the Klein bottle[4], complete graph K_n embeds into a closed surface M if and only if $(n-3)(n-4) \leq 6b_1(M)$, where b_1 is the first $\mathbb{Z}_2$-Betti number of M. See [Hea90, Hef91] for the original statement of the problem and [Rin74] for the history and detailed references to the series of work by Gustin, Guy, Mayer, Ringel, Terry, Welch, and Youngs that solved the problem in 1950–1960.

On the other hand, it is possible to replace complete graphs K_{n+1} with their higher dimensional analogues $\Delta^{(k)}_n$, k-dimensional skeletons of n-dimensional simplices, and ask when they embed into $\mathbb{R}^m$. Since every finite k-dimensional simplicial complex embeds into $\mathbb{R}^{2k+1}$, the first interesting value of m is $2k$. In this case the optimal solution is known: Van Kampen [vK32] and Flores [Flo33] proved that $\Delta^{(k)}_n$ embeds into $\mathbb{R}^{2k}$ if and only if $n \leq 2k+1$.

Two decades ago Kühnel conjectured [Küh94, Conjecture B] that $\Delta^{(k)}_n$ embeds in a compact, $(k-1)$-connected $2k$-dimensional manifold with kth $\mathbb{Z}_2$-Betti number b_k only if the following *generalized Heawood inequality* holds:

$$\binom{n-k-1}{k+1} \leq \binom{2k+1}{k+1} b_k. \tag{1.1}$$

[4]Klein bottle does not allow an embedding of K_7, only of K_6.

This is a common generalization of the case of graph on surfaces ($k = 1$) as well as the Van Kampen-Flores theorem ($b_k = 0$). So far the conjecture remained essentially untouched.

In Chapter 4, we are able to prove the following bound for Kühnel's conjecture:

Theorem 1.1. *Let M be a $2k$-dimensional manifold with kth $\mathbb{Z}_2$-Betti number b_k. If $n \geq 2b_k\binom{2k+2}{k} + 2k + 3$, then $\Delta_n^{(k)}$ does not embed into M.*

Our assumptions are weaker and apply to a much broader class of manifolds than the original conjecture, but our bound on n is approximately kth power of the value proposed by Kühnel.

We note that Volovikov [Vol96b] has also generalized Van Kampen-Flores theorem for manifolds, however, his version does not answer the question what is the largest integer n, for which $\Delta_n^{(k)}$ embeds into a $2k$-dimensional manifold with kth Betti number b_k. Volovikov's theorem concerns[5] $2k$-dimensional compact manifolds M and states that there exists no almost embedding[6] $f\colon \Delta_{2k+2}^{(k)} \to M$, for which the induced homomorphism[7] $f_*\colon H_*\left(\Delta_{2k+2}^{(k)}; \mathbb{Z}_2\right) \to H_*(M, \mathbb{Z}_2)$ is trivial.

We deduce Theorem 1.1 from Volovikov's result by using the following sophisticated reduction:

Theorem 1.2. *Let $n, s, k, b \geq 0$ be integers. Let M be a manifold with k-th reduced $\mathbb{Z}_2$-Betti number at most b. Let $f\colon \Delta_n^{(k)} \to M$ be an almost embedding. If*

$$n \geq \binom{s}{k} b(s - 2k) + s + 1 \quad \textit{and} \quad n \geq s + 1.$$

then there exists an almost embedding $g\colon \Delta_s^{(k)} \to M$ such that the induced homomorphism in homology $g_*\colon H_*\left(\Delta_s^{(k)}; \mathbb{Z}_2\right) \to H_*(M; \mathbb{Z}_2)$ is trivial.

The details of the reduction and its proof can be found in Chapter 4.

1.2.2 Multiple intersections

We have asked when does $\Delta_n^{(k)}$ embed into $\mathbb{R}^m$. We already know the answer for the extremal value $m = 2k$, due to Van Kampen-Flores Theorem [vK32, Flo33]. Since no $(k+1)$-dimensional complex can be embedded into $\mathbb{R}^k$ for dimension reasons, the other extremal value is $k = m$. Even in that case an optimal solution is known: the topological Radon's theorem [BB79] (see also [Mat03, Theorem 5.1.2]) asserts that $\Delta_{k+1}^{(k)}$ does not embed into $\mathbb{R}^k$.

However, there is another direction, in which non-embeddability results can be generalized. We can namely restate "$\Delta_n^{(k)}$ does not embed into $\mathbb{R}^m$" as follows: "For every continuous map $f\colon \Delta_n^{(k)} \to \mathbb{R}^m$, there exist two distinct points $x_1, x_2 \in \Delta_n^{(k)}$ such that $f(x_1) = f(x_2)$." So it is natural to ask which conditions guarantee that for every continuous map $f\colon \Delta_n^{(k)} \to \mathbb{R}^m$ there are r distinct points $x_1, \dots, x_r \in \Delta_n^{(k)}$ with $f(x_1) = f(x_2) = \ldots = f(x_r)$. A generalization of the topological Radon's theorem in this direction is known as topological Tverberg's theorem [Öza87, Vol96a]: If r is a prime power[8] then for every continuous map $\Delta_{(m+1)(r-1)} \to \mathbb{R}^m$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of $\Delta_{(m+1)(r-1)}$ such that $f(\sigma_1) \cap f(\sigma_2) \cap \dots \cap f(\sigma_r) \neq \emptyset$. Consequently, if r is a prime power then for every continuous map $f\colon \Delta_{(m+1)(r-1)} \to \mathbb{R}^m$ there are r distinct points $x_1, \dots, x_r$ satisfying $f(x_1) = f(x_2) = \ldots = f(x_r)$ if and only if $n \geq (m+1)(r-1)$.

For r prime Blagojević, Matschke and Ziegler [BMZ15] proved a version of the topological Tverberg theorem where the position of the points $x_1, \dots, x_r \in \Delta_n^{(n)}$ is somewhat controlled. Their topological 'optimal colored Tverberg theorem' asserts that if $n = (m+1)(r-1)$ and we divide the $n+1$ vertices of

[5]Here we only state a special case of Volovikov's result. It is obtained by setting $j = q = 2$, $m = 2k$, $s = k+1$ and $N = 2k+2$ in item 3 of Volovikov's main result. Moreover, the original result is stated in terms of cohomology, i.e., it asserts that $f^*\colon H_*(M; \mathbb{Z}_2) \to H^*\left(\Delta_{2k+2}^{(k)}; \mathbb{Z}_2\right)$ cannot be trivial. The condition then implies $f_* \neq 0$ by the following argument. By the Universal Coefficient Theorem [Mun84, 53.5], $H_k(\,\cdot\,; \mathbb{Z}_2)$ and $H^k(\,\cdot\,; \mathbb{Z}_2)$ are dual vector spaces, and f^* is the adjoint of f_*, hence triviality of f_* implies that of f^*.

[6]An almost embedding is a continuous map for which disjoint faces of $\Delta_{2k+2}^{(k)}$ have disjoint images.

[7]Since $H_*(\mathbb{R}^{2k}, \mathbb{Z}_2) = 0$, this result can be viewed as a generalization Van Kampen-Flores theorem.

[8]It was an open question whether the result can be extended to general r, see [Mat03, p. 154]. Recently Frick announced a counterexample [Fri15], which is built on methods of Mabillard and Wagner [MU14].

$\Delta_n^{(n)}$ into color classes such that each class contains at most $r-1$ points, then we always find r disjoint rainbow[9] faces $\sigma_1, \dots, \sigma_r \subseteq \Delta_n^{(n)}$ such that $f(\sigma_1) \cap f(\sigma_2) \cap \dots \cap f(\sigma_r) \neq \emptyset$. Matoušek, Tancer and Wagner provided an alternative geometric proof [MTW12].

We replace the convex hulls with affine ones and obtain an algebraic analogue of the 'optimal colored Tverberg theorem' that is valid for all positive integers r and all fields $\mathbb{F}$:

Theorem 1.3. *Let $\mathbb{F}$ be a field, $r \geq 1$ an integer and $\mathbb{A}$ a finitely dimensional affine space over $\mathbb{F}$. If $N \geq (r-1)(\dim \mathbb{A}+1)+1$ is an integer and C is an N-element set partitioned into $m+1$ "color classes"*

$$C = C_0 \uplus \dots \uplus C_m,$$

where $|C_0| \leq r$ and $|C_i| \leq r-1$ for all $i = 1, \dots, m$, then for every map $\psi\colon C \to \mathbb{A}$, it is possible to split C into r sets $F_1, \dots, F_r \subseteq C$ satisfying

(A) $|C_i \cap F_j| \leq 1$ for every $i \in \{0, 1, \dots, m\}$, $j \in \{1, \dots, r\}$, and

(B) $\operatorname{aff}\big(\psi(F_1)\big) \cap \dots \cap \operatorname{aff}\big(\psi(F_r)\big) \neq \emptyset$.

We prove this theorem in Chapter 2. We use it then in Chapter 3 to prove[10] Theorem 1.2.

Since our proof only uses the fact that aff is a closure operator, we also obtain the following matroidal version:

Theorem 1.4. *Let M be a matroid (not necessary finite) with rank function r. Suppose further that the rank $r(M)$ is finite. Let $k \geq 1$ be an integer. If $N > (k-1)r(M)$ is an integer and C is an N-element set partitioned into $m = r(M)$ "color classes"*

$$C = C_0 \uplus \dots \uplus C_m,$$

where $|C_0| \leq r$ and $|C_i| \leq r-1$ for all $i = 1, \dots, m$, then for every map $\psi\colon C \to \mathbb{A}$, it is possible to split C into r sets $F_1, \dots, F_r \subseteq C$ satisfying

(A) $|C_i \cap F_j| \leq 1$ for every $i \in \{0, 1, \dots, m\}$, $j \in \{1, \dots, r\}$, and

(B) $\operatorname{cl}\big(\psi(F_1)\big) \cap \dots \cap \operatorname{cl}\big(\psi(F_r)\big) \neq \emptyset$,

where cl is the closure operator on M.

Observe that Theorem 1.3 differs from "optimal colored Tverberg theorem" [BMZ15] by considering affine hulls and allowing $|C_0|$ to contain r points instead of $r-1$. We also note that without requiring the color constraints (condition (A)), the algebraic Tverberg theorem is easy to prove, see Lemma 2.8.

We relate Theorem 1.4 to the refuted Eckhoff's partition conjecture [Eck00]: Let X be a set and $\operatorname{wcl}\colon 2^X \to 2^X$ a map satisfying $\operatorname{wcl}(\operatorname{wcl}(X)) = \operatorname{wcl}(X)$ and $A \subseteq B \Rightarrow \operatorname{wcl}(A) \subseteq \operatorname{wcl}(B)$. (We call such a map a weak closure operator on X.) Define $t_r(X)$ to be the largest size of a (multi)set in X which cannot be partitioned into r parts whose weak closures have a point in common. Eckhoff asked whether it is true that for every such weak closure operator one has $t_r \leq t_2(r-1)$. An affirmative answer would have implied a combinatorial proof of topological Tverberg theorem. The question was answered negatively[11] by Bukh [Buk10].

However, it is still possible to prove Tverberg type theorems for some classes of weak closure operators: Theorem 1.4 is a very strong Tverberg type theorem for matroids.

We also note that the proof of Theorem 1.4 provides some insight into the difficulties one encounters when trying to prove Rota basis conjecture [HR94, Conjecture 4]. Rota conjectured that given a matroid of rank r and r of its bases, it is possible to arrange them into an $r \times r$ matrix such that the rows are permutations of the given bases and the columns of the matrix are also bases.

[9] A face is called rainbow if all its vertices lie in distinct color classes.

[10] If we used an uncolored version of algebraic Tverberg theorem (Lemma 2.8), we would obtain a slightly worse bounds in Theorems 1.2, 1.1 and 1.5, otherwise the proofs would go through. If the reader does not want to go through the technical proof of Theorem 1.3 and is willing to accept worse bounds, we advise him/her to skip Theorems 1.3 and 1.4 and use Lemma 2.8 instead.

[11] A negative answer is also implied by Frick's example [Fri15] that the topological Tverberg theorem cannot be extended to non-prime values of r.

If we let C_i, $i = 0, \dots, m-1$ be the ith basis, our proof of Theorem 1.4 goes through, yielding rainbow sets F_i, $i = 1, \dots, r$ that are "almost bases": F_i is the basis of $\mathrm{cl}\,(F_i \cup F_{i+1} \cup \ldots \cup F_r)$. A careful investigation of the cases where F_i fails to be a basis of the whole matroid may help to understand Rota basis conjecture better.

At the beginning of this section, we have asked when for every continuous map $f\colon \Delta_n^{(k)} \to \mathbb{R}^m$ there are r points $x_1, \dots, x_r$ satisfying $f(x_1) = f(x_2) = \ldots = f(x_r)$. When r is a prime power, topological Tverberg theorem provides an answer for the extremal case $k = n$, in that case the existence of r such points is ensured for every $n \geq (m+1)(r-1)$.

But what happens if $k \leq n$?

If r is a prime number and $m(r-1) \leq rk$, Sarkaria [Sar00] proved that every continuous map $f\colon \Delta^{(k)}_{rk+2r-2} \to \mathbb{R}^m$ has r distinct points $x_1, \dots, x_r \in \Delta^{(k)}_{rk+2r-2}$ satisfying $f(x_1) = f(x_2) = \ldots = f(x_r)$. Volovikov goes even further and shows that[12] if p is a prime number, $q = p^n$, M is an m-dimensional compact manifold and $m(q-1) \leq qk$, then for every continuous map $f\colon \Delta^{(k)}_{qk+2q-2} \to M$ for which $f_*\colon H_*\left(\Delta^{(k)}_{qk+2q+2}; \mathbb{Z}_p\right) \to H_*(M; \mathbb{Z}_p)$ is trivial, there exist q disjoint points $x_1, \dots, x_q$ with $f(x_1) = f(x_2) = \ldots = f(x_q)$.

Using similar combinatorial reduction to Volovikov's result as for Theorem 1.1, we obtain a version of Theorem 1.1 for multiple intersections.

Theorem 1.5. *Let M be a d-dimensional manifold. Let $q = p^n$ be a prime power. Let b be the kth Betti number of M in the homology with $\mathbb{Z}_p$ coefficients. If $k \geq d\left(1 - \frac{1}{q}\right)$, $N_0 = q(k+1) + q - 2$, and $N \geq \binom{N_0}{k} b\,(N_0 - 2k) + N_0 + 1$ then for every map $f\colon \Delta_N^{(k)} \to M$ there exist q disjoint simplices $\sigma_1, \dots, \sigma_q \subseteq \Delta_N^{(k)}$ with $f(\sigma_1) \cap f(\sigma_2) \cap \cdots \cap f(\sigma_q) \neq \emptyset$.*

Theorem 4.4 has very weak assumptions and the bound is relatively weak.

In contrast, the tight version of the Tverberg Theorem for manifolds by Blagojević, Maschke and Ziegler [BMZ11] provides the optimal bound $N \geq (q-1)(\dim M + 1)$, if one maps the whole simplex: There is no $(q-1)$-almost embedding of Δ_N into any d-dimensional M, provided that $N \geq (q-1)(d+1)$ and q is a prime power.

If q is a prime number, they even get colored version: For every continuous map $f\colon \left|\Delta_N^{(k)}\right| \to M$ and every coloring of vertices of $\Delta_N^{(k)}$ such that no q vertices of Δ_N get the same color, there exist q disjoint rainbow faces $\sigma_1, \dots, \sigma_q \subseteq \Delta_N$ such that $f(|\sigma_1|) \cap \cdots \cap f(|\sigma_q|) \neq \emptyset$.

1.2.3 Homological non-embeddability

Chapter 5 is the last in the non-embeddability part of the thesis. In that chapter we provide analogues of Radon's theorem and Van Kampen-Flores theorem for non-trivial chain maps.

Before we state the results precisely, we define a support of a singular chain $\gamma = \sum_{i=1}^{t} \theta_i \in C_l(M; \mathbb{Z}_2)$, where θ_i are l-dimensional singular simplices in M, by

$$\operatorname{supp} \sum_{i=1}^{t} \theta_i = \bigcup_{i=1}^{t} \theta_i(\Delta_l).$$

(Recall that an l-dimensional singular simplex in M is any continuous map $\theta_i\colon \Delta_l \to M$.)

Theorem 1.6 (Homological Radon's lemma)**.** *If $\varphi\colon C_*\left(\Delta^{(d)}_{d+1}; \mathbb{Z}_2\right) \to C_*(\mathbb{R}^d; \mathbb{Z}_2)$ is a nontrivial chain map, then there exist two disjoint faces $\sigma_1, \sigma_2 \in \Delta^{(d)}_{d+1}$ with $\operatorname{supp}\varphi(\sigma_1) \cap \operatorname{supp}\varphi(\sigma_2) \neq \emptyset$.*

Theorem 1.7 (Homological Van Kampen-Flores)**.** *If $\varphi\colon C_*\left(\Delta^{(k)}_{2k+2}; \mathbb{Z}_2\right) \to C_*(\mathbb{R}^k; \mathbb{Z}_2)$ is a nontrivial chain map, then there exist two disjoint faces $\sigma_1, \sigma_2 \in \Delta^{(k)}_{2k+2}$ satisfying $\operatorname{supp}\varphi(\sigma_1) \cap \operatorname{supp}\varphi(\sigma_2) \neq \emptyset$.*

These theorems provide a valuable tool in our study of Helly types theorems in Chapter 6.

The next section of the introduction shows our Helly type result in the context of related theorems.

[12] In fact, he proved a result which is more general than stated here, however, we do not need the full statement.

1.3 General Helly type theorem

Helly's classical theorem [Hel23] states that a finite family of convex subsets of $\mathbb{R}^d$ must have a point in common if any $d+1$ of the sets have a point in common. Together with Radon's and Caratheodory's theorems, two other "very finite properties" of convexity, Helly's theorem is a pillar of combinatorial geometry. Along with its variants (*e.g.* colorful or fractional), it underlies many fundamental results in discrete geometry, from the centerpoint theorem [Rad46] to the existence of weak ε-nets [ABFK92] or the (p,q)-theorem [AK95].

In the contrapositive, Helly's theorem asserts that any finite family of convex subsets of $\mathbb{R}^d$ with empty intersection contains a sub-family of size at most $d+1$ that already has empty intersection. This inspired the definition of the *Helly number* of a family $\mathcal{F}$ of arbitrary sets. If $\mathcal{F}$ has empty intersection then its Helly number is defined as the size of the largest sub-family $\mathcal{G} \subseteq \mathcal{F}$ with the following properties: $\mathcal{G}$ has empty intersection and any proper sub-family of $\mathcal{G}$ has nonempty intersection; if $\mathcal{F}$ has nonempty intersection then its Helly number is, by convention, 1. With this terminology, Helly's theorem simply states that any finite family of convex sets in $\mathbb{R}^d$ has Helly number at most $d+1$.

Helly already realized that bounds on Helly numbers independent of the cardinality of the family are not a privilege of convexity: his *topological* theorem [Hel30] asserts that a finite family of open subsets of $\mathbb{R}^d$ has Helly number at most $d+1$ if the intersection of any sub-family of at most d members of the family is either empty or a *homology cell.*[13] Such *uniform* bounds are often referred to as *Helly type theorems*. In discrete geometry, Helly type theorems were found in a variety of contexts, from simple geometric assumptions (*e.g.* homothets of a planar convex curve [Swa03]) to more complicated implicit conditions (sets of lines intersecting prescribed geometric shapes [Tve89, GHP+06, CGHP08], sets of norms making a given subset of $\mathbb{R}^d$ equilateral [Pet71, Theorem 5], etc.) and several surveys [Eck93, Wen04, Tan13] were devoted to this abundant literature. These Helly numbers give rise to similar finiteness properties in other areas, for instance in variants of Whitney's extension problem [Shv08] or the combinatorics of generators of certain groups [Far09].

Many Helly numbers are established via ad hoc arguments, and decades sometimes go by before a conjectured bound is effectively proven, as illustrated by Tverberg's proof [Tve89] of a conjecture of Grünbaum [Grü58]. Note that this is true not only for the quantitative question (*what is the best bound?*) but also for the existential question (*is the Helly number uniformly bounded?*); in this example, establishing a first bound [Kat86] was already a matter of decades. Substantial effort was devoted to identify general conditions ensuring bounded Helly numbers, and *topological conditions*, as opposed to more geometric ones like convexity, received particular attention. The general picture that emerges is that requiring that intersections have *trivial* low-dimensional homotopy [Mat97] or have *trivial* high-dimensional homology [CGG12] is sufficient (see below for a more comprehensive account).

In the last part of the thesis, we focus on the existential question and give the following new homological sufficient condition for bounding Helly numbers. Note that we consider homology with coefficients over $\mathbb{Z}_2$, denote by $\tilde{\beta}_i(X)$ the ith reduced Betti number (over $\mathbb{Z}_2$) of a space X, and use the notation $\bigcap \mathcal{F} := \bigcap_{U \in \mathcal{F}} U$ as a shorthand for the intersection of a family of sets.

Theorem 1.8. *For any non-negative integers b and d there exists an integer $h(b,d)$ such that the following holds. If $\mathcal{F}$ is a finite family of subsets of $\mathbb{R}^d$ such that $\tilde{\beta}_i\left(\bigcap \mathcal{G}\right) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and every $0 \leq i \leq \lceil d/2 \rceil - 1$ then $\mathcal{F}$ has Helly number at most $h(b,d)$.*

Our proof hinges on a general principle, which we learned from Matoušek [Mat97] but already underlies the classical proof of Helly's theorem from Radon's lemma, to derive Helly type theorems from results of non-embeddability of certain simplicial complexes. The novelty of our approach is to examine these non-embeddability arguments from a homological point of view. This turns out to be a surprisingly effective idea, as homological analogues of embeddings appear to be much richer and easier to build than their homotopic counterparts. More precisely, our proof of Theorem 1.8 builds on two contributions of independent interest:

- In Chapter 5 we reformulate some non-embeddability results in homological terms. We obtain a homological analogue of the Van Kampen-Flores Theorem (Corollary 5.7) and, as a side-product, a

[13] By definition, a homology cell is a topological space X all of whose (reduced, singular, integer coefficient) homology groups are trivial, as is the case if $X = \mathbb{R}^d$ or X is a single point. Here and in what follows, we refer the reader to standard textbooks like [Hat02, Mun84] for further topological background and various topological notions that we leave undefined.

homological version of Radon's lemma (Lemma 5.9). This is part of a systematic effort to translate various homotopy technique to a more tractable homology setting. It builds on, and extends, previous work on homological minors [Wag11].

- By working with homology rather than homotopy, we can then generalize a technique of Matoušek [Mat97] that uses Ramsey's theorem to find embedded structures. This part is contained in Chapter 6.

Our method also proves a bound of $d+1$ on the Helly number of any family $\mathcal{F}$ such that $\tilde{\beta}_i\left(\bigcap \mathcal{G}\right) = 0$ for all $i \leq d$ and all $\mathcal{G} \subsetneq \mathcal{F}$:

Theorem 1.9. *Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^d$ such that $\widetilde{\beta}_i(\bigcap \mathcal{G}) = 0$ for every $\mathcal{G} \subsetneq \mathcal{F}$ and $i = 0, 1, \ldots, d-1$. Then the Helly number of $\mathcal{F}$ is at most $d+1$.*

Theorem 1.9 is a variant of Helly's topological theorem, where the sets of $\mathcal{F}$ are not assumed to be open.[14] Under the weaker assumption that $\tilde{\beta}_i\left(\bigcap \mathcal{G}\right) = 0$ for all subfamilies $\mathcal{G} \subsetneq \mathcal{F}$ but only for $i \leq \lceil d/2 \rceil - 1$, our method still yields a bound of $d+2$ on the Helly number:

Theorem 1.10. *Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^d$ such that $\widetilde{\beta}_i(\bigcap \mathcal{G}) = 0$ for every $\mathcal{G} \subsetneq \mathcal{F}$ and $i = 0, 1, \ldots, \lceil d/2 \rceil - 1$. Then the Helly number of $\mathcal{F}$ is at most $d+2$.*

In both cases the bounds are tight, as shown by Remark 6.5.

Quantitatively, the bound on $h(b,d)$ that we obtain in the general case is very large as it follows from successive applications of Ramsey's theorems. The conditions of Theorem 1.8 relax the conditions of a Helly type theorem of Amenta [Ame96] (see the discussion below) for which a lower bound of $b(d+1)$ is known [Lar68]; we note that a stronger lower bound is possible for $h(b,d)$ (see Example 6.2) but consider narrowing this gap to be outside the scope of the thesis. Qualitatively, Theorem 1.8 is sharp in the sense that all (reduced) Betti numbers $\tilde{\beta}_i$ with $0 \leq i \leq \lceil d/2 \rceil - 1$ need to be bounded to obtain a bounded Helly number (see Example 6.1).

1.3.1 Relation to previous work.

The search for topological conditions that ensure bounded Helly numbers started with Helly's topological theorem [Hel30] (see also [Deb70] for a modern version of the proof) and organized along several directions related to classical questions in topology. Theorem 1.8 unifies topological conditions originating from two different approaches:

- Helly type theorem can be derived from non-embeddability results, in the spirit of the classical proof of Helly's theorem from Radon's lemma. Using this approach, Matoušek [Mat97] showed that it is sufficient to control the *low-dimensional homotopy* of intersections of sub-families to ensure bounded Helly numbers: for any non-negative integers b and d there exists a constant $c(b,d)$ such that any finite family of subsets of $\mathbb{R}^d$ in which every sub-family intersects in at most b connected components, each $(\lceil d/2 \rceil - 1)$-*connected*,[15] has Helly number at most $c(b,d)$. By Hurewicz' Theorem and the Universal Coefficient Theorem [Hat02, Theorem 4.37 and Corollary 3A.6], a k-connected space X satisfies $\tilde{\beta}_i(X) = 0$ for all $i \leq k$. Thus, our condition indeed relaxes Matoušek's, in two ways: by using $\mathbb{Z}_2$-homology instead of the homotopy-theoretic assumptions of k-connectedness[16], and by allowing an arbitrary fixed bound b instead of $b = 0$.

- Helly's topological theorem can be easily derived from classical results in algebraic topology relating the homology/homotopy of the nerve of a family to that of its union: Leray's *acyclic cover theorem* [Bre97, Sections III.4.13, VI.4 and VI.13] for homology, and Borsuk's *Nerve theorem* [Bor48, Bjö03] for homotopy (in that case one considers finite open *good cover*[17]). More

[14] In the original proof, this assumption is crucial and used to ensure that the union of the sets must have trivial homology in dimensions larger than d; this may fail if the sets are not open.

[15] We recall that a topological space X is k-connected, for some integer $k \geq 0$, if every continuous map $S^i \to X$ from the i-dimensional sphere to X, $0 \leq i \leq k$, can be extended to a map $D^{i+1} \to X$ from the $(i+1)$-dimensional disk to X.

[16] We also remark that our condition can be verified algorithmically since Betti numbers are easily computable, at least for sufficiently nice spaces that can be represented by finite simplicial complexes, say. By contrast, it is algorithmically undecidable whether a given 2-dimensional simplicial complex is 1-connected, see, e.g., the survey [Soa04].

[17] An open good cover is a finite family of open subsets of $\mathbb{R}^d$ such that the intersection of any sub-family is either empty or is contractible (and hence, in particular, a homology cell).

general Helly numbers were obtained via this approach by Dugundji [Dug66], Amenta [Ame96][18], Kalai and Meshulam [KM08], and[19] Colin de Verdière et al. [CGG12]. The outcome is that if a family of subsets of $\mathbb{R}^d$ is such that any sub-family intersects in at most b connected components, each a homology cell (over $\mathbb{Q}$), then it has Helly number at most $b(d+1)$. This therefore relaxes Helly's original assumption by allowing intersections of sub-families to have $\tilde{\beta}_0$'s bounded by an arbitrary fixed bound b instead of $b = 0$. Theorem 1.8 makes the same relaxation for the $\tilde{\beta}_1$'s, $\tilde{\beta}_2$'s, $\ldots \tilde{\beta}_{\lceil d/2 \rceil - 1}$'s and drops *all* assumptions on higher-dimensional homology, including the requirement that the sets are open (which is used to control the ($> d$)-dimensional homology of intersections).

Let us highlight two Helly numbers that stand out in this line of research as *not* subsumed (qualitatively) by Theorem 1.8. On the one hand, Eckhoff and Nischke [EN09] gave a purely combinatorial argument that derives the theorems of Amenta [Ame96] and Kalai and Meshulam [KM08] from Helly's convex and topological theorems. On the other hand, Montejano [Mon13] relaxed Helly's original assumption on the intersection of sub-families of size $k \leq d+1$ from being a homology cell into having trivial $d-k$ homology (so only one Betti number needs to be controlled for each intersection, but it must be zero). These results neither contain nor are contained in Theorem 1.8.

We notice that other non-topological structural conditions, known to ensure bounded Helly numbers, also fall under the umbrella of Theorem 1.8. As already observed by Motzkin [Mot55, Theorem 7] (see also Deza and Frankl [DF87]), any family of real algebraic subvarieties of $\mathbb{R}^d$ defined by polynomials of degree at most k has Helly number bounded by a function of d and k (more precisely, by the dimension of the vector subspace of $\mathbb{R}[x_1, x_2, \ldots, x_d]$ spanned by these polynomials); since the Betti numbers of an algebraic variety in $\mathbb{R}^n$ can be bounded in terms of the degree of the polynomials that define it [Mil63, Tho65], this also follows from Theorem 1.8. We give some other examples in Section 1.3.2, where we easily derive from Theorem 1.8 generalizations of various existing Helly type theorems.

Note that Theorem 1.8 is similar, in spirit, to some of the general relations between the growth of Betti numbers and *fractional* Helly theorems conjectured by Kalai and Meshulam [Kal04, Conjectures 6 and 7]. Kalai and Meshulam, in their conjectures, allow a polynomial growth of the Betti numbers in $|\bigcap \mathcal{G}|$. As the following example shows, Theorem 1.8 is also sharp in the sense that even a linear growth of Betti number, already in $\mathbb{R}^1$, may yield unbounded Helly numbers. In particular, the conjectures of Kalai and Meshulam cannot be strengthened to include Theorem 1.8.

Example 1.11. *Consider a positive integer n and open intervals $I_i := (i-1.1; i+0.1)$ for $i \in [n]$. Let $X_i := [0,n] \setminus I_i$. The intersection of all X_i is empty but the intersection of any proper subfamily is nonempty. In addition, the intersection of k such X_i can be obtained from $[0,n]$ by removing at most k open intervals, thus the reduced Betti numbers of such intersection are bounded by k.*

1.3.2 Further consequences

We conclude this introduction with a few implications of our main result.

New geometric Helly type theorems. The main strength of our result is to show that very weak assumptions on families of sets are enough to guarantee a bounded Helly number. This can be used to identify new Helly type theorems, for instance by detecting easily generalizations of known results, as we now illustrate on two Helly type theorems of Swanepoel.

A first example is given by a Helly type theorem for hollow boxes [Swa99], which generalizes (qualitatively) as follows:

Corollary 1.12. *For any integers s, k, d there exists an integer $h(s,k,d)$ such that the following holds. Let S be a set of s vectors in $\mathbb{R}^d$, and let $\mathcal{F} = \{U_1, U_2, \ldots, U_n\}$ where U_i is the k-skeleton of some polytope in $\mathbb{R}^d$ whose facets all have their normal vector in S. Then $\mathcal{F}$ has Helly number at most $h(s,k,d)$.*

[18] The role of nerves is implicit in Amenta's proof but becomes apparent when compared to an earlier work of Wegner [Weg75] that uses similar ideas.

[19] The result of Colin de Verdière et al. [CGG12] holds in any paracompact topological space; Theorem 1.8 only subsumes the $\mathbb{R}^d$ case.

Swanepoel's result corresponds to the case $k = d-1$ and $S = \{\pm e_1, \pm e_2, \dots, \pm e_d\}$ where $(e_1, e_2, \dots, e_d)$ is a basis of $\mathbb{R}^d$.

Proof. We need to verify the assumptions of Theorem 1.8, that is, we consider a subfamily $\mathcal{G} = \{U_i \colon i \in I\} \subseteq \mathcal{F}$ and we check that $\tilde{\beta}_i(\bigcap \mathcal{G})$ is bounded by a function of s and d for any $i \geq 0$ (according to the assumptions of Theorem 1.8, it would be sufficient to consider $i \leq \lceil d/2 \rceil - 1$, but in this case, there is no difference in reasoning for other values of i).

Let $\mathcal{P} = \mathcal{P}(S)$ be the set of all polytopes which can be obtained as an intersection of half-spaces with the normal vectors to their boundary hyperplanes in S. Let $P_i \in P$ be a polytope such that U_i is a (polyhedral) subcomplex of P_i.

Let us consider the polytope $P = \bigcap_{i \in I} P_i$. From the definition of $\mathcal{P}$ we immediately deduce that $P \in \mathcal{P}$. Moreover, the intersection $U := \bigcap G$ is a polyhedral subcomplex of P. (The faces U are of form $\bigcap_{i \in I} \sigma_i$ where σ_i is a face of U_i; see [RS72, Exercise 2.8(5) + hint].)

Since $P \in \mathcal{P}$ we deduce that it has at most $2s$ facets. By the dual version of the upper bound theorem [Zie95, Theorem 8.23], the number of faces of P is bounded by a function of s and d. Consequently, $\tilde{\beta}_i(U)$ is bounded by a function of s and d, since U is a subcomplex of P. □

A second example concerns a Helly type theorem for families of translates and homothets of a convex curve [Swa03], which are special cases of families of *pseudo-circles*. More generally, a family of *pseudo-spheres* is defined as a set $\mathcal{F} = \{U_1, U_2, \dots, U_n\}$ of subsets of $\mathbb{R}^d$ such that for any $\mathcal{G} \subseteq \mathcal{F}$, the intersection $\bigcap \mathcal{G}$ is homeomorphic to a k-dimension sphere for some $k \in \{0, 1, \dots, d-1\}$ or to a single point. The case $b = 1$ of Theorem 1.8 immediately implies the following:

Corollary 1.13. *For any integer d there exists an integer $h(d)$ such that the Helly number of any finite family of pseudo-spheres in $\mathbb{R}^d$ is at most $h(d)$.*

Note that the case of Euclidean spheres, contained in Corollary 1.13, also received some attention [Mae89, DF87].

Generalized linear programming. Theorem 1.8 also has consequences in the direction of optimization problems. Various optimization problems can be formulated as the minimization of some function $f : \mathbb{R}^d \to \mathbb{R}$ over some intersection $\bigcap_{i=1}^n C_i$ of subsets $C_1, C_2, \dots, C_n$ of $\mathbb{R}^d$. If, for $t \in \mathbb{R}$, we let $L_t = f^{-1}((-\infty, t])$ and $\mathcal{F}_t = \{C_1, C_2, \dots, C_n, L_t\}$ then

$$\min_{x \in \bigcap_{i=1}^n C_i} f(x) = \min \left\{ t \in \mathbb{R} : \bigcap \mathcal{F}_t \neq \emptyset \right\}.$$

If the Helly number of the families $\mathcal{F}_t$ can be bounded *uniformly* in t by some constant h then there exists a subset of $h-1$ constraints $C_{i_1}, C_{i_2}, \dots, C_{i_{h-1}}$ that suffice to define the minimum of f:

$$\min_{x \in \bigcap_{i=1}^n C_i} f(x) = \min_{x \in \bigcap_{j=1}^{h-1} C_{i_j}} f(x).$$

A consequence of this observation, noted by Amenta [Ame94], is that the minimum of f over $C_1 \cap C_2 \cap \ldots \cap C_n$ can[20] be computed in randomized $O(n)$ time by *generalized linear programming* [SW92]. Together with Theorem 1.8, this implies that an optimization problem of the above form can be solved in randomized linear time if it has the property that every intersection of some subset of the constraints with a level set of the function has bounded "topological complexity" (measured in terms of the sum of the first $\lceil d/2 \rceil$ Betti numbers). Let us emphasize that this linear-time bound holds in a real-RAM model of computation, where any constant-size subproblems can be solved in $O(1)$-time; it therefore concerns the *combinatorial difficulty* of the problem and says nothing about its *numerical difficulty*.

Since we use many notions and facts from different areas of mathematics and since some chapters are independent of the others, we provide, for the readers' convenience, the basic definitions and fact at the beginning of the corresponding chapter. However, to omit repetitions, we respect the logical dependency of the chapters. For further convenience, we also provide a list of used symbols at the end of the thesis.

[20] This requires f and $C_1, C_2, \dots, C_n$ to be generic in the sense that the number of minima of f over $\cap_{i \in I} C_i$ is bounded uniformly for $I \subseteq \{1, 2, \dots, n\}$.

Chapter 2

Colorful algebraic Tverberg type theorem

This chapter contains the proof of the colorful algebraic Tverberg type theorem (Theorem 1.3) and its matroidal generalization, Theorem 1.4. The proof is algorithmic but somewhat technical. Hence we also provide Lemma 2.8, a variant of Theorem 1.3, which is more accessible, but provides worse bounds in later applications. Therefore, readers who do not want to go through all the technicalities and are willing to accept a slightly worse bound in Theorems 1.2, 1.1 and 1.5 may jump to Chapter 3 directly after the proof of Lemma 2.8 and hence skip sections 2.3 and 2.4.

Because of the algorithmic character of the proof, we can also address the complexity question and hence state stronger versions of Theorems 1.3 and 1.4 – Theorems 2.12 and 2.13.

2.1 Preliminaries – Affine spaces

Before we state the main results, we recall some definitions and basic facts from linear algebra. The notions that we leave undefined can be found in any linear algebra textbook, see e.g. [MLB88]. We also refer the reader to [Ben95] for detailed proofs of all here mentioned statements, although all of them are relatively straightforward.

Throughout the text we use the symbol $\mathbb{Z}_n$ to denote $\mathbb{Z}/n\mathbb{Z}$, e.g., for p prime $\mathbb{Z}_p$ is the p-element field.

Definition 2.1 (Affine space)**.** *Let $\mathbb{F}$ be a field and V a vector space over $\mathbb{F}$. A subset $\mathbb{A} \subseteq V$ is called an* affine space *over $\mathbb{F}$ (or $\mathbb{F}$-affine space), if $\mathbb{A}$ has the form $\mathbb{A} = \{\mathbf{a} + \mathbf{v} \mid \mathbf{v} \in U\}$, where $\mathbf{a} \in V$ and U is a vector subspace of V. In this case the* dimension $\dim \mathbb{A}$ *is defined as* $\dim U$*.*

If X is an arbitrary set and $\mathbb{F}$ a field, we can turn X into an $\mathbb{F}$-affine space $\mathcal{M}(X;\mathbb{F})$ in the following way:

Definition 2.2. *Let X be a set and $\mathbb{F}$ a field. A* multipoint *μ over $\mathbb{F}$ in X is a formal sum*[1]

$$\mu = \sum_{x \in X} a_x x,$$

where $a_x \in \mathbb{F}$, only finitely many a_x are non-zero and $\sum_{x\in X} a_x = 1$. The set of all multipoints over $\mathbb{F}$ in X *is denoted by $\mathcal{M}(X;\mathbb{F})$, where the field $\mathbb{F}$ may be omitted if it is clear from the context. The* support $\operatorname{supp}\mu$ *of a multipoint $\mu = \sum_{x\in X} a_x x$ is defined as $\operatorname{supp}\mu := \{x \in X \mid a_x \neq 0\}$.*

It can be easily checked that $\mathcal{M}(X;\mathbb{F})$ is an affine subspace of $\mathbb{F}^X$. Because the map $x \mapsto 1 \cdot x$ is an inclusion of X into $\mathcal{M}(X;\mathbb{F})$, we regard $\mathcal{M}(X;\mathbb{F})$ as a superset of X.

[1] We have chosen the name multipoint for the following reason: Imagine we are tourists and we want to go from valley a to mountain peek b and we have three possibilities: either to go through cottage c_1, c_2 or c_3. A trip through a combination $c_1 - c_2 + c_3$ then corresponds to visiting multiple cottages in the following order: climbing the mountain up via the path around c_1, then going down around c_2 and then climbing up around c_3. The "route" then goes through multiple points. It is useful to keep this image in mind during some proofs in Chapter 3.

Definition 2.3. *Let $\mathbb{A}$ be an affine space over $\mathbb{F}$ and $B \subseteq \mathbb{A}$ a set. We define the* affine hull *of B as*

$$\operatorname{aff}_{\mathbb{F}} B := \left\{ \sum_{i \in I} a_i \mathbf{b}_i \mid I \text{ is a finite set, } \mathbf{b}_i \in B, a_i \in \mathbb{F} \text{ and } \sum_{i \in I} a_i = 1 \right\}. \tag{2.1}$$

If $\mathbb{F}$ is an ordered field, we further define the convex hull *of B as*

$$\operatorname{conv}_{\mathbb{F}} B := \left\{ \sum_{i \in I} a_i \mathbf{b}_i \mid I \text{ is a finite set, } \mathbf{b}_i \in B, a_i \in \mathbb{F}, 0 \leq a_i \leq 1 \text{ and } \sum_{i \in I} a_i = 1 \right\}. \tag{2.2}$$

If the field $\mathbb{F}$ is clear from the context, we omit it from the subscript.

It can be easily proven that $\operatorname{aff} B$ is an affine space over $\mathbb{F}$ and the operator aff satisfies the following axioms:

Observation 2.4. *For every $X, Y \subseteq \mathbb{A}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{A}$:*

(CL1) $X \subseteq \operatorname{aff}(X)$,

(CL2) $X \subseteq Y \Rightarrow \operatorname{aff}(X) \subseteq \operatorname{aff}(Y)$,

(CL3) $\operatorname{aff}\big(\operatorname{aff}(X)\big) = \operatorname{aff}(X)$ *and*

(CL4) $\mathbf{y} \in \operatorname{aff}(X \cup \mathbf{x}) \setminus \operatorname{aff}(X) \Rightarrow \mathbf{x} \in \operatorname{aff}(X \cup \mathbf{y})$ *(Mac Lane-Steinitz exchange property)*

Let us briefly demonstrate that aff really satisfies the exchange axiom: If $\mathbf{y} \in \operatorname{aff}\big(X \cup \{\mathbf{x}\}\big) \setminus \operatorname{aff} X$, then $\mathbf{y} = \sum_{i \in I} a_i \mathbf{x}_i$, where all $\mathbf{x}_i \in X \cup \{\mathbf{x}\}$. Without loss of generality, we may assume that all $\mathbf{x}_i$ are distinct. Since $\mathbf{y} \notin \operatorname{aff} X$, one $\mathbf{x}_i$, say $\mathbf{x}_1$, equals $\mathbf{x}$ and the corresponding coefficient a_1 is nonzero. So we may write $\mathbf{y} = a_1 \mathbf{x} + \sum_{i \in I \setminus \{1\}} a_i \mathbf{x}_i$, where $a_1 + \sum_{i \in I \setminus \{1\}} a_i = 1$, hence $\mathbf{x} = \frac{1}{a_1} \mathbf{y} - \sum_{i \in I \setminus \{1\}} \frac{a_i}{a_1} \mathbf{x}_i$, where $\frac{1}{a_1} - \sum_{i \in I \setminus \{1\}} \frac{a_i}{a_1} = 1$ (note that $a_1 \neq 0$) and the exchange axiom follows.

Definition 2.5. *If M is a set and $\operatorname{cl}: 2^M \to 2^M$ a map satisfying (CL1)–(CL4) we say that cl is a* closure operator[2]. *A set with a closure operator is called a (not necessary finite)* matroid.

Definition 2.6. *Let $\mathbb{A}$ be an affine space over a field $\mathbb{F}$. A set $B \subseteq \mathbb{A}$ is called* affinely independent, *if $\mathbf{b} \notin \operatorname{aff}\big(B \setminus \{\mathbf{b}\}\big)$ holds true for every element $\mathbf{b} \in B$. Note that the notion of independence makes sense for any closure operator* cl. *In such a case, we define the* rank $r(C)$ of a set C *as the size of an inclusion maximal independent subset of C.*

Note that (CL4) implies that the rank is well-defined and in fact equals the size of any maximal independent subset in $\operatorname{cl}(C)$. Moreover, for affine spaces we have $r(B) = \dim B + 1$.

Observation 2.7. *Let $\mathbb{A}$ be an affine space over a field $\mathbb{F}$.*

1. *Every affinely independent set $B \subseteq \mathbb{A}$ has at most $\dim \mathbb{A} + 1$ elements.*

2. *If B is affinely independent, then $\dim \operatorname{aff} B + 1 = \operatorname{card} B$.*

3. *If $B \subseteq \mathbb{A}$ is an affinely independent set and $\mathbf{c} \in \mathbb{A} \setminus \operatorname{aff} B$, then $B \cup \{\mathbf{c}\}$ is an affinely independent set.*

4. *For all $B, C \subseteq \mathbb{A}$ the following equality holds: $\operatorname{aff}(B \cup C) = \operatorname{aff}\big(B \cup \operatorname{aff} C\big)$.*

The proof of Observation 2.7 is an easy exercise, Moreover, the properties in fact follow from axioms (CL1)–(CL4). Hence the observation remains valid, even if we replace aff with an arbitrary closure operator and $\dim \mathbb{A}$ with $r(A) - 1$.

Let us briefly demonstrate that Observation 2.7 is indeed implied by (CL1)–(CL4) and does not depend on any other properties of aff:

[2] A typical examples of closure operators are: identity on any set, affine closure on an affine space, linear span in a vector space, algebraic closure, etc.; there are also some closure operators arising in graph theory [Oxl11]

Proof. Properties 1 and 2 are trivial (noting that the definition of dimension (or rank) does not depend on the chosen maximal independent set).

Let us now prove 3: If $B \cup \{\mathbf{c}\}$ is affinely dependent, there exists a point $\mathbf{b} \in B \cup \{\mathbf{c}\}$ such that $\mathbf{b} \in \operatorname{aff}\big((B \cup \{\mathbf{c}\}) \setminus \{\mathbf{b}\}\big)$. Clearly $\mathbf{b} \neq \mathbf{c}$, hence $\mathbf{b} \in B$. Because B is affinely independent, $\mathbf{b} \notin \operatorname{aff}\big(B \setminus \{\mathbf{b}\}\big)$, hence $\mathbf{b} \in \operatorname{aff}((B \cup \{\mathbf{c}\}) \setminus \mathbf{b}) \setminus \operatorname{aff}\big(B \setminus \{\mathbf{b}\}\big)$. Using (CL4), we see that $\mathbf{c} \in \operatorname{aff}((B \setminus \{\mathbf{b}\}) \cup \{\mathbf{b}\}) = \operatorname{aff}(B)$ – a contradiction.

To prove 4, we use axioms (CL1)–(CL3). By (CL1) and (CL2) we have $B \subseteq \operatorname{aff}(B \cup C)$ and $\operatorname{aff}(C) \subseteq \operatorname{aff}(B \cup C)$. Altogether $B \cup \operatorname{aff}(C) \subseteq \operatorname{aff}(B \cup C)$. Using (CL1) and (CL2) once again, we see that $\operatorname{aff}(B \cup C) \subseteq \operatorname{aff}(B \cup \operatorname{aff}(C)) \subseteq \operatorname{aff}(\operatorname{aff}(B \cup C))$. By (CL3) the leftmost and the rightmost term coincide, hence $\operatorname{aff}(B \cup \operatorname{aff}(C)) = \operatorname{aff}(B \cup C)$ as well. □

2.2 Prelude

In this section we prove a simple algebraic Tverberg type result (Lemma 2.8), so far without any color restriction. We include this lemma for two reasons. Firstly, its proof clearly demonstrates the basic idea which we use while proving the full colorful version, whereas in the full proof the idea is somewhat obfuscated by the technical details. Secondly, the readers who do not want to go through the technical details in the proof of the colorful version, may use Lemma 2.8 instead of the full colorful version in the proofs of Chapters 3 and 4. The proofs then go through, although with slightly worse bounds.

Lemma 2.8. *Let $\mathbb{A}$ be an affine space over a field $\mathbb{F}$, $C \subseteq \mathbb{A}$ a (multi)set and $r > 0$ an integer. If $|C| > (\dim \mathbb{A} + 1)(r-1)$, then there exist r disjoint sets $F_1, F_2, \ldots, F_r \subseteq C$ such that $\operatorname{aff} F_1 \cap \operatorname{aff} F_2 \cap \cdots \cap \operatorname{aff} F_r \neq \emptyset$.*

Proof. We prove the statement by induction over r. If $r = 1$, then C is nonempty and we may choose $F_1 = C$.

So assume that $r > 1$. Let $\mathbb{B} = \operatorname{aff} C$. Because $\mathbb{B} \subseteq \mathbb{A}$, the dimension of $\mathbb{B}$ is at most $\dim \mathbb{A}$. Therefore, there exists a set $F_r \subseteq C$ with at most $\dim \mathbb{A} + 1$ points such that $\operatorname{aff} F_r = \mathbb{B}$. We set $C' := C \setminus F_r$. Then $|C'| > (\dim \mathbb{A} + 1)(r-2)$, so we may apply induction and obtain $r-1$ pairwise disjoint sets $F_1, F_2, \ldots, F_{r-1} \subseteq C'$ with $\operatorname{aff} F_1 \cap \operatorname{aff} F_2 \cap \cdots \cap \operatorname{aff} F_{r-1} \neq \emptyset$. Since $F_i \subset C \setminus F_r$, the r sets $F_1, \ldots, F_r$ are also pairwise disjoint. For all $i = 1, \ldots, r-1$ we have $F_i \subseteq C \setminus F_r = C' \subseteq \operatorname{aff} \mathbb{B} = \operatorname{aff} F_r$. It follows that $\operatorname{aff} F_1 \cap \operatorname{aff} F_2 \cap \cdots \cap \operatorname{aff} F_{r-1} \cap \operatorname{aff} F_r \neq \emptyset$ and the proof is finished. □

We note that the bound $|C| > (\dim \mathbb{A} + 1)(r-1)$ in Lemma 2.8 is optimal.

Observation 2.9. *Let $r > 0$ be an integer, $\mathbb{F}$ a field and $\mathbb{A}$ an affine space. Then there exists a set C having exactly $|C| = (\dim A + 1)(r-1)$ elements and a map $\psi \colon C \to \mathbb{A}$, such that every r pairwise disjoint subsets $F_1, \ldots, F_r \subseteq C$ satisfy $\bigcap_{i=1}^{r} \operatorname{aff}_{\mathbb{F}} (\psi(F_i)) = \emptyset$.*

For simplicity we show the observation only for finitely dimensional affine spaces $\mathbb{A}$. We note that if $\dim \mathbb{A}$ is infinite, the same construction goes through, the only difficulty is how to write down infinite sequences properly.

Proof of Observation 2.9. Let $\mathbb{F}$ be a field and let $\mathbb{A}$ be an m-dimensional affine space over $\mathbb{F}$. Without loss of generality, we may assume $\mathbb{A} = \mathbb{F}^m$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be the standard basis vectors of $\mathbb{F}^m$. Let C be a set containing exactly $(m+1)(r-1)$ elements. We group the elements in C into $(m+1)$ groups, each containing $(r-1)$ elements. We define ψ as follows:

$$\psi(x) = \begin{cases} \mathbf{e}_i & \text{if } x \text{ is in the } i\text{th group, where } i \leq m, \\ 0 & \text{if } x \text{ is in the } (m+1)\text{th group.} \end{cases}$$

Since $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m, 0\}$ are affinely independent, and every point $\mathbf{e}_i$ or 0 is missing it at least one F_i, we see that there are no r pairwise disjoint subsets $F_1, F_2, \ldots, F_r \subseteq C$ satisfying

$$\bigcap_{i=1}^{r} \operatorname{aff}_{\mathbb{F}} (\psi(F_i)) \neq \emptyset,$$

which proves the optimality of Lemma 2.8. □

Now we reformulate Lemma 2.8 in the terms of multipoints[3], which provides some geometric intuition for the proofs in Chapter 3. If μ, μ' are two multipoints satisfying $\operatorname{supp}(\mu) \cap \operatorname{supp}(\mu') = \emptyset$, we say that μ and μ' are *disjoint*.

If X is a set, $\mathbb{F}$ a field, $\mathbb{A}$ an affine space over $\mathbb{F}$ and $\psi\colon X \to \mathbb{A}$ a map, then the map ψ can be extended to a map $\psi\colon \mathcal{M}(X;\mathbb{F}) \to \mathbb{A}$ via

$$\psi\left(\sum_{x\in X} a_x x\right) = \sum_{x\in X} a_x \psi(x).$$

Lemma 2.10. *Let $\mathbb{A}$ be an affine space over a field $\mathbb{F}$, $\psi\colon C \to \mathbb{A}$ a set and $r > 0$ an integer. If $|C| > (\dim \mathbb{A} + 1)(r-1)$, then there exists r disjoint multipoints $\mu_1, \mu_2, \ldots, \mu_r \in \mathcal{M}(C;\mathbb{F})$ such that $\psi(\mu_1) = \psi(\mu_2) = \ldots = \psi(\mu_r)$.*

Proof. Let $C' = \varphi(C)$ be a multiset. If we apply Lemma 2.8, we obtain r sets $F'_1, \ldots, F'_r \subseteq C'$ with $\operatorname{aff} F'_1 \cap \operatorname{aff} F'_2 \cap \cdots \cap \operatorname{aff} F'_r \neq \emptyset$.

If $F' \subseteq C$ is a set and $\mathbf{a}$ is a point in $\operatorname{aff}(F')$, then $\mathbf{a} = \sum_{x\in F'} a_x x$, for some a_x, where only finitely many of a_x are non-zero and $\sum a_x = 1$.

If $\mathbf{a}$ is a point in $\operatorname{aff} F'_1 \cap \operatorname{aff} F'_2 \cap \cdots \cap \operatorname{aff} F'_r$, we have

$$\mathbf{a} = \sum_{x\in F_1} a_{x,1}\psi(x) = \sum_{x\in F'_2} a_{x,2}\psi(x) = \ldots = \sum_{x\in F'_r} a_{x,r}\psi(x),$$

where only finitely many $a_{x,i}$ are nonzero and $\sum a_{x,i} = 1$ for all $i = 1, \ldots, r$.

If we set $\mu_i = \sum_{x\in F_i} a_{x,i} x \in \mathcal{M}(C;\mathbb{F})$ for all $i = 1, \ldots, r$, the equality rewrites as $\mathbf{a} = \psi(\mu_1) = \psi(\mu_2) = \ldots = \psi(\mu_r)$; in particular $\psi(\mu_1) = \psi(\mu_2) = \ldots = \psi(\mu_r)$. □

2.3 Statement of the colorful algebraic theorem

Before we formulate the result precisely, we introduce some terminology.

Definition 2.11. *Let $m \geq 0$ be an integer. Let a set C be partitioned into $(m+1)$ pairwise disjoint non-empty sets $C_0, C_1, \ldots, C_m$. We call the sets C_i **color classes**. A subset $S \subseteq C$ satisfying $|S \cap C_i| \leq 1$ for all $i = 0, \ldots, m$, is called **rainbow**. We further define the coloring $c\colon C \to \{0, 1, \ldots, m\}$ via*

$$c(x) = i \quad \text{if and only if } x \in C_i. \tag{2.3}$$

To simplify the notation, we introduce the following convention. If $I \subseteq \mathbb{Z}$ is a subset of $\{0, \ldots, m\}$, we set

$$C_I := \bigcup_{i\in I} C_i. \tag{2.4}$$

We can now restate Theorem 1.3 in more precise form, addressing also the algorithmic aspects.

Theorem 2.12. *Let $m \geq 0, r \geq 1$ be integers, C a set, $\mathbb{F}$ a field, $\mathbb{A}$ a finitely dimensional affine space over $\mathbb{F}$ and $\psi\colon C \to \mathbb{A}$ a map. Let C be partitioned into $(m+1)$ non-empty color classes $C_0, \ldots, C_m$ with $|C_0| \leq r$ and $|C_i| \leq r-1$ for all $i = 1, \ldots, m$. If $|C| > (\dim \mathbb{A} + 1)(r-1)$ then there exist r pairwise disjoint rainbow sets $F_1, \ldots, F_r \subseteq C$ satisfying*

$$\bigcap_{i=1}^{r} \operatorname{aff}_{\mathbb{F}}\left(\psi\left(F_i\right)\right) \neq \emptyset. \tag{2.5}$$

Furthermore, if $\dim \mathbb{A}$ can be computed in time polynomial in m, r and u is the maximal time needed to decide whether a point $p \in C$ and a set $S \subseteq C$ satisfy $\psi(p) \in \operatorname{aff}\big(\psi(S)\big)$, then such sets F_i can be algorithmically found in time polynomial in u,m,r and $|C|$.

[3]See Definition 2.2.

Note that the assumptions $|C| > (\dim \mathbb{A}+1)(r-1)$, $|C_0| \le r$ and $|C_i| \le r-1$ imply that $m \ge \dim \mathbb{A}$.

Since the proof only uses the fact that the operator aff satisfies axioms (CL1)–(CL4), we also obtain the following matroidal version[4]:

Theorem 2.13. *Let M be a matroid (not necessary finite) with rank function r. Let $k \ge 1$ be an integer. If $N > (k-1)r(M)$ is an integer and C an N-element set with a partition into $m+1$ parts ("color classes")*

$$C = C_0 \uplus \ldots \uplus C_m,$$

with $|C_0| \le k$ and $|C_i| \le k-1$ for all $i = 1, \ldots, m$, then for every map $\psi\colon C \to M$, it is possible to split C into k sets $F_1, \ldots, F_k \subseteq C$ satisfying

(A) $|C_i \cap F_j| \le 1$ for every $i \in \{0, 1, \ldots, m\}$, $j \in \{1, \ldots, k\}$, and

(B) $\operatorname{cl}\big(\psi(F_1)\big) \cap \cdots \cap \operatorname{cl}\big(\psi(F_k)\big) \neq \emptyset$,

where cl is the closure operator on M.

Since we only use properties of the closure operator aff that follow from axioms (CL1)–(CL4) in our proof of Theorem 2.12, the proof of Theorem 2.13 follows from the proof of Theorem 2.12, if we replace all occurrences of aff with cl, only the reduction of Theorem 2.12 to Theorem 2.15 needs small adjustment, which we address in Remark 2.16.

Although it would be possible to write the proof down in the more abstract matroidal setting, we prefer the proof with aff for two reasons: Firstly, it provides some geometric intuition, which helps to understand the proof, secondly, we only need the affine version in later constructions.

Before we proceed with the proof, we restate Theorem 2.12 using Definition 2.2 (multipoints). The reformulation provides nice geometric intuition for Chapter 3.

If $m \ge 0$ is an integer and a set C is partitioned into $(m+1)$ color classes $C_0, C_1, \ldots, C_m$ we say that a multipoint $\mu \in \mathcal{M}(C;\mathbb{F})$ is *rainbow*, if $\operatorname{supp}\mu$ is.

Using the definition of multipoints, Theorem 2.12 implies the following:

Theorem 2.14. *Let $m \ge 0, r \ge 1$ be integers, C a set, $\mathbb{F}$ a field, $\mathbb{A}$ a finitely dimensional affine space over $\mathbb{F}$ and $\psi\colon C \to \mathbb{A}$ a map. Let C be partitioned into $(m+1)$ non-empty color classes $C_0, \ldots, C_m$ with $|C_0| \le r$ and $|C_i| \le r-1$ for all $i = 1, \ldots, m$. If $|C| > (\dim \mathbb{A}+1)(r-1)$ then there exist r pairwise disjoint rainbow multipoints $\mu_1, \mu_2, \ldots, \mu_r \subseteq \mathcal{M}(C;\mathbb{F})$ satisfying*

$$\psi(\mu_1) = \psi(\mu_2) = \ldots = \psi(\mu_r). \tag{2.6}$$

Since we are only interested in theoretical applications, we do not address the complexity question in Theorem 2.14.

Proof. In the setting of Theorem 2.14, the assumptions of Theorem 2.12 are satisfied, so there exist r pairwise disjoint rainbow sets $F_1, \ldots, F_r$ with $\operatorname{aff}\big(\psi(F_1)\big) \cap \operatorname{aff}\big(\psi(F_2)\big) \cap \cdots \cap \operatorname{aff}\big(\psi(F_r)\big) \neq \emptyset$. Let $\mathbf{a}$ be a point in the intersection.

Then according to the same considerations as in the proof of Lemma 2.10, $\mathbf{a} = \psi(\mu_1) = \psi(\mu_2) = \ldots = \psi(\mu_r)$, where $\mu_i \in \mathcal{M}(F_i;\mathbb{F}) \subseteq \mathcal{M}(C;\mathbb{F})$ for every $i = 1, 2, \ldots, r$.

Because the sets F_i, $i = 1, \ldots, r$, are pairwise disjoint and rainbow and $\operatorname{supp}\mu_i \subseteq F_i$, the multipoints μ_i are pairwise disjoint and rainbow as well. □

For a better presentation, we reduce Theorem 2.12 to the following statement.

Theorem 2.15. *Let $m \ge 0$ and $r \ge 1$ be integers, C a set, $\mathbb{F}$ a field, $\mathbb{A}$ an m-dimensional affine space over $\mathbb{F}$ and $\psi\colon C \to \mathbb{A}$ a map. Let C be partitioned into $(m+1)$ color classes $C_0, \ldots, C_m$ with $|C_0| \ge r$ and $|C_i| \ge (r-1)$ for all $i = 1, \ldots, m$. Then there exist r pairwise disjoint rainbow sets $F_1, \ldots, F_r \subseteq C$ satisfying*

$$\bigcap_{i=1}^{r} \operatorname{aff}_{\mathbb{F}}\left(\psi\left(F_i\right)\right) \neq \emptyset.$$

Furthermore, if u is the maximal time needed to decide whether a point $p \in C$ and a set $S \subseteq C$ satisfy $\psi(p) \in \operatorname{aff}\big(\psi(S)\big)$, then such sets F_i can be algorithmically found in time $O\big((m+r)|C|m^2u + |C|\big)$.

[4]See Definition 2.5.

Note that in Theorem 2.15 we replace the conditions $|C_0| \le r$ and $|C_i| \le r-1$ from Theorem 2.12 by $|C_0| \ge r$ and $|C_i| \ge r-1$ and add a condition that there are exactly $(\dim \mathbb{A} + 1)$ colors. The main purpose of the conditions $|C_0| \le r$, $|C_i| \le r-1$ in Theorem 2.12 is to ensure that we have enough color classes, in Theorem 2.15 this is stated explicitly as the additional condition; what becomes implicit is the fact that $|C| > (r-1)(\dim \mathbb{A} + 1)$.

The reduction of Theorem 2.12 to Theorem 2.15 follows a well known pattern, a similar reduction previously appeared in the proof of the optimal colored Tverberg theorem [BMZ15] or in Sarkaria's proof for the prime power Tverberg theorem [Sar00, 2.7.3], see also de Longueville's exposition [dL02, Prop. 2.5]. Nevertheless, there is a subtle difference, since we do not now how the affine space $\mathbb{A}$ is represented and it may be unpractical to search for an isomorphism of $\mathbb{A}$ and $\mathbb{F}^d$.

Observation 2.9 shows that the bound $|C| > (\dim A + 1)(r-1)$ in Theorem 2.12 is optimal, which also proves optimality of Theorem 2.13 and Theorem 2.15.

In general, the assumption $|C_0| \le r$ in Theorem 2.12 is also necessary. For example if $\mathbb{A} = \mathbb{F} = \mathbb{R}$, $r = 3$, $m = 1$, $C_0 := \{0, 1, 2, 3\}$, $C_1 := \{4\}$ and ψ is the identity map, then $|C| > (m+1)(r-1)$, but there are no three pairwise disjoint rainbow subsets satisfying (2.5). We note that it is not true that the condition is necessary in all cases. E.g. if $\mathbb{A} = \mathbb{F} = \mathbb{Z}_2$, Theorem 2.12 without any restriction on $|C_0|$ easily follows from the pigeonhole principle.

2.4 Proof of Theorem 2.12

The reduction of Thm. 2.12 to Thm. 2.15. Let $d = \dim \mathbb{A}$. Let 0 be the origin in $\mathbb{F}^{m-d}$ and $\mathbf{e}_1, \dots, \mathbf{e}_{m-d}$ be the standard basis of $\mathbb{F}^{m-d}$. Then $\mathbb{A} \cong \mathbb{A} \times \{0\} \subseteq \mathbb{A} \times \mathbb{F}^{m-d}$.

If $|C| > (d+1)(r-1) + 1$, we throw the superfluous elements of C away. This does not increase the size of any color class, therefore all the assumptions of Theorem 2.12 remain preserved. So we may assume that $|C| = (d+1)(r-1) + 1$.

Now we add $(m+1)(r-1) + 1 - |C| = (m-d)(r-1)$ points to C to obtain set C'. We partition C' into color classes $C'_0, \dots, C'_m$ in such a way that $C_0 \subseteq C'_0$, $|C'_0| = r$, $C_i \subseteq C'_i$ and $|C'_i| = r-1$ for all $i = 1, \dots, m$. This is clearly possible.

We group the $(m-d)(r-1)$ added points into $(m-d)$ groups of $(r-1)$ points. Now we construct map $\psi' \colon C' \to \mathbb{A} \times \mathbb{F}^{m-d}$ as follows: Let o be a fixed element from C. We set

$$\psi'(x) := \begin{cases} \{\psi(o)\} \times \mathbf{e}_i & \text{for all elements } x \text{ in the } i\text{th group } (i = 1, \dots, m-d), \\ \psi(x) \times \{0\} & \text{for } x \in C. \end{cases}$$

Since $\dim \mathbb{A} \times \mathbb{F}^{m-d} = m$, the assumptions of Theorem 2.15 are satisfied, so there are r rainbow sets $F'_1, \dots, F'_r \subseteq C'$ satisfying $\bigcap_{i=1}^r \operatorname{aff}_{\mathbb{F}} (\psi'(F'_i)) \neq \emptyset$.

Observe that by the construction only $(r-1)$ points got mapped onto a fixed point $\{\psi(o)\} \times \mathbf{e}_i$; in particular every point $\{\psi(o)\} \times \mathbf{e}_i$ is missing in at least one $\psi'(F'_i)$. Moreover, the points $\{\psi(o)\} \times \mathbf{e}_i$, $i = 1, \dots, m-d$, are affinely independent and $\operatorname{aff}\Big\{\{\psi(o)\} \times \mathbf{e}_i \mid i = 1, \dots, m-d\Big\} \cap \Big(\mathbb{A} \times \{0\}\Big) = \emptyset$. It follows that $\bigcap_{i=1}^r \operatorname{aff}_{\mathbb{F}} (\psi'(F'_i)) \subseteq \mathbb{A} \times \{0\}$. Moreover, because $\psi' \restriction C = \psi \times \{0\}$, we immediately see that the sets $F_i := F'_i \cap C$ are rainbow subsets of C that satisfy

$$\bigcap_{i=1}^r \operatorname{aff}_{\mathbb{F}} (\psi(F_i)) \neq \emptyset.$$

If d can be computed in polynomial time, the reduction time is also polynomial. Let u be the time needed to decide whether a point $p \in C$ satisfies $\psi(p) \in \operatorname{aff}\big(\psi(S)\big)$, where $S \subseteq C$. Then the time u' needed to decide whether a point $p' \in C'$ satisfies $\psi(p') \in \operatorname{aff}\big(\psi(S')\big)$, where $S' \subseteq C'$, is polynomial in u,d. □

Remark 2.16. *The proof generalizes to matroids as follows. Instead of increasing dimension, we add $(m-d)$ points $p_1, p_2, \dots, p_{m-d}$ to the matroid M and make them mutually independent and also independent on all others. Since we add $(r-1)(m-d)$ points to C, we may simply extend the map by mapping first $r-1$ added elements to p_1, second $r-1$ added elements to p_2, etc. The rest of the proof follows.*

Proof of Theorem 2.15

We show that there is a recursive algorithm that performs the task in time $O\big((m+r)|C|m^2u + |C|\big)$. Its inputs are: set C partitioned into color classes $C_0, \dots, C_m$, map $\psi\colon C \to \mathbb{A}$ and r. Its output is a collection of r pairwise disjoint rainbow sets $F_1, \dots, F_r \subseteq C$ satisfying (2.5). First we describe the algorithm and show that whenever it stops it outputs a correct answer. Then we provide a pseudo-code for the algorithm, bound the running time and provide some optimizations.

Correctness

The recursion

The algorithm runs recursively, constructing inductively sets $F_r, F_{r-1}, \dots, F_1$. If $F_r, \dots, F_{i+1}$ are already constructed, the algorithm either constructs set F_i and calls itself recursively to construct $F_{i-1}, \dots, F_1$, or it decreases the dimension of $\mathbb{A}$ and calls itself recursively to construct $F_i, \dots, F_1$.

We may throw away the additional points, so we may assume that $|C_0| = r$ and $|C_i| = r-1$ for all $i = 1, \dots, m$.

If $r = 1$, we are searching for one nonempty set F_1. Since $|C_0| = 1$ by assumption, we set $F_1 := C_0$. If $m = 0$, $\dim \mathbb{A} = 0$, hence $\psi(x) \in \mathbb{A}$ is the same for all points $x \in C$. In this case $|C_0| = r$. We split C_0 into r disjoint sets $F_1, \dots, F_r$, each containing one point. Then clearly $\operatorname{aff}\big(\psi(F_1)\big) \cap \operatorname{aff}\big(\psi(F_2)\big) \cap \dots \cap \operatorname{aff}\big(\psi(F_r)\big) \neq \emptyset$.

So we may assume that $r > 1$ and $m > 0$. We try to iteratively build sets $G_j \subseteq C$, $j = 0, 1, \dots, m$, satisfying the following three conditions:

(I) $|G_j| = j+1$,

(II) G_j is rainbow and

(III) $\dim \operatorname{aff}\big(\psi(G_j)\big) = j$.

The idea behind the sets G_j is the following: If $\operatorname{aff}\psi(G_j) \supseteq \psi(C)$, G_j is rainbow and $|G_j| \leq \dim \mathbb{A} + 1$, we may put $F_r := G_j$, as we did in the proof of Lemma 2.8. We try to construct such set G_j by starting with $j = 0$, requiring conditions (I)–(III) and increasing j by one at each step, while still maintaining (I)–(III). Then if we succeed in constructing G_j with $\operatorname{aff}\psi(G_j) \supseteq \psi(C)$, the set $F_r = G_j$ is rainbow with at most $\dim \mathbb{A} + 1$ elements and we may continue by induction as in the proof of Lemma 2.8. Unfortunately, it might happen that we do not succeed. In that case, we show that there is a subspace $\mathbb{A}' \subsetneq \mathbb{A}$ so that if we restrict our attention to $\mathbb{A}'$ and $C' = C \cap \psi^{-1}(\mathbb{A}')$, we may apply induction there.

Let us now build sets G_j.

First loop

The first step is easy. Because $|C_0| = r \geq 1$, there exists an element $p \in C_0$. We set $G_0 := \{p\}$. This assignment clearly satisfies conditions (I),(II) and (III).

So suppose $j \geq 0$, we already have set G_j satisfying all the conditions and we want to construct set G_{j+1}.

There are two possibilities what can happen:

1. $\psi(C) \subseteq \operatorname{aff}\big(\psi(G_j)\big)$,
2. $\psi(C) \not\subseteq \operatorname{aff}\big(\psi(G_j)\big)$.

Before going into technical details we sketch the overall idea how we deal with the particular cases, To make the reasoning easier, we use c as defined in Equation (2.3).

In the first case, we have found the desired set G_j satisfying $\operatorname{aff}\psi(G_j) \supseteq \psi(C)$. We set $F_r := G_j$ and apply recursion. The second case is more complicated. We would like to find a point p with $c(p) \notin c(G_j)$ and $\psi(p) \notin \operatorname{aff}\big(\psi(G_j)\big)$, so that we could form G_{j+1} by adding p to G_j. Unfortunately, this may not be possible without replacing some points in G_j.

Now we provide the details. Let us start with possibility 1.

Case 1: $\psi(C) \subseteq \operatorname{aff}\big(\psi(G_j)\big)$

In this case we set $F_r := G_j$. Since G_j is rainbow, so is F_r. We further define $C' := C \setminus F_r$, $\psi' := \psi \restriction C'$ and $r' = r - 1$. We partition C' into $m+1$ color classes $C'_i := C_i \setminus F_r$ for $i = 0, \dots, m$. Now we run the algorithm recursively on C' and ψ'. We obtain $(r-1)$ pairwise disjoint rainbow sets $F_1, \dots, F_{r-1} \subseteq C'$ that satisfy

$$\emptyset \neq \bigcap_{i=1}^{r-1} \operatorname{aff}\big(\psi'(F_i)\big) = \bigcap_{i=1}^{r-1} \operatorname{aff}\big(\psi(F_i)\big).$$

Since the sets F_i, $i = 1, \dots, r-1$, are rainbow subsets of C', they are rainbow in C as well. It follows that $F_1, \dots, F_r$ are pairwise disjoint, since for $i = 1, \dots, r-1$, $F_i \subseteq C' = C \setminus F_r$ and F_i are pairwise disjoint.

Because

$$\bigcap_{i=1}^{r-1} \operatorname{aff}\big(\psi(F_i)\big) \subseteq \operatorname{aff}\big(\psi(C)\big) \subseteq \operatorname{aff}\big(\psi(G_j)\big) = \operatorname{aff}\big(\psi(F_r)\big),$$

we have

$$\bigcap_{i=1}^{r} \operatorname{aff}\big(\psi(F_i)\big) \neq \emptyset.$$

Since the sets $F_1, \dots, F_r$ form the desired system, we output $F_1, \dots, F_r$ and stop the algorithm.

Now we deal with the more complicated situation.

Case 2: $\psi(C) \not\subseteq \operatorname{aff}\big(\psi(G_j)\big)$

In this case we want to find point p with $c(p) \notin c(G_j)$ and $\psi(p) \notin \operatorname{aff}\big(\psi(G_j)\big)$ so that we could set $G_{j+1} = G_j \cup \{p\}$. In general, this may not be possible, it may happen that we need to replace some points in G_j before we can add p. In order to know which points to replace and how, the algorithm uses a second loop. During the loop, the algorithm keeps track of "replacement rules", which makes this part somewhat technical.

Moreover, there are three possibilities in each iteration: Either we have collected enough information and we can construct the desired set G_{j+1}, or we adjust the replacement rules, or we obtain a proper subspace $\mathbb{A}' \subsetneq \mathbb{A}$ such that we can find the desired sets $F_1, \dots, F_r$ by a recursive call of our algorithm to $C' := C \cap \psi^{-1}(\mathbb{A}')$, $\mathbb{A}'$ and $\psi \restriction C'$.

Second loop

In the kth step of the second loop the replacement rules consist of the following data: sets $K_k \subseteq \{0, 1, \dots, m\}$, $R_k \subseteq G_j$ and $R_k^p \subseteq C$, where p ranges over all elements[5] $p \in C_{K_k}$ for which $\psi(p) \notin \operatorname{aff}\big(\psi(R_k)\big)$.

Furthermore, we want that these sets satisfy the following conditions:

(i) $c(R_k) \subsetneq K_k$,

(ii) $c(R_k^p) = c(R_k) \cup \{c_k^p\}$, for some $c_k^p \in K_k \setminus c(G_j)$,

(iii) $|R_k^p| = |R_k| + 1$,

(iv) $p \in R_k^p$ and $\operatorname{aff}\big(\psi(R_k^p \setminus \{p\})\big) = \operatorname{aff}\big(\psi(R_k)\big)$,

(v) $G_j \cap C_{K_k} = R_k$ and $K_k \not\subseteq c(G_j)$.

Note that condition (ii) states that R_k^p only contains points that have the same colors as points in R_k and one additional point that has color c_k^p, which is not yet present in $c(G_j)$.

The intuition behind the sets K_k, R_k and R_k^p is the following. The set R_k represents the subset of G_j that we want to replace. The set K_k represents the colors that we might use while replacing R_k. The

[5] Recall that the symbol C_{K_k} stands for $\bigcup_{i \in K_k} C_k$, i.e., C_{K_k} are all the points with color in K_k. We will also use the equivalence $p \in C_{K_k} \Leftrightarrow c(p) \in K_k$.

set $R_k^p \setminus \{p\}$ is the replacement of R_k if we want to add point p. More precisely, R_k^p is a rainbow set, the sets $\psi(R_k)$ and $\psi\big(R_k^p \setminus \{p\}\big)$ have the same affine hull and $c(R_k^p) \subseteq K_k$.

We will start with R_0 as small as possible and a suitable set K_0. In each step, we will enlarge the sets R_k and K_k until we find an element $p \in C_{K_k}$ with $\psi(p) \notin \operatorname{aff}\big(\psi(G_j)\big)$, or until $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_k)\big)$. If we fin an element $p \in C_{K_k}$ with $\psi(p) \notin \operatorname{aff}\big(\psi(G_j)\big)$, we will construct the desired set G_{j+1}. If $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_k)\big)$, then the affine space $\mathbb{A}' = \operatorname{aff}\big(\psi(R_k)\big) \subsetneq \mathbb{A}$ and $C' = C \cap \psi^{-1}(\mathbb{A}')$ satisfy $|C'| > (\dim \mathbb{A}' + 1)(r-1)$, so we will apply recursion to obtain the desired sets $F_1, \ldots, F_r$.

Let us now carry out the technical details.

The first step ($k = 0$) is easy. We set $R_0 := \emptyset$ and $K_0 := \{0, 1, \ldots, m\} \setminus c(G_j)$. If we now take a point $p \in C_{K_0}$, then $\psi(p)$ is not contained in $\operatorname{aff}\big(\psi(R_0)\big) = \emptyset$, so we need to define the set R_0^p for every such p. We simply put $R_0^p := \{p\}$.

Now we check the above defined sets satisfy all the prescribed conditions. Note $m = \dim \mathbb{A}$, $C \subseteq \mathbb{A}$ and we are in case 2, where $\psi(C) \not\subseteq \operatorname{aff}\big(\psi(G_j)\big)$. This, together with $|G_j| = j + 1$ (condition (I)) and $\dim \operatorname{aff} \psi(G_j) = j$ (condition (III)) implies that $|G_j| < m+1$. Consequently, the set $K_0 = \{0, 1, \ldots, m\} \setminus c(G_j)$ is nonempty. Hence conditions (i)–(v) are satisfied trivially (with $c_k^p = c(p)$ in condition (ii)).

So we may suppose that the sets K_k, R_k and R_k^p are already constructed for all relevant $p \in C_{K_k}$ and we want to continue with our construction. Since $R_k \subseteq G_j$ there are three cases that may occur, as announced:

2a) $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_k)\big)$,

2b) $\psi(C_{K_k}) \not\subseteq \operatorname{aff}\big(\psi(G_j)\big)$ or

2c) $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(G_j)\big)$ and $\psi(C_{K_k}) \not\subseteq \operatorname{aff}\big(\psi(R_k)\big)$.

Before going into technical details we give a short overview how we deal with the particular situations. In the first case, the restriction of ψ to C_{K_k} leads to an affine space $\mathbb{A}' := \operatorname{aff}\big(\psi(R_k)\big)$. We show that this space has dimension lower than m and $|C_{K_k}| > (\dim \mathbb{A}' + 1)(r-1)$. After that we adjust the color classes in C_{K_k} so that we can apply the algorithm recursively for $m' < m$. We obtain r disjoint rainbow sets $F_1, \ldots, F_r$ (with $\psi(F_i) \subseteq \mathbb{A}'$) satisfying the desired conditions and can stop our algorithm. In the second case, we show how to use sets R_k^p to construct set G_{j+1} so that we can continue in the first loop. In the third case, we prove that we can continue in the second loop.

We are now ready to deal with the particular cases:

Case 2a): $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_k)\big)$

In this case, we show that the affine space $\mathbb{A}' = \operatorname{aff}\big(\psi(R_k)\big)$ is a proper subset of $\mathbb{A}$ and there is a set $C' \subseteq C \cap \psi^{-1}(\mathbb{A}')$ satisfying $|C'| > (\dim \mathbb{A}' + 1)(r-1)$, so that we can apply induction to $\mathbb{A}'$, C' and obtain the desired sets $F_1, \ldots, F_r$.

Let us verify it now. Let $\mathbb{A}' := \operatorname{aff}\big(\psi(R_k)\big)$ and $m' = \dim \mathbb{A}'$. $\mathbb{A}$ has dimension m and because we are in Case 2, we know that $\psi(C) \not\subseteq \operatorname{aff}\big(\psi(G_j)\big)$. It follows that $\dim \operatorname{aff}\big(\psi(G_j)\big) < m$. Since $R_k \subseteq G_j$, we also have $m' = \dim \operatorname{aff}\big(\psi(R_k)\big) < m$.

Condition (i) implies $c(R_k) \subsetneq K_k$, so there is a point $p \in C_{K_k} \setminus C_{c(R_k)}$.

Because R_k is rainbow and $\dim \operatorname{aff}\big(\psi(R_k)\big) = m'$ we can choose $m'+1$ distinct elements $k_0, k_1, \ldots, k_{m'}$ in $c(R_k)$. We define $C' := C_{\{k_0, \ldots, k_{m'}\}} \cup \{p\}$ and partition C' into color classes $C'_0 := C_{k_0} \cup \{p\}$ and $C'_i := C_{k_i}$ for $i = 1, \ldots m'$.

Because $C' \subseteq C_{K_k}$, the assumption $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_k)\big)$ (Case 2a)) implies $\psi(C') \subseteq \mathbb{A}'$. Also $|C'_0| \geq r$ and $|C'_i| \geq r - 1$ for all $i = 1, \ldots, m'$. It follows that we can apply the algorithm recursively on C' and $\psi \restriction C' \colon C' \to \mathbb{A}'$. We obtain r pairwise disjoint rainbow sets $F_1, \ldots, F_r \subseteq C'$ that satisfy $\bigcap_{i=1}^r \operatorname{aff}\big(\psi(F_i)\big) \neq \emptyset$. From the definition of color classes C'_i easily follows that any set $F \subseteq C'$ that is rainbow in C' is also rainbow as a subset of C. We conclude that the sets $F_1, \ldots, F_r$ form the desired system.

Case 2b): $\psi(C_{K_k}) \not\subseteq \operatorname{aff}\big(\psi(G_j)\big)$

In this case, we may construct the set G_{j+1} as follows: We pick a point $p \in C_{K_k}$ with $\psi(p) \notin \operatorname{aff}\big(\psi(G_j)\big)$ and set $G_{j+1} := (G_j \setminus R_k) \cup R_k^p$.

Before we show that such G_{j+1} satisfies conditions (I)–(III), we prove the following auxiliary equality:

$$\operatorname{aff}\big(\psi(G_{j+1})\big) = \operatorname{aff}\big(\psi\left(G_j \cup \{p\}\right)\big). \tag{2.7}$$

Indeed,

$$\begin{aligned} \operatorname{aff}\big(\psi(G_{j+1})\big) &= \operatorname{aff}\big(\psi\left((G_j \setminus R_k) \cup R_k^p\right)\big) \\ &= \operatorname{aff}\big(\psi\left((G_j \setminus R_k) \cup (R_k^p \setminus \{p\}) \cup \{p\}\right)\big), \end{aligned}$$

where the last equality uses the fact that $p \in R_k^p$ from condition (iv). Because the operator aff satisfies $\operatorname{aff}\big(B \cup C\big) = \operatorname{aff}\big(B \cup \operatorname{aff} C\big)$ for every two sets $B, C \subseteq \mathbb{A}$ (we refer to Observation 2.7) we may rewrite the expression further to

$$\operatorname{aff}\big(\psi(G_{j+1})\big) = \operatorname{aff}\big(\psi\left(G_j \setminus R_k\right) \cup \operatorname{aff}\big(\psi(R_k^p \setminus \{p\})\big) \cup \psi(\{p\})\big).$$

By condition (iv) $\operatorname{aff}\big(\psi(R_k^p \setminus \{p\})\big) = \operatorname{aff}\big(\psi(R_k)\big)$, which reduces the equality to:

$$\operatorname{aff}\big(\psi(G_{j+1})\big) = \operatorname{aff}\Big(\psi\big(G_j \setminus R_k\big) \cup \operatorname{aff}\big(\psi(R_k)\big) \cup \psi\big(\{p\}\big)\Big).$$

Using Observation 2.7 again, we obtain

$$\operatorname{aff}\big(\psi(G_{j+1})\big) = \operatorname{aff}\Big(\psi\big((G_j \setminus R_k) \cup R_k \cup \{p\}\big)\Big)$$

Since $R_k \subseteq G_j$, Equation (2.7) follows.

Using the fact that $R_k \subseteq G_j$, we are now ready to verify that G_{j+1} satisfies conditions (I)–(III).

- **Condition (I) ($|G_{j+1}| = j+2$):** $|G_{j+1}| = |\,(G_j \setminus R_k) \cup R_k^p|$. Because G_j is rainbow, condition[6] (ii) implies that the sets $G_j \setminus R_k$ and R_k^p do not share any color. In particular, they are disjoint and $|G_{j+1}| = |G_j \setminus R_k| + |R_k^p|$. Since $|R_k^p| = |R_k| + 1$ (condition (iii)), $|G_{j+1}| = |G_j \setminus R_k| + |R_k| + 1$. Because $R_k \subseteq G_j$, we have $|G_{j+1}| = |G_j| + 1$. Condition (I) for G_j then implies $|G_{j+1}| = j + 2$. We conclude that G_{j+1} satisfies condition (I).
- **Condition (II) (G_{j+1} is rainbow):** We have $G_{j+1} = (G_j \setminus R_k) \cup R_k^p$. As we have already shown while verifying condition (I), the sets $G_j \setminus R_k$ and R_k^p do not share any color. Hence it suffices to show that both $G_j \setminus R_k$ and R_k^p are rainbow. Since G_j is rainbow by condition (II), so are $G_j \setminus R_k$ and $R_k \subseteq G_j$. Conditions (iii) ($|R_k^p| = |R_k| + 1$) and (ii) ($c(R_k^p) = c(R_k) \cup \{c_k^p\}$ for some $c_k^p \in K_k \setminus c(G_j)$) then imply that R_k^p cannot use any color twice, so R_k^p is rainbow as well and the condition follows.
- **Condition (III) ($\dim \operatorname{aff}\big(\psi(G_{j+1})\big) = j + 1$):** From the equality (2.7) we get $\operatorname{aff}\big(\psi(G_{j+1})\big) = \operatorname{aff}\big(\psi\left(G_j \cup \{p\}\right)\big)$. Moreover, we have chosen a point p which satisfies $\psi(p) \notin \operatorname{aff}\big(\psi(G_j)\big)$, so $\dim \operatorname{aff}\big(\psi(G_{j+1})\big) = \dim \operatorname{aff}\big(\psi(G_j)\big) + 1$. By induction hypothesis this equals $j+1$ and G_{j+1} satisfies condition (III).

It follows that we have constructed set G_{j+1} satisfying the desired conditions, therefore, we can continue with the first loop.

Case 2c): $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(G_j)\big)$ and $\psi(C_{K_k}) \not\subseteq \operatorname{aff}\big(\psi(R_k)\big)$

In this case, we show how to construct sets K_{k+1}, R_{k+1} and R_{k+1}^p for all points $p \in C_{K_{k+1}}$ for which $\psi(p) \notin \operatorname{aff}\big(\psi(R_{k+1})\big)$.

We choose an inclusion minimal subset $S \subseteq G_j$ satisfying $\psi\big(C_{K_k}\big) \subseteq \operatorname{aff}\big(\psi(S)\big)$ and define $R_{k+1} := S$. Because we assume that $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(G_j)\big)$, such set R_{k+1} does exist. We further define

$$K_{k+1} := K_k \cup c(R_{k+1}). \tag{2.8}$$

Before we construct R_{k+1}^p, we prove the following two auxiliary claims:

[6] $c(R_k^p) = c(R_k) \cup \{c_k^p\}$, for some $c_k^p \in K_k \setminus c(G_j) \subseteq K_k \setminus c(R_k)$

Claim 2.1. *$\psi \restriction G_j$ is injective, and $\psi(G_j)$ is affinely independent.*

Proof. The claim easily follows from $|G_j| = j+1$ (condition (I)) and $\dim \operatorname{aff}\big(\psi(G_j)\big) = j$ (condition (III)). □

Claim 2.2.

$$R_k \subsetneq R_{k+1}. \tag{2.9}$$

Proof. By condition (i) $R_k \subsetneq C_{K_k}$, so $\psi(R_k) \subseteq \psi(C_{K_k})$. Since we have chosen R_{k+1} as a set satisfying $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_{k+1})\big)$, we have $\operatorname{aff}\big(\psi(R_k)\big) \subseteq \operatorname{aff}\big(\psi(R_{k+1})\big)$. Because $\psi(R_k)$ and $\psi(R_{k+1})$ are subsets of the affinely independent set $\psi(G_j)$ (Claim 2.1), we have $\psi(R_k) \subseteq \psi(R_{k+1})$. Since $\psi \restriction G_j$ is injective and $R_k, R_{k+1} \subseteq G_j$, we have $R_k \subseteq R_{k+1}$. Since $\psi(R_k) \subseteq \psi(C_{K_k}) \not\subseteq \operatorname{aff}\big(\psi(R_k)\big)$ by condition (i) and the fact that we are in case 2c), we can use $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_{k+1})\big)$ to deduce that that $R_{k+1} \neq R_k$. □

Now we construct sets R^p_{k+1} for all points $p \in C_{K_{k+1}}$ satisfying $\psi(p) \notin \operatorname{aff}\big(\psi(R_{k+1})\big)$. Let p be such a point. By definition of R_{k+1}, $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_{k+1})\big)$, so p cannot lie in C_{K_k}. Equation (2.8) then implies $c(p) \in (K_{k+1} \setminus K_k) \subseteq c(R_{k+1})$. Because $R_{k+1} \subseteq G_j$ is a rainbow set[7], there exists a unique element $r \in R_{k+1}$ with $c(r) = c(p)$. Since we assume $p \notin C_{K_k}$, we have $c(r) = c(p) \notin K_k \supseteq c(R_k)$, where the last inclusion follows from condition (i). In particular, $c(r) \notin c(R_k)$, hence

$$r \in R_{k+1} \setminus R_k. \tag{2.10}$$

Since R_{k+1} is an inclusion minimal subset of G_j for which $\psi\big(C_{K_k}\big) \subseteq \operatorname{aff}\big(\psi(R_{k+1})\big)$, there exists an element $q \in C_{K_k}$ such that $\psi(q) \notin \operatorname{aff}\big(\psi(R_{k+1} \setminus \{r\})\big)$. Since $\psi(q) \in \psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_{k+1})\big)$, the exchange principle implies $\psi(r) \in \operatorname{aff}\big(\psi(R_{k+1} \setminus \{r\}) \cup \psi(\{q\})\big)$.

It easily follows that

$$\operatorname{aff}\big(\psi(R_{k+1})\big) = \operatorname{aff}\Big(\psi\big((R_{k+1} \setminus \{r\}) \cup \{q\}\big)\Big). \tag{2.11}$$

Claim 2.2 together with (2.10) imply that $R_k \subseteq R_{k+1} \setminus \{r\}$. Since q was chosen to satisfy $\psi(q) \notin \operatorname{aff}\big(\psi(R_{k+1} \setminus \{r\})\big)$, we have $\psi(q) \notin \operatorname{aff}\big(\psi(R_k)\big)$ as well. Together with $q \in C_{K_k}$, this implies that R^q_k is defined. We set[8]

$$R^p_{k+1} := R_{k+1} \setminus \big(R_k \cup \{r\}\big) \cup R^q_k \cup \{p\}. \tag{2.12}$$

It remains to show that our assignment satisfies conditions (i)–(v).

- **Condition (i):** By definition of K_{k+1}, we have $c(R_{k+1}) \subseteq K_{k+1}$. $R_{k+1} \subseteq G_j$ and $K_k \not\subseteq c(G_j)$ (condition (v)) then imply that $K_{k+1} \not\subseteq c(R_{k+1})$, in particular $K_{k+1} \neq c(R_{k+1})$ and condition (i) follows.

- **Condition (ii):** Condition (ii) states that $c(R^q_k) = c(R_k) \cup \{c^q_k\}$ for some $c^q_k \in K_k \setminus G_j$, in particular $c(R_k) \subseteq c(R^q_k)$. Together with the fact that elements p and r have the same color ($c(p) = c(r)$), Equation (2.12) then yields $c(R^p_{k+1}) = c\left(R_{k+1} \setminus R_k\right) \cup c(R^q_k)$. If we now apply condition (ii) for R^q_k and Claim 2.2, we see that $c(R^p_{k+1}) = c(R_{k+1}) \cup \{c^p_{k+1}\}$, where $c^p_{k+1} = c^q_k$. Note that $K_k \subseteq K_{k+1}$, hence $c^p_{k+1} \in K_{k+1} \setminus c(G_j)$. Condition (ii) follows.

- **Condition (iii):** By definition $R^p_{k+1} = R_{k+1} \setminus \big(R_k \cup \{r\}\big) \cup R^q_k \cup \{p\}$. Because R_{k+1} is a subset of G_j, R_{k+1} is rainbow (condition (II)). Together with $c(R^q_k) = c(R_k) \cup \{c^q_k\}$, where $c^q_k \notin c(G_j) \supseteq c(R_{k+1})$, it implies that the sets $R_{k+1} \setminus R_k$ and R^q_k are disjoint. Since $r \in R_{k+1} \setminus R_k$ (Equation (2.10)), $c(p) = c(r) \in c(R_{k+1}) \setminus c(R_k)$ and $c(R^q_k) \cap c(G_j) = c(R_k)$ (conditions (ii) and (v)), we have $p, r \notin R^q_k$ and $p, r \notin R_k$. From $\psi(p) \notin \operatorname{aff}\big(\psi(R_{k+1})\big)$ follows $p \notin R_{k+1}$. Since $r \in R_{k+1}$, we have $|R^p_{k+1}| = |R_{k+1} \setminus R_k| - |\{r\}| + |\{p\}| + |R^q_k| = |R_{k+1} \setminus R_k| + |R_k| + 1$, where the last equality uses the induction hypothesis for k. From Claim 2.2 then easily follows that $|R^p_{k+1}| = |R_{k+1}| + 1$ as desired.

[7] G_j is rainbow by condition (II).

[8] We note that R^p_{k+1} does depend on our choice of q, i.e., if we choose another $q \in C_{K_k}$ that satisfies $\psi(q) \notin \operatorname{aff}\big(\psi(R_{k+1} \setminus \{r\})\big)$, we obtain a different set R^p_{k+1}.

- **Condition (iv):** By definition $p \in R^p_{k+1}$, so we only need to verify that $\operatorname{aff}\big(\psi(R^p_{k+1} \setminus \{p\})\big) = \operatorname{aff}\big(\psi(R_{k+1})\big)$. Let us compute. Using the fact that $q \in R^q_k$ from condition (iv) and the equality $\operatorname{aff}\big(B \cup C\big) = \operatorname{aff}\big(B \cup \operatorname{aff}(C)\big)$ from Observation 2.7 we may rewrite $\operatorname{aff}\big(\psi(R^p_{k+1} \setminus \{p\})\big)$ as follows:

$$\begin{aligned}\operatorname{aff}\big(\psi(R^p_{k+1} \setminus \{p\})\big) &= \operatorname{aff}\Big(\psi\big(R_{k+1} \setminus (R_k \cup \{r\})\big) \cup \psi(R^q_k)\Big)\\ &= \operatorname{aff}\Big(\psi\big(R_{k+1} \setminus (R_k \cup \{r\})\big) \cup \psi(R^q_k \setminus \{q\} \cup \{q\})\Big)\\ &= \operatorname{aff}\Big(\psi\big(R_{k+1} \setminus (R_k \cup \{r\})\big) \cup \operatorname{aff}\big(\psi(R^q_k \setminus \{q\})\big) \cup \psi(\{q\})\Big).\end{aligned}$$

  Now we use condition (iv) for k $\big(\operatorname{aff}\big(\psi(R^q_k \setminus \{q\})\big) = \operatorname{aff}\big(\psi(R_k)\big)\big)$. We obtain

$$\begin{aligned}\operatorname{aff}\big(\psi(R^p_{k+1} \setminus \{p\})\big) &= \operatorname{aff}\Big(\psi\big(R_{k+1} \setminus (R_k \cup \{r\})\big) \cup \operatorname{aff}\big(\psi(R_k)\big) \cup \psi(\{q\})\Big)\\ &= \operatorname{aff}\Big(\psi\big((R_{k+1} \setminus \{r\}) \cup \{q\}\big)\Big)\\ &= \operatorname{aff}\big(\psi(R_{k+1})\big),\end{aligned}$$

  where the last equality follows from (2.11).

- **Condition (v):** By definition $K_{k+1} = K_k \cup c(R_{k+1})$. This implies $C_{K_{k+1}} = C_{K_k} \cup C_{c(R_{k+1})}$. Hence $G_j \cap C_{K_{k+1}} = (G_j \cap C_{K_k}) \cup (G_j \cap C_{c(R_{k+1})})$. By the induction assumption $G_j \cap C_{K_k} = R_k$. Because $G_j \supseteq R_{k+1}$ is rainbow (condition (II)), $G_j \cap C_{c(R_{k+1})} = R_{k+1}$. Claim 2.2 then implies $G_j \cap C_{K_{k+1}} = R_{k+1}$ as desired. Because $K_k \not\subseteq c(G_j)$ and $K_k \subseteq K_{k+1}$, we have $K_{k+1} \not\subseteq c(G_j)$ as well.

It follows that we may continue in the second loop. This ends our description of the algorithm.

Running time

We have already shown that if the algorithm stops, it always outputs r sets $F_1, \ldots, F_r$ that satisfy the desired conditions. It remains to argue that it always stops in polynomial time.

Let us recapitulate how the algorithm works. First, if $|C_0| > r$ it deletes the additional points and similarly if $|C_i| > i$. This can be clearly done in $O(|C|)$-time. So we may assume that $|C| = (m+1)(r-1)+1$.

Let $t(m,r)$ denote the running time of the algorithm under the conditions that

$$|C| = (m+1)(r-1)+1. \tag{2.13}$$

If $m = 0$, the algorithm splits C_0 into r sets $F_1, \ldots, F_r$ and outputs them in $O(r)$-time, hence $t(0,r) = O(r)$. If $r = 1$, the algorithm outputs C_0 in time $O(|C_0|) = O(r) = O(1)$, hence $t(m,1) = O(1)$.

Since $r < |C|$, it follows that the total running time of our algorithm equals

$$t(m,r) + O(|C|). \tag{2.14}$$

If $m > 0$, $r > 1$ and $|C| = (m+1)(r-1)+1$ the algorithm runs as follows:

function TVERBERG_DECOMPOSITION
  Let p be an element in C_0. Set $G_0 := \{p\}$.
  for $j = 0, 1, \ldots$ **do** ▷ First loop
    if $\psi(C) \subseteq \operatorname{aff}\big(\psi(G_j)\big)$ **then** ▷ Case 1
      Set $F_r := G_j$
      Set $C' := C \setminus F_r$, $C'_i := C_i \setminus F_r$, $i = 0, \ldots, m$, $\psi' := \psi \restriction C'$
      Call TVERBERG_DECOMPOSITION on m, $r-1$ and C', C'_i, ψ'
      We obtain $r-1$ sets $F'_1, \ldots, F'_{r-1}$.
      Output $F'_1, \ldots, F'_{r-1}, F_r$ and stop
    else ▷ Case 2
      Set $R_0 := \emptyset$, $K_0 := \{0, 1, \ldots, m\} \setminus c(G_j)$ and $R^p_0 := \{p\}$ for all $p \in C_{K_0}$

12: **for** $k = 0, \dots$ **do** ▷ Second loop
13: **if** $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_k)\big)$ **then** ▷ Case 2a)
14: Choose a point $p \in C_{K_k} \setminus C_{c(R_k)}$. Let $c(R_k) = \{k_0, \dots, k_{m'}\}$.
15: Set $C' := C_{c(R_k)} \cup \{p\}$.
16: Partition C' into color classes $C'_0 := C_{k_0} \cup \{p\}$
17: and $C'_i := C_{k_i}$ for $i = 1, \dots, m'$. Set $\psi' := \psi \restriction C'$.
18: Call TVERBERG_DECOMPOSITION on m', r and C', C'_i.
19: We obtain r sets $F'_1, \dots, F'_r$.
20: Output $F'_1, \dots, F'_r$ and stop
21: **else if** $\psi(C_{K_k}) \not\subseteq \operatorname{aff}\big(\psi(G_j)\big)$ **then** ▷ Case 2b)
22: Choose a point $p \in C_{K_k}$ with $\psi(p) \notin \operatorname{aff}\big(\psi(G_j)\big)$
23: Set $G_{j+1} := (G_j \setminus R_k) \cup R_k^p$.
24: Continue in the first loop
25: **else** ▷ Case 2c)
26: Find inclusion minimal $R_{k+1} \subseteq G_j$ with $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_{k+1})\big)$
27: Set $K_{k+1} := K_k \cup c(R_{k+1})$.
28: **for all** $p \in C_{K_{k+1}}$ for which $\psi(p) \notin \operatorname{aff}\big(\psi(R_{k+1})\big)$ **do**
29: Let $r \in R_{k+1} \setminus R_k$ be the unique element with $c(r) = c(p)$.
30: Choose an element $q_r \in C_{K_k}$ with
31: $\psi(q_r) \notin \operatorname{aff}\Big(\psi\big(R_{k+1} \setminus \{r\}\big)\Big)$.
32: Set $R_k^p := R_{k+1} \setminus (R_k \cup \{r\}) \cup R_k^{q_r} \cup \{p\}$.
33: **end for**
34: **end if**
35: **end for**
36: **end if**
37: **end for**
38: **end function**

Before we start with the time analysis of the algorithm, we recall that u is the maximal time needed to decide whether a point $p \in C$ and a set $S \subseteq C$ satisfy $\psi(p) \in \operatorname{aff}\big(\psi(S)\big)$.

The time that we need to decide whether $\psi(C) \subseteq \operatorname{aff}\big(\psi(G_j)\big)$ (line 4) is $O(|C|u)$: We simply test whether all elements of $\psi(C)$ lie in $\operatorname{aff}\big(\psi(G_j)\big)$.

Case 1 (lines 5 – 9) deletes points from G_j in all sets C_i. This task can be performed in $O(|C|)$-time. Then the algorithm call itself recursively on $C \setminus G_j$ and stops. Since $|C \setminus G_j| = (m+1)(r-1)+1-(j+1)$, the recursive call has to delete $m-j$ points from $|C \setminus G_j|$ before the set satisfies condition (2.13). Hence the total running time of Case 1 is $t(m, r-1) + O(m) + O(|C|) = t(m, r-1) + O(|C|)$.

Let us now analyze Case 2 (lines 10 – 36). To avoid some technicalities during the analysis, we set $K_{-1} = \emptyset$. The decision whether we are in Case 2a), Case 2b) or Case 2c) can be done in $O(|C|u)$-time for the same reason as the decision whether we are in Case 1 or Case 2.

Line 11 can be performed in $O(|C|)$-time. Case 2a) (lines 13 – 20) calls the algorithm recursively and then stops. If $0 \in c(R_k)$, the set C' contains one superfluous point and the recursively called instance has to delete it. Otherwise $|C'| = (m'+1)(r-1)+1$ from the beginning. Since lines 14 – 17 can be performed in $O(|C|)$ time and $m' < m$, the total running time of Case 2a) is at most $t(m-1, r) + O(|C|)$ regardless whether $0 \in c(R_k)$ or not.

Case 2b) (lines 21 – 24) finds point $p \in C_{K_k}$ with $\psi(p) \notin \operatorname{aff}\big(\psi(G_j)\big)$. Such a point can be found as follows: We go through all elements of C_{K_k} and test whether $\psi(p) \in \operatorname{aff}\big(\psi(G_j)\big)$. This requires $O(|C_{K_k} \setminus C_{K_{k-1}}| \cdot u) = O(|C|u)$-time. Constructing G_{j+1} from p (line 23) can then be done in constant time. Hence Case 2b) finishes in $O(|C|u)$ time.

Case 2c) (lines 25 – 34) is the most difficult to analyze. It contains the nontrivial task of finding inclusion minimal subset $R_{k+1} \subseteq G_j$ for which $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_{k+1})\big)$ (line 26).

Claim 2.3. *An inclusion minimal set $R_{k+1} \subseteq G_j$ for which $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_{k+1})\big)$ can be found in $O\big(mu \cdot |C_{K_k} \setminus C_{K_{k-1}}|\big)$ time.*

We postpone the proof after the analysis of the remaining lines.

Line 27 can be performed in $O(|C|)$ time. The cycle on lines 28 – 33 runs through all elements $p \in C_{K_{k+1}}$ for which $\psi(p) \notin \operatorname{aff}\big(\psi(R_{k+1})\big)$. Line 26 implies that $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_{k+1})\big)$, hence it suffices to go through all elements $p \in C_{K_{k+1}} \setminus C_{K_k}$ and the cycle is repeated at most $|C_{K_{k+1}} \setminus C_{K_k}|$-times. Testing whether p satisfies $\psi(p) \notin \operatorname{aff}\big(\psi(R_{k+1})\big)$ (line 28) is performed in time u.

Finding an element $r \in R_{k+1} \setminus R_k$ that has the same color as p (line 29) can be done as follows: We go through all elements $r \in R_{k+1} \setminus R_k$ and check their colors. Equation (2.10) implies that we find an element r with $c(r) = c(p)$. This takes at most $O(|R_{k+1} \setminus R_k|)$ time in total. We choose an element $q_r \in C_{K_k}$ with $\psi(q_r) \notin \operatorname{aff}\Big(\psi\big(R_{k+1} \setminus \{r\}\big)\Big)$ (lines 30–31) as follows: Such element cannot lie in $C_{K_{k-1}}$, because $C_{K_{k-1}} \subseteq R_k$ (lines 26 and 11), hence we go through all elements $q_r \in C_{K_k} \setminus C_{K_{k-1}}$ and test whether $\psi(q_r) \in \operatorname{aff}\Big(\psi\big(R_{k+1} \setminus \{r\}\big)\Big)$. Because this condition only depends on r, we remember the points q_r and reuse them for all points p with $c(p) = c(r)$.

The last line (32) can be performed in constant time. We conclude that the cycle on lines $28 - 33$ can be performed in time

$$O\big(|C_{K_{k+1}} \setminus C_{K_k}|(u + |R_{k+1} \setminus R_k|) + |C_{K_k} \setminus C_{K_{k-1}}|u\big).$$

Since $R_{k+1} \subseteq G_j$ and $|G_j| = j + 1 \leq m + 1$, this lies in

$$O\big(|C_{K_{k+1}} \setminus C_{K_k}|(u + m) + |C_{K_k} \setminus C_{K_{k-1}}|u\big).$$

Observe that $R_k \subsetneq R_{k+1}$ (Claim 2.2), $R_{k+1} \subseteq G_j$, $|G_j| = j+1$ and Cases 2a) and 2b) both terminate the second loop (lines 12–35). It follows that we go through Case 2c) at most $j + 1 < m + 1$-times.

Now we use $\bigcup\big(C_{K_{k+1}} \setminus C_{K_k}\big) \subseteq C$. Using amortized complexity, we see that the total time that the first loop (lines $12 - 35$) spends on lines $25 - 34$ is bounded by $O(|C|mu)$.

If we sum the running times up, we obtain that one iteration of Case 2 can be done in time $t(m - 1, r) + O(|C|mu)$.

From the condition $\dim \operatorname{aff}\big(\psi(G_j)\big) = j$ follows that Case 2 (lines $10 - 36$) can happen at most $(\dim \mathbb{A})$-times. Since $\dim \mathbb{A} \leq m$, it follows that the total running time of Case 2 is bounded by $t(m - 1, r) + O(|C|m^2u)$. (We recall that the term $t(m-1,r)$ is obtained from Case 2a), which immediately runs the algorithm recursively and then stops the algorithm. In particular the term $t(m-1,r)$ appears at most once.)

In combinations with the running time of Case 1 it gives $t(m,r) \leq \max\{t(m-1,r), t(m,r-1)\} + O(|C|m^2u)$.

If we solve the recursion, we get $t(m,r) = O\big(m+r)|C|m^2u\big)$, hence the total running time of our algorithm equals $O\big((m+r)|C|m^2u + |C|\big)$, see Equation 2.14.

The last thing we need to verify is Claim 2.3.

Proof of Claim 2.3. The task of finding some minimal $R_{k+1} \subseteq G_j$ such that $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(R_{k+1})\big)$ can be achieved as follows: We start with set $S' := G_j$ and go through all elements $r \in G_j \setminus R_k$ ($R_k \subseteq R_{k+1}$ by Claim 2.2). For each such r we test whether $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(S' \setminus \{r\})\big)$. Since $\psi(C_{K_{k-1}}) \subseteq \operatorname{aff}\big(\psi(R_k)\big)$, it suffices to test whether $\psi(C_{K_k} \setminus C_{K_{k-1}}) \subseteq \operatorname{aff}\big(\psi(S' \setminus \{r\})\big)$. One such test can be done in time $O\big(|C_{K_k} \setminus C_{K_{k-1}}|u\big)$. If $\psi(C_{K_k}) \subseteq \operatorname{aff}\big(\psi(S' \setminus \{r\})\big)$, we delete r from S'. When we have tested all elements, we set $R_{k+1} := S'$. It follows that finding set R_{k+1} requires time $O\big(|G_j| \cdot |C_{K_k} \setminus C_{K_{k-1}}| \cdot u\big) = O\big(mu \cdot |C_{K_k} \setminus C_{K_{k-1}}|\big)$. ☐

Chapter 3

Ramsey type result for simplicial chain maps

In this chapter we prove Theorem 1.2. In fact, we prove a slightly stronger version (Theorem 6.8): Let $s, k, n \geq 1$ be integers, M a manifold and $f\colon \Delta_n^{(k)} \to M$ a continuous map. If

$$n \geq \max\left\{s+1, \binom{s}{k}\widetilde{\beta}_k(M;\mathbb{Z}_p)(s-2k)+s+1\right\},$$

then there exists an almost embedding $g\colon \Delta_s^{(k)} \to \Delta_n^{(k)}$, such that the induced map

$$(f\circ g)_*\colon H_*\left(\Delta_s^{(k)};\mathbb{Z}_p\right) \to H_*\left(M;\mathbb{Z}_p\right)$$

is trivial and the g-image of every face is a union of faces in $\Delta_n^{(k)}$.

In this chapter we work with ordered simplicial and singular chains and derived homology groups, since the argumentation in the proofs is then much easier. Because not all of these terms are commonly used nowadays and our proofs rely on precise definitions, we recall them first.

Moreover, until now we have omitted the difference between an abstract simplicial complex K and its geometric realization $|K|$. Since in some of our constructions the distinction is necessary, we will carefully distinguish these two throughout this chapter.

3.1 Preliminaries

3.1.1 Simplicial complexes

If we write simplicial complex, we mean a finite abstract simplicial complex:

Definition 3.1 (Simplicial complexes)**.** *A* finite simplicial complex *is a finite family K of finite sets such that $\tau \subseteq \sigma \in K$ implies $\tau \in K$. A simplicial complex $L \subseteq K$ is called a* subcomplex *of K.*

If $\sigma \in K$ and $\operatorname{card}\sigma = l$, we call σ an $(l-1)$-dimensional face *of K. The kth dimensional skeleton $K^{(k)}$ of K is the family of all $\leq k$-dimensional faces of K. The vertex set $V(K)$ of K is the set of all elements appearing in faces of K, i.e., $V(K) = \bigcup K$. We furthermore assume that the vertex set $V(K)$ is linearly ordered.*

The abstract d-dimensional simplex *Δ_d is the family of all subsets of $\{0, 1, \ldots, d\}$. This simplex inherits its vertex ordering from the integers.*

Since we are only working with finite simplicial complexes, we will usually omit the word finite.

Definition 3.2 (Geometric realization)**.** *A geometric realization of a finite abstract simplicial complex K is an assignment $\sigma \mapsto |\sigma|$, $\sigma \in K$, that satisfies:*

1. *If σ is a k-dimensional face, then $|\sigma|$ is a k-dimensional simplex and*

2. $|\sigma| \cap |\sigma'| = |\sigma \cap \sigma'|$.

In such case we define $|L| := \bigcup_{\sigma \in L} |\sigma|$ *for every subcomplex* $L \subseteq K$.

By a slight abuse of notation, we also call $|K|$ a geometric realization of K. Note that $|K|$ is a topological space.

We implicitly assume that every (finite abstract) simplicial complex K comes equipped with a fixed geometric realization. If K does not come with a geometric realization, we equip it with the standard one.

Definition 3.3 (Standard geometric realization)**.** *The standard geometric realization of an abstract simplicial complex K is defined as follows: If v is the ith vertex in the ordering of $V(K)$, we define $|v|$ as $\mathbf{e}_i$, the ith vector from the standard canonical basis of $\mathbb{R}^{\mathrm{card}(V(K))+1}$. For an l-dimensional face $\sigma = \{v_0, v_1, \ldots, v_l\}$ of K, we set $|\sigma| := \mathrm{conv}\{|v_0|, |v_1|, \ldots, |v_l|\}$. Finally, for a subcomplex $L \subseteq K$ we set $|L| := \bigcup_{\sigma \in L} |\sigma|$. In particular, $|K| := \bigcup_{\sigma \in K} |\sigma| \subseteq \mathbb{R}^{\mathrm{card}\left(V(K)\right)+1}$.*

Unless stated otherwise, we endow every abstract simplicial complex with the standard geometric realization described above.

Note that for $s < n$ the standard inclusion of $|\Delta_s|$ into $|\Delta_n|$ is the restriction of the standard inclusion $\mathbb{R}^{s+2} \to \mathbb{R}^{n+2}$ onto $|\Delta_s|$.

In our constructions, we need the following special case of stellar subdivision:

Definition 3.4 (Stellar subdivision)**.** *Let K be a finite abstract simplicial complex with geometric realization $|K|$ and $\mathbf{a} \notin V(K)$ be a point with $|\mathbf{a}|$ contained in the interior of $|\sigma|$ for some maximal face $\sigma \in K$. The stellar subdivision $\mathrm{sd}(K, \mathbf{a})$ is the abstract simplicial complex K' that is obtained from K by the following procedure:*

- *Remove σ from K*
- *for every $\tau \subsetneq \sigma$ add $\{\mathbf{a}\} \cup \tau$ to K'.*

For technical reasons, we equip $K' := \mathrm{sd}(K, \mathbf{a})$ with a geometric realization that slightly differs from the standard geometric realization. We have assumed that $|\mathbf{a}|$ is given and it stays the same in K'. For faces of K' containing $\mathbf{a}$, we set $|\{\mathbf{a}, v_0, \ldots, v_{l-1}\}| := \mathrm{conv}\{|\mathbf{a}|, |v_0|, \ldots, |v_{l-1}|\}$. The remaining faces inherit their realizations from $|K|$. Finally, for every subcomplex $L \subseteq K'$, we set $|L| := \bigcup_{\sigma \in L} |\sigma|$. In particular, $|\mathrm{sd}(K, \mathbf{a})| := \bigcup_{\sigma \in \mathrm{sd}(K, \mathbf{a})} |\sigma|$. Since we require the vertices of a simplicial complex to be ordered, we shall now define the ordering on vertices of $\mathrm{sd}(K, \mathbf{a})$. The ordering of $V\left(\mathrm{sd}(K, \mathbf{a})\right)$ puts $\mathbf{a}$ before all other vertices and orders the remaining vertices according to their original ordering in K.

Observe that the geometric realization of $\mathrm{sd}(K, \mathbf{a})$ is homeomorphic to the standard geometric realization of $\mathrm{sd}(K, \mathbf{a})$ and also to $|K|$.

3.1.2 Chain complexes

Choosing appropriate homology theories in this chapter makes our argumentation much easier.[1] It turns out that the ordered versions of simplicial and singular homology are suitable for our purposes. In both cases we consider augmented chain complexes and hence reduced homology groups.

Since the corresponding definitions are scattered throughout the literature, we provide all the necessary notions here. In order to introduce the homology groups efficiently, we define some abstract machinery first. Moreover, since we are only interested in homology with field coefficients, we state the definitions in this simpler setting[2].

[1] Since we are only working with sufficiently nice spaces, the resulting homology groups will be naturally isomorphic, regardless of the definition we choose. The difference only appears at the level of chain groups.

[2] If the reader prefers more abstract setting, he/she can replace the field $\mathbb{F}$ with a commutative ring R and vector spaces with (free) R-modules.

Definition 3.5 (Chain complexes). *Let $\mathbb{F}$ be a field. An infinite sequence*

$$\dots \xrightarrow{\partial_{l+2}} C_{l+1} \xrightarrow{\partial_{l+1}} C_l \xrightarrow{\partial_l} C_{l-1} \xrightarrow{\partial_{l-1}} \dots$$

of $\mathbb{F}$-vector spaces C_l and their homomorphisms $\partial_l \colon C_l \to C_{l-1}$, where l ranges over the integers, is called an $\mathbb{F}$-chain complex, *if $\partial_l \circ \partial_{l+1} = 0$ for every integer l. In that case we define the graded $\mathbb{F}$-module C_* as the direct sum $C_* := \bigoplus_{l\in\mathbb{Z}} C_l$ and set $\partial := \bigoplus_{l\in\mathbb{Z}} \partial_l$. An element $c \in C_l \subseteq C_*$ is called an element of degree l.*

If $\varphi \colon C_* \to D_*$ is a linear map between two graded $\mathbb{F}$-modules, such that $\varphi(C_l) \subseteq D_{l+i}$ for some integer i, then $\varphi = \bigoplus_{l\in\mathbb{Z}} \varphi_l$, where each $\varphi_l \colon C_l \to D_{l+i}$ is an $\mathbb{F}$-linear map. We call such a map φ a graded map of degree i.

A chain map between two $\mathbb{F}$-chain complexes

$$\dots \xrightarrow{\partial^C_{l+2}} C_{l+1} \xrightarrow{\partial^C_{l+1}} C_l \xrightarrow{\partial^C_l} C_{l-1} \xrightarrow{\partial^C_{l-1}} \dots$$

and

$$\dots \xrightarrow{\partial^D_{l+2}} D_{l+1} \xrightarrow{\partial^D_{l+1}} D_l \xrightarrow{\partial^D_l} D_{l-1} \xrightarrow{\partial^D_{l-1}} \dots$$

is any 0-degree $\mathbb{F}$-linear map $\varphi \colon C_* \to D_*$, satisfying $\partial^D \circ \varphi = \varphi \circ \partial^C$.

By a slight abuse of notation, we also write C_* as a shorthand for

$$\dots \xrightarrow{\partial^C_{l+2}} C_{l+1} \xrightarrow{\partial^C_{l+1}} C_l \xrightarrow{\partial^C_l} C_{l-1} \xrightarrow{\partial^C_{l-1}} \dots,$$

i.e., for C_* equipped with a (-1)-degree $\mathbb{F}$-linear map $\partial^C \colon C_* \to C_*$ satisfying $\partial^C \circ \partial^C = 0$. Hence we may say that C_* is an $\mathbb{F}$-chain complex.

If we write $\varphi \colon C_* \to D_*$, we mean that φ is a chain map between two $\mathbb{F}$-chain complexes C_* and D_*.

Also note that we mostly use augmented chain complexes[3], i.e., chain complexes C_*, where $C_{-1} \neq \emptyset$. It is a custom to denote such chain complexes with a tilde, i.e. $\widetilde{C}_*$ instead of C_*.

We are now ready to define homology groups. We also define boundaries and cycles, which provide some geometric intuition about the properties captured in homology, at least for simplicial and singular chain complexes that we define later.

Definition 3.6 (Homology groups). *Let $\mathbb{F}$ be a field and*

$$C_* = \quad \dots \xrightarrow{\partial_{l+2}} C_{l+1} \xrightarrow{\partial_{l+1}} C_l \xrightarrow{\partial_l} C_{l-1} \xrightarrow{\partial_{l-1}} \dots$$

*an $\mathbb{F}$-chain complex. The l*th group of cycles Z_l with $\mathbb{F}$-coefficients *is defined as $Z_l := \operatorname{Ker} \partial_l$ and the l*th group of boundaries B_l with $\mathbb{F}$-coefficients *is defined as $B_l := \operatorname{Im} \partial_{l+1}$. The l*th homology group H_l with $\mathbb{F}$-coefficients *is then defined as the quotient Z_l/B_l. The dimension of H_l as an $\mathbb{F}$-vector space is called the l*th Betti number β_l.

If we consider the trivial morphisms between the homology groups $0 \colon H_l \to H_{l-1}$, $l \in \mathbb{Z}$, we obtain the following chain complex

$$H_* = \quad \dots \xrightarrow{0} H_{l+1} \xrightarrow{0} H_l \xrightarrow{0} H_{l-1} \xrightarrow{0} \dots$$

It is a custom to call the homology groups of augmented complexes reduced and denote them with a tilde, i.e. $\widetilde{H_l}$ instead of H_l and $\widetilde{H_*}$ instead of H_*, similarly we have reduced Betti numbers $\widetilde{\beta_l}$.

Definition 3.7 (Induced map). *Let $\mathbb{F}$ be a field and $\varphi \colon C_* \to D_*$ be a chain map between two $\mathbb{F}$-chain complexes. Let B^C_l be the boundaries in C_*, H^C_l be the homology groups for C_*, B^D_l the boundaries for D_* and H^D_l the homology groups for D_*.*

The induced map in homology $\varphi_* \colon H^C_* \to H^D_*$ is defined as follows:

$$\varphi_*(z + B^C_l) := \varphi(z) + B^D_l \quad \text{if } z \text{ is a cycle in } C_l.$$

It can be easily checked that φ_* is a well-defined chain map.

[3]In simplicial and singular homology, we may assign two reasonable chain complexes to a simplicial complex (or a topological space), one of them having $C_{-1} = 0$ and the other not. The latter is called augmented.

3.1.3 Simplicial homology

When one works with simplicial complexes in algebraic topology, one usually assumes that some ordering of their vertices is implicitly given. One also assumes that subcomplexes inherit this ordering.

Our calculations turned out to be easier, if we change the ordering of vertices in certain situations. It means that we have to state the vertex ordering explicitly. Let us now look how this affects the definition of simplicial homology. We note that the constructions are relatively standard[4], see e.g. [Mun84, Chapter 1, §13] or [Bro, Jon11].

Definition 3.8 (Ordered simplices)**.** *Let K be a (finite abstract) simplicial complex and $l \geq -1$ an integer. A sequence $(v_0, v_1, \dots, v_l)$ with $\{v_0, v_1, \dots, v_l\} \in K$ is called an* ordered l-simplex in K. *The elements $v_0, \dots, v_l$ are its* vertices. *An ordered l-simplex $\sigma = (v_0, v_1, \dots, v_l)$ is* degenerated, *if two of its vertices coincide, i.e., if there exist i,j, $0 \leq i < j \leq l$ for which $v_i = v_j$. If $\pi \in S(\{0, 1, \dots, l\})$ is a permutation of the set $\{0, 1, \dots, l\}$ and $\sigma = (v_0, \dots, v_l)$ an ordered l-simplex in K, we define*[5] *$\pi(\sigma) := (v_{\pi^{-1}(0)}, v_{\pi^{-1}(1)}, \dots, v_{\pi^{-1}(l)})$. If $\sigma = (v_0, \dots, v_l)$ is an l-simplex in K and $a \in V(K)$ is a vertex of K, then $a \wedge \sigma$ denotes the ordered $(l+1)$-simplex $(a, v_0, v_1, \dots, v_l)$. Note that we cannot assume that $a \wedge \sigma$ is an $(l+1)$-face of K, so we regard $a \wedge \sigma$ as an $(l+1)$-face of the simplicial complex $2^{V(K)}$.*

We are now ready to define the simplicial chain groups.

Definition 3.9 (Simplicial chain groups)**.** *Let K be a (finite abstract) simplicial complex, $\mathbb{F}$ a field and $l \geq -1$ an integer. We define the* lth augmented ordered simplicial chain group $\widetilde{O}_l(K;\mathbb{F})$ *to be the $\mathbb{F}$-vector space with basis consisting of all ordered l-simplices in K. In other words, $\widetilde{O}_l(K;\mathbb{F})$ is the set of all finite sums $\sum a_i\sigma_i$, where $a_i \in \mathbb{F}$ and σ_i are ordered l-simplices in K. The addition of two such sums and multiplication of such sum by an element of $\mathbb{F}$ is defined in the natural way.*

We further define the lth augmented simplifying[6] simplicial chain group $\widetilde{T}_l(K;\mathbb{F})$ *to be the vector subspace of $\widetilde{O}_l(K;\mathbb{F})$ generated by degenerated singular l-simplices and elements of the form $\big(\sigma - \operatorname{sgn}(\pi)\pi(\sigma)\big)$, where σ ranges over all ordered l-simplices in K and π ranges over all permutations $\pi \in S\big(\{0, 1, \dots, l\}\big)$.*

The lth augmented simplicial chain group $\widetilde{C}_l(K;\mathbb{F})$ *is the quotient-space*

$$\widetilde{C}_l(K;\mathbb{F}) := \widetilde{O}_l(K;\mathbb{F})/\widetilde{T}_l(K;\mathbb{F}).$$

If $l < -1$, we put $\widetilde{C}_l(K;\mathbb{F}) = \widetilde{O}_l(K;\mathbb{F}) = \widetilde{T}_l(K;\mathbb{F}) = 0$.

We note that $\widetilde{O}_{-1}(K;\mathbb{F})$ is generated by the empty sequence ω and hence isomorphic to $\mathbb{F}$. Furthermore, there are no ordered degenerated (-1)-simplices. Since there is only one permutation of the empty set, and its sign is by definition 1, $T_{-1}(K;\mathbb{F})$ is generated by $\omega - \omega = 0$, hence $\widetilde{C}_{-1}(K;\mathbb{F}) \cong \widetilde{O}_{-1}(K;\mathbb{F}) \cong \mathbb{F}$.

Now we define the boundary operators in order to turn simplicial chain groups into chain complexes:

Definition 3.10 (Simplicial boundary operators)**.** *Let K be a (finite abstract) simplicial complex and $\mathbb{F}$ be a field. If $l \geq 0$ is an integer, the* lth ordered simplicial boundary operator $\partial^o_l \colon \widetilde{O}_l(K;\mathbb{F}) \to \widetilde{O}_{l-1}(K;\mathbb{F})$

[4]The ordered homology is not new. As noted in [Bar95] the discussion whether to use ordered or oriented chain groups for singular homology dates back to Lefschetz [Lef33], Eilenberg [Eil44] and Steenrod [ES52]. In short: in oriented homology one regards two l-dimensional singular simplices σ and σ' as equal, if there exists an order preserving linear transformation τ of $|\Delta_l|$ such that $\sigma = \sigma' \circ \tau$ and one regards σ as equal to $-\sigma'$ if there is an order reversing linear transformation τ' such that $\sigma = \sigma' \circ \tau$. One further throws away "degenerated" simplices satisfying $\sigma = -\sigma$. In ordered homology one considers degenerated simplices, but factors out the subgroup they generate, see [ST80]. Regardless whether we choose the chain group of all singular simplices, ordered singular simplices or oriented singular simplices [Bar95], the resulting homology functors are naturally isomorphic.

The situation for ordered and oriented simplicial homology is analogous.

[5]Observe that $\pi(\sigma)$ is again an ordered l-simplex in K. Also note that it is common to define $\pi(\sigma)$ using π^{-1} to permute vertices, since in this way v_i maps onto $v_{\pi(i)}$. Moreover, without the inverse the equality $\pi'(\pi(\sigma)) = (\pi' \circ \pi)(\sigma)$ would not hold.

[6]The letter T stands for trivial, but the term trivial chain group would collide with the term trivial group, hence we decided to call $\widetilde{T}_l(K;\mathbb{F})$ simplifying group, because we use it to simplify our calculations.

is given[7] *on the basis of* $\widetilde{O}_l(K;\mathbb{F})$ *by*[8]

$$\partial_l^o\big((v_0,v_1,\dots,v_l)\big) := \sum_{i=0}^{l}(-1)^i(v_0,v_1,\dots,\widehat{v_i},\dots,v_l)$$

and extended linearly to the whole space $\widetilde{O}_l(K;\mathbb{F})$.

It is easy to check that $\partial_l^o\big(\widetilde{T}_l(K;\mathbb{F})\big) \subseteq \widetilde{T}_{l-1}(K;\mathbb{F})$, *so we may define the* lth simplicial boundary operator $\partial_l\colon \widetilde{C}_l(K;\mathbb{F}) \to \widetilde{C}_{l-1}(K;\mathbb{F})$ *by* $\partial_l\big(o+\widetilde{T}_l(K;\mathbb{F})\big) := \big(\partial_l^o o\big) + \widetilde{T}_{l-1}(K;\mathbb{F})$ *for every* $o \in \widetilde{O}_l(K;\mathbb{F})$.

For $l<0$ *we put* $\partial_l^o = 0$, $\partial_l = 0$.

It can be easily verified that $\partial_l^o \circ \partial_{l+1}^o = 0$ and $\partial_l \circ \partial_{l+1} = 0$ for every integer l.

Definition 3.11 (Simplicial chain complexes)**.** *Let K be a (finite abstract) simplicial complex and $\mathbb{F}$ be a field. The* augmented ordered simplicial chain complex $\widetilde{O}_*(K;\mathbb{F})$ *is the following infinite sequence of $\mathbb{F}$-vector spaces and maps between them*

$$\widetilde{O}_*(K;\mathbb{F}) := \qquad \dots \xrightarrow{\partial_{l+2}^o} \widetilde{O}_{l+1}(K;\mathbb{F}) \xrightarrow{\partial_{l+1}^o} \widetilde{O}_l(K;\mathbb{F}) \xrightarrow{\partial_l^o} \widetilde{O}_{l-1}(K;\mathbb{F}) \xrightarrow{\partial_{l-1}^o} \dots$$

According to Definition 3.6, the augmented ordered simplicial chain complex gives rise to ordered simplicial boundary groups $B_l^o(K;\mathbb{F})$, ordered simplicial cycle groups $Z_l^o(K;\mathbb{F})$ *and* reduced ordered simplicial homology groups $\widetilde{H}_l^o(K;\mathbb{F})$.

The augmented simplicial chain complex $\widetilde{C}_*(K;\mathbb{F})$ *is the infinite sequence*

$$\widetilde{C}_*(K;\mathbb{F}) := \qquad \dots \xrightarrow{\partial_{l+2}} \widetilde{C}_{l+1}(K;\mathbb{F}) \xrightarrow{\partial_{l+1}} \widetilde{C}_l(K;\mathbb{F}) \xrightarrow{\partial_l} \widetilde{C}_{l-1}(K;\mathbb{F}) \xrightarrow{\partial_{l-1}} \dots$$

It gives rise to simplicial boundary groups $B_l(K;\mathbb{F})$, simplicial cycle groups $Z_l(K;\mathbb{F})$ *and* reduced simplicial homology groups $\widetilde{H}_l(K;\mathbb{F})$.

If the field $\mathbb{F}$ is clear from the context, we omit it from the notation and only write $\widetilde{H}_l(K)$, $B_l^o(K)$, $Z_l^o(K)$, $\widetilde{C}_l(K)$, etc. If we explicitly state the source and target space of the boundary operator, (or when it does not matter), we also omit the indexes l and o for ∂, i.e., we write $\widetilde{O}_l(K) \xrightarrow{\partial} \widetilde{O}_{l-1}(K)$, instead of $\widetilde{O}_l(K;\mathbb{F}) \xrightarrow{\partial_l^o} \widetilde{O}_{l-1}(K;\mathbb{F})$, etc.

3.1.4 Singular homology

In order to allow easy transition from simplicial to singular homology groups, we present a definition of singular homology that matches the ordered approach.

Let us define several maps first.

Definition 3.12 (Standard simplices)**.** *Let $l > -1$. The* standard l-dimensional simplex $|\Delta_l|$ *is defined as*[9] $|\Delta_l| := \operatorname{conv}\big(\{\mathbf{e}_1,\dots,\mathbf{e}_{l+1}\}\big) = \{(t_0,t_1,\dots,t_l) \mid 0 \le t_i \le 1, \sum_{i=1}^{l} t_i = 1\} \subseteq \mathbb{R}^{l+1}$.

For $0 \le i \le l$ *we define the* ith face map $\delta_l^i\colon |\Delta_{l-1}| \to |\Delta_l|$ *by*

$$\delta_l^i(t_0,t_1,\dots,t_{l-1}) := (t_0,t_1,\dots t_{i-1},0,t_i,t_{i+1},\dots,t_{l-1}).$$

If $0 \le i \le l$ *and* $0 \le j \le l$ *are two distinct integers* ($i \ne j$)*, we define the* degeneracy map $\sigma_l^{i,j}\colon |\Delta_l| \to |\Delta_l|$ *by*[10]

$$\sigma_l^{i,j}(t_0,t_1,\dots,t_l) := \begin{cases}(t_0,\dots,t_{i-1},0,t_{i+1},\dots,t_{j-1},t_j+t_i,t_{j+1},\dots,t_l), & \text{if } i<j,\\ (t_0,\dots,t_{j-1},t_j+t_i,t_{j+1},\dots,t_{i-1},0,t_{i+1},\dots,t_l) & \text{if } i>j.\end{cases}$$

If $\pi \in S\big(\{0,1,\dots,l\}\big)$ *is an permutation, we define the* permutation map $p_l^\pi\colon |\Delta_l| \to |\Delta_l|$ *by* $p_l^\pi(t_0,t_1,\dots,t_l) := (t_{\pi^{-1}(0)},t_{\pi^{-1}(1)},\dots,t_{\pi^{-1}(l)})$. *Furthermore, if π is the identity on $\emptyset$, we let p_{-1}^π be the identity on $\emptyset$ as well.*

[7] The boundary operator depends on $\mathbb{F}$ and K, but to keep the notation simple, these are usually omitted.

[8] We use the symbol $(v_0,v_1,\dots,\widehat{v_i},\dots,v_l)$ as a shorthand for $(v_0,v_1,\dots,v_l)$ with v_i removed. That is, $(v_0,v_1,\dots,\widehat{v_i},\dots,v_l) = (v_0,v_1,\dots,v_{i-1},v_{i+1},v_{i+2},\dots,v_l)$.

[9] See Definitions 3.1 and 3.3.

[10] With some fantasy the case $i>j$ can be viewed as the proper interpretation of the first formula for $j<i$.

In other words, δ_l^i is the orientation preserving inclusion of $|\Delta_{l-1}|$ on the facet of $|\Delta_l|$ that does not contain the ith vertex, the map $\sigma_l^{i,j}$ maps the ith vertex to the jth vertex and leaves other vertices unchanged[11] and p_l^π permutes the vertices of $|\Delta_l|$ according to the permutation π.

In the proof of Lemma 3.32 we will need the following observation concerning maps $\sigma_l^{i,j}$, δ_l^i and p_l^π:

Observation 3.13. *Let k, l, m be integers satisfying $0 \le k \le l$, $0 \le m \le l-1$. Then*

$$\sigma_l^{m+1,m} \circ \delta_l^k = \begin{cases} \delta_l^k \circ \sigma_{l-1}^{m,m-1} & \text{if } k < m, \\ \delta_l^{m+1} & \text{if } k \in \{m, m+1\}, \\ \delta_l^k \circ \sigma_{l-1}^{m+1,m} & \text{if } k > m+1 \end{cases} \tag{3.1}$$

and

$$\delta_{l+1}^{k+1} \circ \delta_l^0 = \delta_{l+1}^0 \circ \delta_l^k.$$

Moreover, if π is the transposition (01) viewed as an element of $S(\{0, 1, \dots, l\})$ and π' is the transposition (01) viewed as an element of $S(\{0, 1, \dots, l-1\})$, then

$$p_l^\pi \circ \delta_l^k = \begin{cases} \delta_l^1 & \text{if } k = 0 \\ \delta_l^0 & \text{if } k = 1 \\ \delta_l^k \circ p_{l-1}^{\pi'} & \text{if } k > 1 \end{cases} \tag{3.2}$$

and $\sigma_l^{2,1} \circ p_l^\pi = p_l^\pi \circ \sigma_l^{2,0}$.

Proof. The maps δ_i^j and $\sigma_i^{j,n}$ are linear for all non-negative integers i, j, n. Hence it suffices to verify that for every $i = 1, \dots, l+1$ the left and right hand-side of Equation (3.1) map $\mathbf{e}_i$ onto the same point and similarly for the remaining equations. This can be easily checked using the equalities

$$\begin{aligned} \delta_l^j(\mathbf{e}_i) &= \begin{cases} \mathbf{e}_i & \text{for } 0 \le i < j, \\ \mathbf{e}_{i+1} & \text{for } l \ge i \ge j, \end{cases} \\ \sigma_l^{j+1,j}(\mathbf{e}_i) &= \begin{cases} \mathbf{e}_i & \text{for } 0 \le i \le l, i \ne j+1, \\ \mathbf{e}_j & \text{for } i = j+1, \end{cases} \\ p_l^\pi(\mathbf{e}_i) &= \mathbf{e}_{\pi(i)}. \end{aligned}$$

□

We can now define singular simplices.

Definition 3.14 (Ordered singular simplices)**.** *Let $l \ge -1$ be an integer and X a topological space. An* ordered singular l-simplex *in X is any continuous map $\gamma\colon |\Delta_l| \to X$. An ordered singular l-simplex $\gamma\colon |\Delta_l| \to X$ is* degenerated *if $\gamma = \gamma' \circ \sigma_l^{i,j}$ for some $i, j \in \{0, \dots, l\}$, $i \ne j$ and some ordered singular l-simplex $\gamma'\colon \Delta_l \to X$.*

In other words, γ is degenerated if it factors linearly through some face of $|\Delta_l|$.

Next we describe the singular chain groups.

Definition 3.15 (Singular chain groups)**.** *Let X be a topological space, $\mathbb{F}$ a field and $l \ge -1$ an integer. We define the l*th augmented ordered singular chain group *$\widetilde{O}_l(X;\mathbb{F})$ to be the $\mathbb{F}$-vector space with basis consisting of all ordered singular l-simplices in X. In other words, $\widetilde{O}_l(X;\mathbb{F})$ is the set of all finite sums $\sum a_i\gamma_i$, where $a_i \in \mathbb{F}$ and γ_i are ordered singular l-simplices in X. The addition of two such sums and multiplication of such sum by an element of $\mathbb{F}$ is defined in the natural way.*

*We further define the l*th augmented simplifying[12] singular chain group *$\widetilde{T}_l(X;\mathbb{F})$ to be the vector subspace of $\widetilde{O}_l(X;\mathbb{F})$ generated by degenerated singular l-simplices and elements of the form $(\sigma -$*

[11] Usually the degeneracy map is defined as a map from $|\Delta_l|$ to $|\Delta_{l-1}|$. Our calculations are simpler if we have a map from $|\Delta_l|$ to $|\Delta_l|$. Hence we have decided to compose the usual jth degeneracy map with the ith face map to obtain our map $\sigma_l^{i,j}$.

[12] As in the simplicial case, the letter T stands for trivial, but the term trivial chain group would collide with the term trivial group, hence we decided to call $\widetilde{T}_l(X;\mathbb{F})$ simplifying group, because it is used to simplify our calculations.

$\operatorname{sgn}(\pi)(\sigma \circ p_l^\pi)\big)$, *where* σ *ranges over all singular* l*-simplices in* X *and* π *ranges over all permutations* $\pi \in S\big(\{0, 1, \ldots, l\}\big)$*.*

The lth augmented singular chain group $\widetilde{C}_l(X; \mathbb{F})$ *is the quotient-space*

$$\widetilde{C}_l(X; \mathbb{F}) := \widetilde{O}_l(X; \mathbb{F})/\widetilde{T}_l(X; \mathbb{F}).$$

If $l < -1$*, we put* $\widetilde{C}_l(X; \mathbb{F}) = \widetilde{O}_l(X; \mathbb{F}) = \widetilde{T}_l(X; \mathbb{F}) = 0$*.*

A (-1)-dimensional singular simplex in X is the unique, empty map from $\emptyset$ to X. Furthermore we have $\widetilde{O}_{-1}(X; \mathbb{F}) \cong \mathbb{F}$, $\widetilde{T}_{-1}(X; \mathbb{F}) \cong 0$ and $\widetilde{C}_{-1}(X; \mathbb{F}) \cong \mathbb{F}$ by the same argument as for the simplicial chain groups.

In order to turn singular chain groups into chain complexes, we define the boundary operators.

Definition 3.16 (Singular boundary operators)**.** *Let* X *be a topological space and* $\mathbb{F}$ *a field. If* $l \geq 0$ *is an integer, the* lth ordered singular boundary operator $\partial_l^O \colon \widetilde{O}_l(X; \mathbb{F}) \to \widetilde{O}_{l-1}(X; \mathbb{F})$ *is given on the basis of* $\widetilde{O}_l(X; \mathbb{F})$ *by*[13]

$$\partial_l^O(\gamma) = \sum_{i=0}^{l} (-1)^i (\gamma \circ \delta_l^i)$$

and extended linearly to the whole space $\widetilde{O}_l(X; \mathbb{F})$*.*

It can be checked that $\partial_l^O\big(\widetilde{T}_l(X; \mathbb{F})\big) \subseteq \widetilde{T}_{l-1}(X; \mathbb{F})$*, so we may further define the* lth singular boundary operator $\partial_l \colon \widetilde{C}_l(X; \mathbb{F}) \to \widetilde{C}_{l-1}(X; \mathbb{F})$ *by* $\partial_l\big(o + \widetilde{T}_l(X; \mathbb{F})\big) := \big(\partial_l^O o\big) + \widetilde{T}_{l-1}(X; \mathbb{F})$ *for every* $o \in \widetilde{O}_l(X; \mathbb{F})$*.*

For $l < 0$ *we put* $\partial_l^O = 0$*,* $\partial_l = 0$*.*

It can be easily verified that $\partial_l^O \circ \partial_{l+1}^O = 0$ and $\partial_l \circ \partial_{l+1} = 0$ for every integer l.

Definition 3.17 is an analogue of Definition 3.11 for singular chains.

Definition 3.17 (Singular chain complexes)**.** *Let* X *be a topological space and* $\mathbb{F}$ *a field. The* augmented ordered singular chain complex $\widetilde{O}_*(X; \mathbb{F})$ *is the following infinite sequence of* $\mathbb{F}$*-vector spaces and maps between them*

$$\widetilde{O}_*(X; \mathbb{F}) := \qquad \ldots \xrightarrow{\partial_{l+2}^O} \widetilde{O}_{l+1}(X; \mathbb{F}) \xrightarrow{\partial_{l+1}^O} \widetilde{O}_l(X; \mathbb{F}) \xrightarrow{\partial_l^O} \widetilde{O}_{l-1}(X; \mathbb{F}) \xrightarrow{\partial_{l-1}^O} \ldots$$

According to Definition 3.6, the augmented ordered singular chain complex gives rise to ordered singular boundary groups $B_l^O(X; \mathbb{F})$*,* ordered singular cycle groups $Z_l^O(X; \mathbb{F})$ *and* reduced ordered singular homology groups $\widetilde{H}_l^O(X; \mathbb{F})$*.*

The augmented singular chain complex $\widetilde{C}_*(X; \mathbb{F})$ *is the infinite sequence*

$$\widetilde{C}_*(X; \mathbb{F}) := \qquad \ldots \xrightarrow{\partial_{l+2}} \widetilde{C}_{l+1}(X; \mathbb{F}) \xrightarrow{\partial_{l+1}} \widetilde{C}_l(X; \mathbb{F}) \xrightarrow{\partial_l} \widetilde{C}_{l-1}(X; \mathbb{F}) \xrightarrow{\partial_{l-1}} \ldots$$

It gives rise to singular boundary groups $B_l(X; \mathbb{F})$*,* singular cycle groups $Z_l(X; \mathbb{F})$ *and* reduced singular homology groups $\widetilde{H}_l(X; \mathbb{F})$ *and* reduced Betti numbers $\widetilde{\beta}_l(X; \mathbb{F})$*.*

As for the simplicial homology, if the field $\mathbb{F}$ is clear from the context, we omit it from the notation and only write $\widetilde{H}_l(X)$, $Z_l^O(X)$, $\widetilde{C}_l(X)$, etc. If we explicitly state the source and target space of the boundary operator, (or when it does not matter), we also omit the indexes l and O for ∂, i.e., we write $\widetilde{O}_l(X) \xrightarrow{\partial} \widetilde{O}_{l-1}(X)$, instead of $\widetilde{O}_l(X; \mathbb{F}) \xrightarrow{\partial_l^O} \widetilde{O}_{l-1}(X; \mathbb{F})$, etc.

Note that if K is a simplicial complex, $\widetilde{H}_l(K)$ stands for the lth simplicial homology group, whereas for a topological space X, the symbol $\widetilde{H}_l(X)$ means the lth singular homology group. In particular, $\widetilde{H}_l(|K|)$ is the lth *singular* homology group corresponding to simplicial complex K.

Any continuous map $f \colon X \to Y$ between two topological spaces X, Y induces chain maps between the corresponding chain complexes and hence also between the corresponding homology groups. The following definition shows which symbols we use for various induced maps.

[13] See Definition 3.12, which introduces the maps δ_l^i.

Definition 3.18. *Let $f\colon X \to Y$ be a continuous map between two topological spaces X, Y and $\mathbb{F}$ be a field. The induced chain map $f^O_\sharp\colon \widetilde{O}_*(X;\mathbb{F}) \to \widetilde{O}_*(Y;\mathbb{F})$ is prescribed on the generators of $\widetilde{O}_*(X;\mathbb{F})$ by*

$$f^O_\sharp(\gamma) := f \circ \gamma \quad \text{for every ordered singular simplex } \gamma$$

and extended linearly onto the whole $\widetilde{O}_*(X;\mathbb{F})$.

The induced chain map $f_\sharp\colon \widetilde{C}_*(X;\mathbb{F}) \to \widetilde{C}_*(Y;\mathbb{F})$ is prescribed on the generators of $\widetilde{C}_*(X;\mathbb{F})$ by

$$f_\sharp\big(\gamma + \widetilde{T}_l(X;\mathbb{F})\big) := (f\circ\gamma) + \widetilde{T}_l(Y;\mathbb{F}) \quad \text{if } \gamma \text{ is an ordered singular } l\text{-simplex.}$$

We further set $f^O_* := (f^O_\sharp)_*$, $f_* := (f_\sharp)_*$.

Note that $f^O_\sharp\big(\widetilde{T}_l(X;\mathbb{F})\big) \subseteq \widetilde{T}_l(Y;\mathbb{F})$, hence $f_\sharp$ is well-defined.

As before, we may omit the superscript O, if it is clear from the context.

Definition 3.19 (Inclusion of simplicial chains into singular chains)**.** *Let K be a (finite abstract) simplicial complex with geometric realization $|K|$. Let $\mathbb{F}$ be a field. There exists a* natural inclusion *ι^o_K of $\widetilde{O}_*(K;\mathbb{F})$ into $\widetilde{O}_*(|K|;\mathbb{F})$. It is defined as follows: Let $\sigma = (v_0, v_1, \ldots, v_l)$ be an l-dimensional ordered simplex in K. Let the geometric realization of its vertices be $|v_0|,\ldots, |v_l|$. The value of $\iota^o_K(\sigma)$ is the singular l-simplex[14] $\gamma\colon |\Delta_l| \to |K|$ given by $(t_0, t_1, \ldots, t_l) \mapsto t_0\,|v_0| + \ldots + t_l\,|v_l|$.*

The natural inclusion *$\iota_K\colon \widetilde{C}_*(K;\mathbb{F}) \to \widetilde{C}_*(|K|;\mathbb{F})$ is defined as follows:*

$$\iota_K\big(\sigma + \widetilde{T}_l(K;\mathbb{F})\big) := \iota^o_K(\sigma) + \widetilde{T}_l(|K|;\mathbb{F}) \quad \text{if } \sigma \text{ is an ordered } l\text{-simplex in } K.$$

We conclude this subsection with a comparison of the defined homology groups.

Theorem 3.20 (Equivalence of defined homologies)**.** *Let X be a topological space and $\mathbb{F}$ a field. Then the factorization $\pi^{sg}\colon \widetilde{O}_*(X;\mathbb{F}) \to \widetilde{O}_*(X;\mathbb{F})/T_*(X;\mathbb{F}) = \widetilde{C}_*(X;\mathbb{F})$ induces an isomorphism π^{sg}_* of $\widetilde{H}^O_*(X;\mathbb{F})$ and $\widetilde{H}_*(X;\mathbb{F})$.*

Let K be a (finite abstract) simplicial complex and $\mathbb{F}$ a field. Then the factorization $\pi^\Delta\colon \widetilde{O}_*(K;\mathbb{F}) \to \widetilde{O}_*(K;\mathbb{F})/T_*(K;\mathbb{F}) = \widetilde{C}_*(K;\mathbb{F})$ induces an isomorphism π^Δ_* of $\widetilde{H}^o_*(K;\mathbb{F})$ and $\widetilde{H}_*(K;\mathbb{F})$.

Also the induced maps $(\iota^o_K)_*\colon \widetilde{H}^o_*(K;\mathbb{F}) \to \widetilde{H}^O_*(|K|;\mathbb{F})$ and $(\iota_K)_*\colon \widetilde{H}_*(K;\mathbb{F}) \to \widetilde{H}_*(|K|;\mathbb{F})$ are isomorphisms.

Proof. The isomorphisms $\widetilde{H}^o_*(K;\mathbb{F}) \cong \widetilde{H}_*(K;\mathbb{F}) \cong \widetilde{H}^O_*(|K|;\mathbb{F})$ follow from [Pra07, Thm. 2.1, Thm. 4.7]. The isomorphism $\widetilde{H}^O_*(|K|;\mathbb{F}) \cong \widetilde{H}_*(|K|;\mathbb{F})$ is provided in [Bar95] or can be obtained by an inspection of the proof of equivalence of simplicial and singular homology in Hatcher's textbook [Hat02, Thm. 2.27]. □

We note that for simplicial homology the definition of reduced homology in Hatcher's textbook [Hat02] agrees with $\widetilde{H}_*$, whereas for singular homology his definition agrees with $\widetilde{H}^O_*$.

Remark 3.21. *For a simplicial complex K, we have defined the lth simplicial chain group $\widetilde{C}_l(K)$ as the factor $\widetilde{O}_l(K)/\widetilde{T}_l(K)$. Every ordered l-face $(u_0, u_1, \ldots, u_l)$ of K is modulo $\widetilde{T}_l$ equivalent to 0 or $(-1)^t(v_0, v_1, \ldots, v_l)$, where $v_0 < v_1 < \ldots, v_l$ in the ordering of $V(K)$ and t is either 0 or 1. Since chains of this form generate a subgroup of $\widetilde{O}_l(K)$, the short exact sequence $0 \to \widetilde{T}_l(K) \to \widetilde{O}_l(K) \to \widetilde{C}_l(K) \to 0$ splits and $\widetilde{O}_l(K) = \widetilde{C}_l(K) \oplus \widetilde{T}_l(K)$. So we can either view $\widetilde{C}_l$ as a free group generated by the simplices of the form $(v_0, v_1, \ldots, v_l)$, where $v_0 < v_1 < \ldots < v_l$, or as the factor $\widetilde{O}_l(K)/\widetilde{T}_l(K)$. The second approach only differs from the first one by introducing new names for some chains in $\widetilde{C}_l(K)$, e.g. $(v_0, v_2, v_1) = -(v_0, v_1, v_2)$, $(v_0, v_1, v_1) = 0$, etc. The new names are very useful during our calculations.*

*To simplify our terminology, we say that σ is a k-*face *of K, if $\sigma = (v_0, v_1, \ldots, v_l)$, where $v_0 < v_1 < \ldots < v_l$ and $\{v_0, v_1, \ldots, v_l\} \in K$.*

[14] Recall that $|v_i|$ is a point in the simplex $|\sigma|$, hence $t_0\,|v_0| + \ldots + t_l\,|v_l| \in |\sigma| \subseteq |K|$ and $\iota^o_K(\sigma)$ is indeed a singular simplex in $\widetilde{O}_*(|K|;\mathbb{F})$.

3.1.5 Almost embeddings

Definition 3.22. *Let K be an abstract simplicial complex and $\mathbb{F}$ a field. A* support *of an ordered l-simplex $\sigma = (v_0, v_1, \dots, v_l)$ in K is defined as follows*

$$\operatorname{supp}(\sigma) := \begin{cases} \emptyset & \text{if } \sigma \text{ is degenerated,} \\ \{v_0, v_1, \dots, v_l\} & \text{otherwise.} \end{cases}$$

Let $\gamma \in \widetilde{C}_l\,(K;\mathbb{F})$ be a simplicial chain. Then γ can be expressed as $\gamma = \sum_{i\in I} a_i\sigma_i + \widetilde{T}_l\,(K;\mathbb{F})$, where $\sigma_i \in K$ are non-degenerated ordered simplices in K, all a_i are nonzero and σ_i and σ_j have distinct supports for $i \neq j$. We then define the support $\operatorname{supp}(\gamma)$ of γ *as*

$$\operatorname{supp}(\gamma) := \bigcup_{i\in I} \sigma_i.$$

We note that the expression of γ as $\sum_{i\in I} a_i\sigma_i + \widetilde{T}_l\,(K;\mathbb{F})$ is not unique. For example, $(v_0, v_1) + \widetilde{T}_l(K;\mathbb{F}) = -(v_1, v_0) + \widetilde{T}_l(K;\mathbb{F})$. However, the result $\operatorname{supp}(\gamma)$ is independent of the way we express γ.

Before we proceed further, let us recall that we assume that any (abstract) simplicial complex K comes equipped with a fixed geometrical realization $|K|$. That is, if we write $|K|$, then we always mean the same geometric realization of K.

Definition 3.23. *Let K be an abstract simplicial complex with geometric realization $|K|$, $j \geq 1$ an integer and X a topological space. We say that f is a j-*almost embedding of K into X*, if $f\colon |K| \to X$ is a continuous map such that for every $(j+1)$ pairwise disjoint faces $\sigma_0, \sigma_1, \dots, \sigma_j \in K$*

$$f(|\sigma_0|) \cap f(|\sigma_1|) \cap \dots \cap f(|\sigma_j|) = \emptyset.$$

The term almost embedding *is used as a shorthand for 1-almost embedding.*

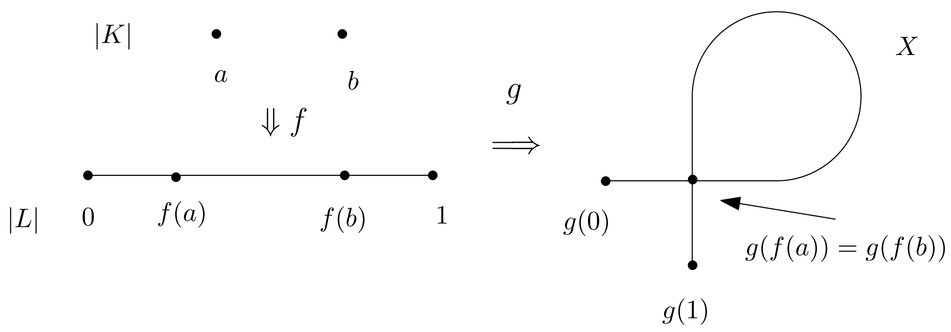
If K and L are two abstract simplicial complexes and $\mathbb{F}$ a field, then a chain map $\varphi\colon \widetilde{C}_*\,(K;\mathbb{F}) \to \widetilde{C}_*\,(L;\mathbb{F})$ is called an almost embedding *if for every two disjoint faces $\sigma, \sigma' \in K$*

$$\operatorname{supp}\big(\varphi(\sigma)\big) \cap \operatorname{supp}\big(\varphi(\sigma')\big) = \emptyset.$$

The definition of a j-almost embedding f ensures that for every $(j+1)$-tuple of disjoint faces $\sigma_1, \dots, \sigma_{j+1}$ the intersection of their f images is empty, i.e., $f(|\sigma_0|) \cap f(|\sigma_1|) \cap \dots \cap f(|\sigma_j|) = \emptyset$, however, there may exist up to j disjoint faces, whose f images intersect.

Note that every (continuous) embedding $f\colon |K| \to X$ is an almost embedding as well.

If K, L are two simplicial complexes and X a topological space, f an almost embedding of K into $|L|$ and g an almost embedding of L into X, it may happen that $g \circ f$ is not an almost embedding, see Fig. 3.1.



The composition of almost embeddings is not always an almost embedding. Here an example for $K = \{\emptyset, \{a\}, \{b\}\}$, $L = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$ and almost embeddings f of K into $|L|$ and g of L into $X = \mathbb{R}^2$.

Figure 3.1: Bad composition of almost embeddings

But this cannot happen, if f behaves nicely:

Observation 3.24. *If K, L are two simplicial complexes, X a topological space, $k > 0$ an integer, f an almost embedding of K into $|L|$, g a k-almost embedding of L into X and if for every $\sigma \in K$ the image $f(|\sigma|)$ has the form*

$$f(|\sigma|) = \bigcup_{i \in I} |\rho_i'|, \quad \text{where } \rho_i' \text{ are simplices in } L,$$

then $g \circ f$ is a k-almost embedding of K into X.

The proof is straightforward.

3.2 Statement of the main result

Let us recall the main idea behind Theorems 1.1, 1.5 and 1.8: Suppose that L is a finite simplicial complex and $f\colon \left|\Delta_n^{(k)}\right| \to X$ is a continuous map of $\left|\Delta_n^{(k)}\right|$ into a sufficiently nice topological space X. If p is a prime number and n is big enough (depending on X, L and k), we can, using Ramsey theory and the additive structure of the chain group $C_*\left(\Delta_n^{(k)}; \mathbb{Z}_p\right)$, find an almost embedding $\varphi\colon C_*\left(L; \mathbb{Z}_p\right) \to C_*\left(\Delta_n^{(k)}; \mathbb{F}\right)$ such that the composition $f_\sharp \circ \varphi$ is homologically trivial.

One of our main results in this direction[15] is Theorem 3.25, which provides a reasonably good bound on n and serves as our main technical tool in proving Theorems 1.1 and 1.5 in Chapter 4.

Theorem 3.25. *Let $b \geq 0$ and $n, s, k > 0$ be integers and p a prime number. Let M be a manifold[16] with k-th reduced $\mathbb{Z}_p$-Betti number at most b. Let $f\colon \left|\Delta_n^{(k)}\right| \to M$ be a continuous map.*

If

$$n \geq \binom{s}{k} b(s - 2k) + s + 1 \quad \text{and} \quad n \geq s + 1, \tag{3.3}$$

then there exists an almost embedding $g\colon \left|\Delta_s^{(k)}\right| \to \left|\Delta_n^{(k)}\right|$ such that the induced homomorphism in homology $(f \circ g)_*\colon H_*\left(\left|\Delta_s^{(k)}\right|; \mathbb{Z}_p\right) \to H_*(M; \mathbb{Z}_p)$ is trivial. Moreover, the image $g(|\sigma|)$ of every face $\sigma \in \Delta_s^{(k)}$ has the form $\bigcup_{i \in I_\sigma} |\rho_i'|$, where ρ_i' are some simplices from $\Delta_n^{(k)}$.

Actually, we prove the following stronger form of Theorem 3.25:

Theorem 3.26. *Let $b \geq 0$ and $n, s, k > 0$ be integers and p a prime number. Let M be a manifold with k-th reduced $\mathbb{Z}_p$-Betti number at most b. Let $\theta\colon \widetilde{C}_*\left(\left|\Delta_n^{(k)}\right|; \mathbb{Z}_p\right) \to \widetilde{C}_*\left(M; \mathbb{Z}_p\right)$ be a chain map.*

If

$$n \geq \binom{s}{k} b(s - 2k) + s + 1 \quad \text{and} \quad n \geq s + 1, \tag{3.4}$$

then there exists an almost embedding $g\colon \left|\Delta_s^{(k)}\right| \to \left|\Delta_n^{(k)}\right|$ such that the induced homomorphism in homology $\theta_* \circ g_*\colon H_*\left(\left|\Delta_s^{(k)}\right|; \mathbb{Z}_p\right) \to H_*(M; \mathbb{Z}_p)$ is trivial. Moreover, the image $g(|\sigma|)$ of every face $\sigma \in \Delta_s^{(k)}$ has the form $\bigcup_{i \in I_\sigma} |\rho_i'|$, where ρ_i' are some simplices from $\Delta_n^{(k)}$.

Theorem 3.25 then follows from Theorem 3.26 by setting $\theta := f_\sharp$.

Remark 3.27. *The reader who does not want to use colorful algebraic Tverberg theorem (Theorem 2.14), may still go through the proofs in this chapter and use Lemma 2.10 instead, but he/she obtains a bound $n \geq \binom{s}{k} b(s - 2k) + 2s - 2k$ instead of $n \geq \binom{s}{k} b(s - 2k) + s + 1$ in Theorems 3.25 and 3.26.*

[15] For the other results in this direction see Chapter 6.

[16] The theorem remains valid if we replace manifold M with an arbitrary topological space X and consider its singular homology when computing Betti numbers.

$$
\begin{array}{ccccc}
\widetilde{H}_*\left(\Delta_s^{(k)};\mathbb{Z}_p\right) & \xrightarrow{\psi_*} & \widetilde{H}_*\left(\Delta_n^{(k)};\mathbb{Z}_p\right) & & \\
\downarrow{\scriptstyle (\iota_s)_*} & & \downarrow{\scriptstyle (\iota_n)_*} & \searrow{\scriptstyle \varphi_*} & \\
\widetilde{H}_*\left(\left|\Delta_s^{(k)}\right|;\mathbb{Z}_p\right) & \xrightarrow{g_*} & \widetilde{H}_*\left(\left|\Delta_n^{(k)}\right|;\mathbb{Z}_p\right) & \xrightarrow{\theta_*} & \widetilde{H}_*(M;\mathbb{Z}_p)
\end{array}
$$

Figure 3.2: Triviality of $\theta_* \circ (\iota_n)_* \circ \psi_*$ implies triviality of $\theta_* \circ g_*$. Note that $(\iota_s)_*$ is an isomorphism.

3.3 Proof of the main result

We split the proof of Theorem 3.26 into several lemmas representing separate ideas. Throughout the chapter we assume that $\mathbb{F}$ is a field and $k, n, s \geq 1$ are fixed integers, $m = \binom{s+1}{k+1}$ is the number of all k-faces of $\Delta_s^{(k)}$, $\sigma_1, \ldots, \sigma_m$ are all of them and $\theta\colon \widetilde{C}_*\left(\left|\Delta_n^{(k)}\right|;\mathbb{F}\right) \to \widetilde{C}_*(M;\mathbb{F})$ is a fixed chain map. Moreover, because we want the proof to be as understandable as possible, we will avoid an unnecessary symbol for the inclusion map $\Delta_s^{(k)} \hookrightarrow \Delta_n^{(k)}$ and consider $\Delta_s^{(k)}$ as the subcomplex on the first $s+1$ vertices of $\Delta_n^{(k)}$, in accordance Section 3.1.1. The general plan looks as follows:

Let ι_n be the natural inclusion of $\widetilde{C}_*\left(\Delta_n^{(k)};\mathbb{F}\right)$ into $\widetilde{C}_*\left(\left|\Delta_n^{(k)}\right|;\mathbb{F}\right)$, similarly for ι_s. For brevity we set $\varphi := \theta \circ (\iota_n)$. First we show how to construct a certain chain map $\psi\colon \widetilde{C}_*\left(\Delta_s^{(k)};\mathbb{F}\right) \to \widetilde{C}_*\left(\Delta_n^{(k)};\mathbb{F}\right)$ such that the composed map $\varphi_* \circ \psi_*\colon H_*(\Delta_s^{(k)};\mathbb{F}) \to H_*(M;\mathbb{F})$ is trivial, that is, $\varphi_* \circ \psi_* = 0$. Then for $\mathbb{F} = \mathbb{Z}_p$, we use ψ to construct an almost embedding $g\colon \left|\Delta_s^{(k)}\right| \to \left|\Delta_n^{(k)}\right|$ satisfying $(\iota_n)_* \circ \psi_* = g_* \circ (\iota_s)_*$. Then $0 = \varphi_* \circ \psi_* = \theta_* \circ (\iota_n)_* \circ \psi_* = \theta_* \circ g_* \circ (\iota_s)_*$ and because $(\iota_s)_*$ is an isomorphism [Hat02, Theorem 2.21], it implies triviality of $\theta_* \circ g_*$. Commutative diagram 3.2 illustrates the situation.

The proof of Theorem 3.26 is split into separate lemmas as follows: Observation 3.28 describes the following idea behind the construction of ψ: If modify the inclusion

$$\widetilde{C}_*\left(\Delta_s^{(k)};\mathbb{F}\right) \hookrightarrow \widetilde{C}_*\left(\Delta_n^{(k)};\mathbb{F}\right)$$

by subtracting any linear map that maps all maximal dimensional faces σ_i of $\Delta_s^{(k)}$ to cycles z_{σ_i} and all other faces to zero, the resulting linear map $\psi\colon \widetilde{C}_*\left(\Delta_s^{(k)};\mathbb{F}\right) \to \widetilde{C}_*\left(\Delta_n^{(k)};\mathbb{F}\right)$ determined by

$$\psi(\sigma) := \begin{cases} \sigma & \text{if } \sigma \in \Delta_s^{(k)} \text{ is a face of dimension strictly less than } k, \\ \sigma_i - z_{\sigma_i} & \text{if } \sigma = \sigma_i. \end{cases}$$

is a chain map again.

Then we restrict our attention to cycles z_{σ_i} of a special form[17] $\partial(x_i \wedge \sigma_i)$, where $x_i \in V\left(\Delta_n^{(k)}\right)$. If $k = 1$ and $\sigma_i = (v_0, v_1)$ is a k-face, then σ_i "travels" from v_0 to v_1 directly, whereas $\psi(\sigma) = \sigma - z_{\sigma_i} = (v_0, x_i) + (x_i, v_1)$ makes a detour through point x_i, see Fig. 3.3. To obtain better bounds later on we allow x_i to be multipoints[18], see Figure 3.3 again.

Lemma 3.29 shows a condition on the multipoints x_i which forces the induced homomorphism $\varphi_* \circ \psi_*$ to be trivial. Lemma 3.30 provides sufficient condition for ψ to be an almost embedding. Lemma 3.31 states that the condition of Lemma 3.30 can be satisfied with a relatively small number of multipoints. Lemma 3.32 is used to construct $f\colon \left|\Delta_s^{(k)}\right| \to \left|\Delta_n^{(k)}\right|$ satisfying $(\iota_n)_* \circ \psi_* = f_* \circ (\iota_s)_*$. At the end we use Theorem 2.14 to find the multipoints satisfying all the prescribed conditions.

We note that Theorem 3.25 does hold even for $k = 0$, stating that if we have $s(b+1)+2$ points in a topological space having $(b+1)$ path-connected components, we can always find $s+1$ of them lying in a common path-connected component. Lemma 3.34 shows that our approach can be viewed as a natural generalization of the case $k = 0$ to higher values of k.

For the next observation, recall that σ_i are all k-dimensional faces of $\Delta_s^{(k)}$.

[17] See Definition 3.8 for the introduction of $x_i \wedge \sigma_i$.
[18] See Definition 2.2 for the introduction of multipoints.

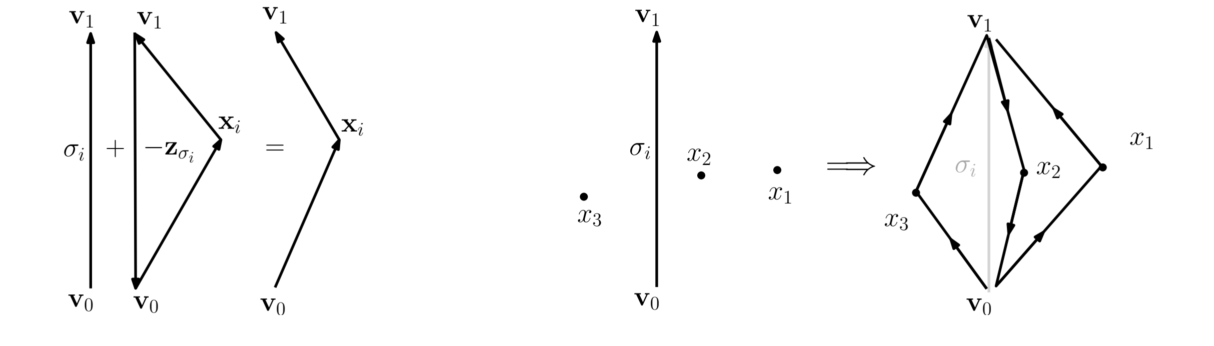
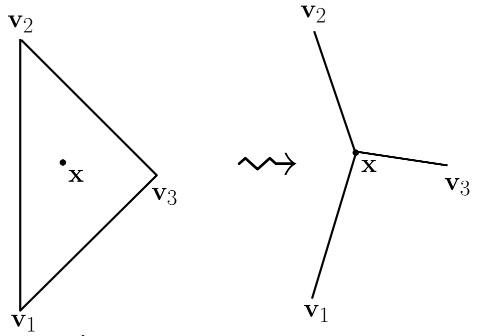


Figure 3.3: Edge $\sigma = (v_1, v_2)$ "rerouted" through point x_i and through multipoint $x_3 - x_2 + x_1$.



The cycle $v_1v_2 + v_2v_3 + v_3v_1$ turns into $v_1x + xv_2 + v_2x + xv_3 + v_3x + xv_1$, which can be contracted to x. Other triangles behave the same way. Similarly in higher dimensions.

Figure 3.4: Setting $z_{\sigma_i} = \partial(x \wedge \sigma_i)$ for all σ_i implies triviality of ψ_*

Observation 3.28. *Let $\mathbb{F}$ be a field and $z_{\sigma_1}, z_{\sigma_2}, \ldots, z_{\sigma_m}$ arbitrary cycles in $\widetilde{C}_*\left(\Delta_n^{(k)}; \mathbb{F}\right)$. If we have a linear map $\psi\colon \widetilde{C}_*\left(\Delta_s^{(k)}; \mathbb{F}\right) \to \widetilde{C}_*\left(\Delta_n^{(k)}; \mathbb{F}\right)$ which satisfies*

$$\psi(\sigma) := \begin{cases} \sigma & \text{if } \sigma \in \Delta_s^{(k)} \text{ is a face of dimension strictly less than } k, \\ \sigma_i - z_{\sigma_i} & \text{if } \sigma = \sigma_i. \end{cases}$$

Then ψ is a chain map.

Proof. Since the map ψ is linear, we only need to verify that $\psi \circ \partial = \partial \circ \psi$. The boundary map ∂ is also linear and the set of all faces of $\Delta_s^{(k)}$ freely generates $\widetilde{C}_*\left(\Delta_s^{(k)}; \mathbb{F}\right)$, see Remark 3.21. So it suffices to check that $\psi \circ \partial(\sigma) = \partial \circ \psi(\sigma)$ for every face $\sigma \in \Delta_s^{(k)}$.

Let σ be a face of $\Delta_s^{(k)}$. Its boundary $\partial\sigma$ is a chain that contains only faces of dimension less than k. Since ψ is identical on such faces, its linearity gives $\psi(\partial\sigma) = \partial\sigma$.

If σ is a face of dimension less than k, $\psi\sigma = \sigma$ by definition, hence $\partial\psi\sigma = \partial\sigma$ in that case.

For a k-dimensional face σ_i we have $\psi(\sigma_i) = \sigma_i - z_{\sigma_i}$. By linearity $\partial(\psi(\sigma_i)) = \partial\sigma_i - \partial z_{\sigma_i}$. Because z_{σ_i} is a cycle, we have $\partial z_{\sigma_i} = 0$, hence $\partial(\psi(\sigma_i)) = \partial\sigma_i$ as well. This finishes the proof. □

To simplify the expressions in the following text we introduce some notational conventions now. If K is a simplicial complex and z a cycle in $\widetilde{C}_k(K; \mathbb{F})$, we let $[z]$ denote its homology class, i.e., $[z] = z + B_k(K; \mathbb{F})$. Furthermore, we write $\varphi_*[z]$ as a shorthand for $\varphi_*([z])$.

Informal sketch of Lemma 3.29. If we picked up a vertex $x \in \Delta_n^{(k)}$ and set $z_{\sigma_i} = \partial(x \wedge \sigma_i)$ for all k-dimensional faces σ_i, the map ψ from Observation 3.28 would satisfy $\psi_* = 0$, see Fig. 3.4 for an illustration in the case $k = 1$. Most likely, ψ would not be an almost embedding. But there is a way around this: If we had m different points[19] $x_1, \dots, x_m$ satisfying $\varphi_*\big[\partial(x_1 \wedge \sigma_i)\big] = \varphi_*\big[\partial(x_j \wedge \sigma_i)\big]$ for all i and j, we could set $z_{\sigma_i} = \partial(x_i \wedge \sigma_i)$. The map $\varphi_* \circ \psi_*$ would be trivial, since the different points x_j all behave as x_1 with respect to φ_*. Note that it is also possible to replace the points $x_1, \dots, x_r$ with multipoints[20] $\mu_1, \dots, \mu_r$.

Now we formalize the idea for multipoints and arbitrary k: Let $X = V\left(\Delta_n^{(k)}\right)$ be the vertex set of $\Delta_n^{(k)}$, $\mu = \sum_{x \in X} a_x x \in \mathcal{M}(X;\mathbb{F})$ be a multipoint and σ be a k-face in $\Delta_s^{(k)}$. We set

$$\partial(\mu \wedge \sigma) := \sum_{x \in X} a_x \partial(x \wedge \sigma). \tag{3.5}$$

Clearly $\partial(\mu \wedge \sigma) \in \widetilde{C}_*\left(\Delta_n^{(k)}, \mathbb{F}\right)$ is a cycle.

Recall that $\sigma_1, \dots, \sigma_m$ are all k-dimensional faces of K and we have a chain map $\varphi\colon \widetilde{C}_*\left(\Delta_n^{(k)};\mathbb{F}\right) \to \widetilde{C}_*(M;\mathbb{F})$. We define

$$\mathbf{v}(\mu) := \Big(\varphi_*\big[\partial(\mu \wedge \sigma_1)\big], \varphi_*\big[\partial(\mu \wedge \sigma_2)\big], \dots, \varphi_*\big[\partial(\mu \wedge \sigma_m)\big]\Big) \tag{3.6}$$

for every $\mu \in \mathcal{M}(X;\mathbb{F})$.

Lemma 3.29. *Let $\mathbb{F}$ be a field, $X = V\left(\Delta_n^{(k)}\right)$ be the vertex set of $\Delta_n^{(k)}$, $\mu_{\sigma_1}, \mu_{\sigma_2}, \dots, \mu_{\sigma_m}$ be (not necessarily distinct) multipoints from $\mathcal{M}(X;\mathbb{F})$ with $\mathbf{v}(\mu_{\sigma_1}) = \mathbf{v}(\mu_{\sigma_2}) = \ldots = \mathbf{v}(\mu_{\sigma_m})$. Let $\psi\colon \widetilde{C}_*\left(\Delta_s^{(k)};\mathbb{F}\right) \to \widetilde{C}_*\left(\Delta_n^{(k)};\mathbb{F}\right)$ be the linear map determined by its value on generators of $\widetilde{C}_*\left(\Delta_s^{(k)};\mathbb{F}\right)$:*

$$\psi(\sigma) := \begin{cases} \sigma & \text{if } \sigma \in \Delta_s^{(k)} \text{ is a face of dimension strictly less than } k, \\ \sigma_i - \partial(\mu_{\sigma_i} \wedge \sigma_i) & \text{if } \sigma = \sigma_i. \end{cases} \tag{3.7}$$

Then ψ satisfies $\varphi_* \circ \psi_* = 0$.

Proof. Because $\partial(\mu_{\sigma_i} \wedge \sigma_i)$ is a cycle, ψ is a well-defined chain map by Observation 3.28. The only condition that needs to be checked is $\varphi_* \circ \psi_* = 0$. The homology group $H_*\left(\Delta_s^{(k)};\mathbb{F}\right)$ is generated[21] by the elements of the form $[\partial\tau]$, where τ is a $(k+1)$-dimensional face of $\Delta_s^{(k+1)}$. Therefore, we only need to verify that $\varphi_*\big(\psi_*[\partial\tau]\big) = 0$ for all faces $\tau \in \Delta_s$ of dimension $k+1$.

Nevertheless, in order to verify it, we need to work with $(k+2)$-dimensional faces. It makes the calculations somewhat tricky, since we can only apply φ_* to cycles of dimension at most k.

So let $\tau = (v_0, v_1, \dots, v_{k+1})$ be a $(k+1)$-dimensional face of Δ_s. To keep the equations short we set $\tau_i = (v_0, v_1, \dots, \widehat{v_i}, \dots, v_{k+1})$. By definition

$$\partial\tau = \sum_{i=0}^{k+1} (-1)^i \tau_i. \tag{3.8}$$

Using (3.7) and observing that all τ_i are k-faces, we obtain

$$\psi(\partial\tau) = \sum_{i=0}^{k+1} (-1)^i \big(\tau_i - \partial(\mu_{\tau_i} \wedge \tau_i)\big),$$

[19] That means one point for every k-face.

[20] See Definition 2.2.

[21] This is a well known fact, which can be verified for example as follows: We know the number of l-dimensional faces of $\Delta_s^{(k)}$ for every l, so we may compute its Euler characteristic. The non-reduced homology of $\Delta_s^{(k)}$ is concentrated in degrees 0 and k, so we may use the Euler characteristic to compute $\tilde{\beta}_k\left(\Delta_s^{(k)};\mathbb{Z}_2\right) = \binom{s+1}{k+1}$. Then we verify that $[\partial\tau]$, where τ is a $(k+1)$-dimensional face of $\Delta_s^{(k+1)}$ containing its first vertex, are linearly independent and hence form a basis of $H_*\left(\Delta_s^{(k)};\mathbb{Z}_2\right)$.

henceforth

$$
\begin{aligned}
(\varphi_* \circ \psi_*)(\partial\tau) &= \varphi_*\left[\sum_{i=0}^{k+1}(-1)^i\big(\tau_i - \partial(\mu_{\tau_i}\wedge\tau_i)\big)\right] \\
&= \varphi_*\left[\sum_{i=0}^{k+1}(-1)^i\tau_i\right] - \sum_{i=0}^{k+1}(-1)^i\varphi_*\big[\partial(\mu_{\tau_i}\wedge\tau_i)\big]
\end{aligned}
$$

The multipoints μ_{τ_i} satisfy $\mathbf{v}(\mu_{\tau_i}) = \mathbf{v}(\mu_{\tau_1})$, hence by Equation (3.6), $\varphi_*\big[\partial(\mu_{\tau_i}\wedge\sigma)\big] = \varphi_*\big[\partial(\mu_{\tau_1}\wedge\sigma)\big]$ for every k-dimensional face σ and we may continue our calculations:

$$
\begin{aligned}
(\varphi_* \circ \psi_*)(\partial\tau) &= \varphi_*\left[\sum_{i=0}^{k+1}(-1)^i\tau_i\right] - \sum_{i=0}^{k+1}(-1)^i\varphi_*\big[\partial(\mu_{\tau_1}\wedge\tau_i)\big] \\
&= \varphi_*\left[\sum_{i=0}^{k+1}(-1)^i\big(\tau_i - \partial(\mu_{\tau_1}\wedge\tau_i)\big)\right].
\end{aligned}
$$

Because μ_{τ_1} is a multipoint in $X = V\left(\Delta_n^{(k)}\right)$, we can express it as an affine combination $\mu_{\tau_1} = \sum_{x\in X} a_x \cdot x$, where $\sum_{x\in X} a_x = 1$. Definition (3.5) of $\partial(\mu\wedge\tau_i)$ then yields

$$
(\varphi_* \circ \psi_*)(\partial\tau) = \varphi_*\left[\sum_{i=0}^{k+1}(-1)^i\left(\tau_i - \sum_{x\in X} a_x\partial(x\wedge\tau_i)\right)\right].
$$

Because $\sum_{x\in X} a_x = 1$, we may rewrite the sum further to

$$
(\varphi_* \circ \psi_*)(\partial\tau) = \varphi_*\left[\sum_{i=0}^{k+1}(-1)^i\left(\sum_{x\in X} a_x\big(\tau_i - \partial(x\wedge\tau_i)\big)\right)\right].
$$

Both sums are finite, we can rearrange them, use the definition of $\partial\tau$ (3.8) and linearity of ∂:

$$
\begin{aligned}
(\varphi_* \circ \psi_*)(\partial\tau) &= \varphi_*\left[\sum_{x\in X} a_x\left(\partial\tau - \sum_{i=0}^{k+1}(-1)^i\partial(x\wedge\tau_i)\right)\right] \\
&= \varphi_*\left[\sum_{x\in X} a_x\left(\partial\left(\tau - \sum_{i=0}^{k+1}(-1)^i(x\wedge\tau_i)\right)\right)\right].
\end{aligned}
$$

Now we observe that by definition $\left(\tau - \sum_{i=0}^{k+1}(-1)^i(x\wedge\tau_i)\right) = \partial(x\wedge\tau)$ and finally arrive at

$$
(\varphi_* \circ \psi_*)(\partial\tau) = \varphi_*\left[\sum_{x\in X} a_x\big(\partial\partial(x\wedge\tau)\big)\right] = \varphi_*\left[\sum_{x\in X} a_x\cdot 0\right] = 0,
$$

which finishes the proof. □

Our task now is to find conditions on the multipoints $\mu_{\sigma_1}, \mu_{\sigma_2}, \ldots, \mu_{\sigma_m}$ that will guarantee that ψ is an almost embedding.

Lemma 3.30. *Let $\mathbb{F}$ be a field. Let U be the set of vertices of $\Delta_n^{(k)}$ that do not lie in $V\left(\Delta_s^{(k)}\right)$. Let $U_{\sigma_1}, U_{\sigma_2}, \ldots, U_{\sigma_m} \subseteq U$ be (not necessarily distinct) sets. Let $X = V\left(\Delta_n^{(k)}\right)$ be the set of vertices of $\Delta_n^{(k)}$. Let $\mu_{\sigma_1}, \mu_{\sigma_2}, \ldots, \mu_{\sigma_m} \in \mathcal{M}(X;\mathbb{F})$ be multipoints. If*

$$
\begin{aligned}
&\operatorname{supp}\mu_{\sigma_i} \subseteq (\sigma_i \cup U_{\sigma_i}) && \textit{for every } i = 1,\ldots,m \textit{ and} && (3.9)\\
&\sigma\cap\tau = \emptyset \Rightarrow U_\sigma \cap U_\tau = \emptyset && \textit{for every two } k\textit{-faces } \sigma,\tau \in \Delta_s^{(k)}, && (3.10)
\end{aligned}
$$

then the linear map $\psi\colon \widetilde{C}_*\left(\Delta_s^{(k)};\mathbb{F}\right) \to \widetilde{C}_*\left(\Delta_n^{(k)};\mathbb{F}\right)$ *defined (as before) by*

$$\psi(\sigma) := \begin{cases} \sigma & \text{if } \sigma \in \Delta_s^{(k)} \text{ is a face of dimension strictly less than } k, \\ \sigma_i - \partial(\mu_{\sigma_i} \wedge \sigma_i) & \text{if } \sigma = \sigma_i \end{cases} \tag{3.11}$$

is an almost embedding.

Proof. According to Definition 3.23 we need to verify that $\operatorname{supp}\psi(\sigma) \cap \operatorname{supp}\psi(\tau) = \emptyset$ whenever σ and τ are disjoint faces in $\Delta_s^{(k)}$.

For every k-face σ_i we have

$$\begin{aligned} \operatorname{supp}\psi(\sigma_i) = \operatorname{supp}\big(\sigma_i - \partial(\mu_{\sigma_i} \wedge \sigma_i)\big) \quad &\subseteq \quad \operatorname{supp}\sigma_i \cup \operatorname{supp}\big(\partial(\mu_{\sigma_i} \wedge \sigma_i)\big) \\ &\subseteq \quad \sigma_i \cup \big(\sigma_i \cup \operatorname{supp}\mu_{\sigma_i}\big) \\ &\subseteq \quad \sigma_i \cup U_{\sigma_i}, \end{aligned}$$

where the last inclusion follows from (3.9).

Our calculations will be simplified if we moreover set $U_\sigma := \emptyset$ for every face $\sigma \in \Delta_s^{(k)}$ of dimension less than k. Because $\psi(\sigma) = \sigma$ in that case, we then have

$$\operatorname{supp}\psi(\sigma) \subseteq \sigma \cup U_\sigma \tag{3.12}$$

for every face σ of $\Delta_s^{(k)}$.

We will now show that ψ is an almost embedding as follows: Assume that $\sigma \cap \tau = \emptyset$. By (3.12) we have

$$\operatorname{supp}\psi(\sigma) \cap \operatorname{supp}\psi(\tau) \subseteq (\sigma \cup U_\sigma) \cap (\tau \cup U_\tau)\,.$$

Using distributivity of $\cup$ and $\cap$ and noting that $\sigma \cap U_\tau \subseteq V\left(\Delta_s^{(k)}\right) \cap U = \emptyset$ and $\tau \cap U_\sigma \subseteq V\left(\Delta_s^{(k)}\right) \cap U = \emptyset$ the right hand-side reduces to

$$(\sigma \cap \tau) \cup (U_\sigma \cap U_\tau).$$

We assume that $\sigma \cap \tau = \emptyset$. If σ or τ has dimension less than k, U_σ or U_τ, respectively, is empty, which implies $U_\sigma \cap U_\tau = \emptyset$. If both σ,τ have dimension k, the second intersection is empty by (3.10). Hence $\operatorname{supp}\psi(\sigma) \cap \operatorname{supp}\psi(\tau) = \emptyset$ in all cases. □

Now we show that conditions of Lemma 3.30 can be satisfied with relatively small number of distinct multipoints. As mentioned above, we only need that φ is an almost-embedding, so we can use the same multipoint for several k-faces provided they pairwise intersect. Optimizing the number of multipoints used reformulates as the following hypergraph coloring problem:

> Assign to each k-face σ_i of Δ_s some color $c(i) \in \mathbb{N}$ such that $\operatorname{card}\{c(i) : 1 \le i \le m\}$ is minimal and disjoint faces use distinct colors.

This question is classically known as Kneser's hypergraph coloring problem and for $s - 2k + 1 \ge 1$ an optimal solution uses $s - 2k + 1$ colors [Lov78, Mat03]. Let us spell out one such coloring (proving its optimality is considerably more difficult, but we do not need to know that it is optimal). Let us assume that the vertex set of $\Delta_s^{(k)}$ equals $\{v_0, v_1, \ldots, v_s\}$. For every k-face σ_i we let $\min\sigma_i$ denote the smallest index of a vertex in σ_i. When $\min\sigma_i \le s - 2k - 1$ we set $c(i) = \min\sigma_i$, otherwise we set $c(i) = s - 2k$. Observe that any k-face with color $c \le s - 2k - 1$ contains vertex v_c. Moreover, the k-faces with color $s - 2k$ consist of $k+1$ vertices each, all from a set of $2k+1$ vertices. It follows that any two k-faces with the same color have some vertex in common.

For $s - 2k + 1 \le 0$, every two k-faces intersect, hence we may use the same color for all of them.

We conclude

Lemma 3.31. *If* $s - 2k + 1 > 0$*, there exists an assignment* c *of* $s - 2k + 1$ *colors to* k*-dimensional faces of* $\Delta_s^{(k)}$ *such that disjoint faces use distinct colors. If* $s - 2k + 1 \le 0$ *such assignment* c *uses only one color.*

Lemma 3.32. *Let $k > 0$ be an integer and p a prime number. Let $\mathbb{F} = \mathbb{Z}_p$. Let $X = V\left(\Delta_n^{(k)}\right)$ be the set of vertices of $\Delta_n^{(k)}$. Let $\mu_{\sigma_1}, \dots, \mu_{\sigma_m}$ be (not necessarily distinct) multipoints from $\mathcal{M}(X;\mathbb{F})$. If, as before, we define $\psi\colon C_*\left(\Delta_s^{(k)};\mathbb{F}\right) \to C_*(M;\mathbb{Z}_p)$ by (3.11), i.e.,*

$$\psi(\sigma) := \begin{cases} \sigma & \text{if } \sigma \in \Delta_s^{(k)} \text{ is a face of dimension strictly less than } k, \\ \sigma_i - \partial(\mu_{\sigma_i} \wedge \sigma_i) & \text{if } \sigma = \sigma_i, \end{cases} \tag{3.13}$$

then there exists a continuous map $f\colon \left|\Delta_s^{(k)}\right| \to \left|\Delta_n^{(k)}\right|$ such that

$$(\iota_n)_* \circ \psi_* = f_* \circ (\iota_s)_*. \tag{3.14}$$

If ψ is an almost embedding, then f is also an almost embedding. Moreover, for every face $\sigma \in \Delta_s^{(k)}$, the image $f\big(|\sigma|\big)$ has the form $\bigcup_{i \in I_\sigma} |\tau_i|$, where τ_i are some simplices in $\Delta_n^{(k)}$.

We first present the main idea of the proof, without going into technical details.

Proof idea. Suppose that $\sigma \in \Delta_s^{(k)}$ is a k-face and $\mu_\sigma = \sum_{x \in X} a_x x$ is a multipoint. By definitions and the linearity of ι_n, we have

$$\iota_n\big(\psi(\sigma)\big) = \iota_n\left(\sum_{x \in X} a_x(\sigma - \partial(x \wedge \sigma))\right) = \sum_{x \in X} a_x \iota_n\left(\sigma - \partial(x \wedge \sigma)\right).$$

Let $B := B_k\left(\left|\Delta_n^{(k)}\right|;\mathbb{F}\right) \subseteq \widetilde{C}_k\left(\left|\Delta_n^{(k)}\right|;\mathbb{F}\right)$ be the subgroup of boundaries. First, for every $x \in X$ and a k-face $\sigma \in \Delta_s^{(k)}$ we will construct a singular simplex $\gamma_{\sigma,x}\colon |\Delta_k| \to \left|\Delta_n^{(k)}\right|$ such that[22] $\gamma_{\sigma,x} \equiv \iota_n\big(\sigma - \partial(x \wedge \sigma)\big) \pmod{B}$. The construction generalizes the concatenation of two paths into higher dimensions.

Then we will construct a singular simplex $\gamma_\sigma\colon |\Delta_k| \to \left|\Delta_n^{(k)}\right|$ with $\gamma_\sigma \equiv \sum_{x \in X} a_x \gamma_{\sigma,x} \pmod{B}$. This construction inductively[23] uses higher dimensional analogue of the following idea: If we have three paths γ_1, γ_2, γ_3 between points a and b, we may take a long walk and go from a to b along γ_1, returning to a via γ_2 and finally arrive to b by γ_3.

Moreover, we construct the maps $\gamma_{\sigma,x}$ and γ_σ so that they agree on boundaries with $\iota_n(\sigma)$, i.e., such that for every $i = 0, 1, \dots, k$ the following holds[24]: $\iota_n(\sigma) \circ \delta_k^i = \gamma_{\sigma,x} \circ \delta_k^i = \gamma_\sigma \circ \delta_k^i$.

It follows that we may define a map $f\colon \left|\Delta_s^{(k)}\right| \to \left|\Delta_n^{(k)}\right|$ by the following procedure[25]:

For a point $x \in \left|\Delta_s^{(k)}\right|$ we fix a k-face $|\sigma_x|$ containing x and define

$$f(x) := \left(\gamma_{\sigma_x} \circ \big(\iota_s(\sigma_x)\big)^{-1}\right)(x).$$

We can read the expression in the following way: $\iota_s(\sigma_x)$ is the order-preserving linear map from $|\Delta_k|$ onto $|\sigma_x|$, we look at the preimage of x in $|\Delta_k|$ and map this preimage via our map γ_{σ_x}.

It can be checked that f is continuous.

Then if $\sigma \in \Delta_s^{(k)}$ is a k-face, we have

$$\begin{aligned} f \circ \iota_s(\sigma) &= \gamma_\sigma \circ \left(\iota_s(\sigma)^{-1}\right) \circ \iota_s(\sigma) = \gamma_\sigma \\ &\equiv \sum_{x \in X} a_x \gamma_{\sigma,x} \equiv \sum_{x \in X} a_x \iota_n\big(\sigma - \partial(x \wedge \sigma)\big) \equiv \iota_n\big(\psi(\sigma)\big) \pmod{B}, \end{aligned}$$

hence $f \circ \iota_s(\sigma) \equiv \iota_n(\psi(\sigma)) \pmod{B}$.

From the definition of homology groups then immediately follows that $f_* \circ (\iota_s)_* = (\iota_n)_* \circ \psi_*$, as desired.

The other claims of the lemma can also be easily checked. ☐

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[22] The notation $x \equiv y \pmod{B}$ means $x + B = y + B$, that is, $\{x + b \mid b \in B\} = \{y + b \mid b \in B\}$.

[23] This is the only place where we need that $\mathbb{F} = \mathbb{Z}_p$ for some prime number p.

[24] See Definition 3.12 for the introduction of δ_k^i.

[25] Recall that ι_s is the natural inclusion of $C_*\left(\Delta_s^{(k)};\mathbb{F}\right)$ into $C_*\left(\left|\Delta_s^{(k)}\right|;\mathbb{F}\right)$. In particular, $\iota_s(\sigma)$ is a bijection of $|\Delta_k|$ onto $|\sigma|$.

The full proof is relatively long and complicated and contains no essential ideas. For this reasons we have decided to postpone it to the end of the current chapter.

Now we address the question how to satisfy assumptions of Lemma 3.29 in our setting. Recall that

$$\mathbf{v}(\mu) := \Big(\varphi_*\big[\partial(\mu\wedge\sigma_1)\big], \varphi_*\big[\partial(\mu\wedge\sigma_2)\big], \dots, \varphi_*\big[\partial(\mu\wedge\sigma_m)\big]\Big),$$

where $\varphi\colon C_*\left(\Delta_n^{(k)};\mathbb{F}\right) \to C_*(M;\mathbb{F})$ is some fixed chain map, M has kth $\mathbb{F}$-Betti number b and $\sigma_1,\dots,\sigma_m$ are all k-faces of $\Delta_s^{(k)}$.

According to Lemma 3.30 we need some multipoints μ with the same value of $\mathbf{v}(\mu)$. We would like to use colored algebraic Tverberg theorem (Theorem 2.14) or Lemma 2.10 to obtain these multipoints. To that end we need to know the dimension of $\operatorname{Im}\mathbf{v}$.

Lemma 3.33. *If $\varphi\colon C_*\left(\Delta_n^{(k)};\mathbb{F}\right) \to C_*(M;\mathbb{F})$ is given, the image $\operatorname{Im}\mathbf{v}$ lies in an $\mathbb{F}$-affine space that has dimension at most $\binom{s}{k}b$.*

Proof. Let $X = V\left(\Delta_n^{(k)}\right)$ be the set of vertices of $\Delta_n^{(k)}$. Let σ_j, $j\in J$, be all k-faces of $\Delta_s^{(k)}$ that contain the first vertex v_0.

We will show that for every k-face $\tau\in\Delta_s^{(k)}$ there exist a constant $c_\tau\in H_k(M;\mathbb{F})$ and coefficients $c_{\tau,j}\in\mathbb{F}$ such that for every $\mu\in\mathcal{M}(X,\mathbb{F})$

$$\varphi_*\big[\partial(\mu\wedge\tau)\big] = c_\tau + \sum_{j\in J} c_{\tau,j}\varphi_*\big[\partial(\mu\wedge\sigma_j)\big]. \tag{3.15}$$

Because all the values $\varphi_*\big[\partial(\mu\wedge\sigma_j)\big]$ belong to $H_k(M;\mathbb{F})\cong\mathbb{F}^b$ and there are $\binom{s}{k}$ faces σ_j containing v_0, that will finish the proof.

We start with the case when $\mu = x$ is an ordinary point.

Let $\tau = (w_0, w_1, \dots, w_k)$ be a k-face and $\tau_i = (w_0, w_1, \dots, \widehat{w_i}, \dots, w_k)$ for every $i = 0,\dots,k$.

We have $\partial\partial((x, v_0, w_0, w_1, \dots, w_k)) = 0$, hence $\varphi_*\big[\partial\partial(x, v_0, w_0, w_1, \dots, w_k)\big] = 0$. Expanding the innermost ∂ by definition, we have

$$\varphi_*\left[\partial\left(v_0\wedge\tau - x\wedge\tau + \sum_{i=0}^{k}(-1)^i x\wedge v_0\wedge\tau_i\right)\right] = 0.$$

By linearity of ∂ and φ_* we get

$$\varphi_*\big[\partial\left(v_0\wedge\tau\right)\big] - \varphi_*\big[\partial\left(x\wedge\tau\right)\big] + \sum_{i=0}^{k}(-1)^i\varphi_*\big[\partial\left(x\wedge v_0\wedge\tau_i\right)\big] = 0,$$

hence

$$\varphi_*\big[\partial\left(x\wedge\tau\right)\big] = \varphi_*\big[\partial\left(v_0\wedge\tau\right)\big] + \sum_{i=0}^{k}(-1)^i\varphi_*\big[\partial\left(x\wedge v_0\wedge\tau_i\right)\big], \tag{3.16}$$

which is the desired equality (3.15).

If $\mu = \sum_{x\in X} a_x x$ is a multipoint, we sum up the equalities $\varphi_*\big[\partial(x\wedge\tau)\big] = c_\tau + \sum_{j\in J} c_{\tau,j}\varphi_*\big[\partial(x\wedge\sigma_j)\big]$ and obtain

$$\sum_{x\in X} a_x\varphi_*\big[\partial(x\wedge\tau)\big] = \sum_{x\in X} a_x c_\tau + \sum_{j\in J} c_{\tau,j}\Big(\sum_{x\in X} a_x\varphi_*\big[\partial(x\wedge\sigma_j)\big]\Big),$$

which, together with (3.5) and the fact that $\sum_{x\in X} a_x = 1$, gives (3.15). □

We are now finally ready to prove Theorem 3.26.

Proof of Theorem 3.26. Let us first deal with the case $s-2k\geq 0$. In that case $n\geq\binom{s}{k}b(s-2k)+s+1\geq s+1$. Set $d := s-2k$. Let $Y = \{v_0, v_1, \dots, v_{d-1}\}$ be the set of the first d vertices of $\Delta_s^{(k)}$ and X be the set of vertices of $\Delta_n^{(k)}$ that do not lie in $\Delta_s^{(k)}$. We partition $C = X \cup Y$ into disjoint color classes $C_0 := Y, C_1, C_2, \dots, C_{|X|}$, where each C_i, $i = 1, \dots, |X|$ contains exactly one point from X.

Let us recall that $\varphi = \theta \circ \iota_n$ and let $\mathbf{v}$ be the affine map from (3.6). We have $|C| = (n+1) - (s+1) + d$. By the assumptions of the theorem $n \geq \binom{s}{k} bd + s + 1$. Hence $|C| \geq \binom{s}{k} bd + s + 1 - s + d = \left(\binom{s}{k} b + 1\right) d + 1$. The dimension of $\operatorname{Im} \mathbf{v}$ is at most $\binom{s}{k} b$ by Lemma 3.33. Hence by Theorem 2.14 there exist $(d+1)$ pairwise disjoint rainbow multipoints $\eta_0, \eta_1, \dots, \eta_{d-1}, \eta_d$ satisfying

$$\mathbf{v}(\eta_i) = \mathbf{v}(\eta_1) \qquad \text{for all } i = 0, \dots, d. \tag{3.17}$$

If we rearrange the multipoints, we may assume that[26]

$$\operatorname{supp} \eta_i \subset \{v_i\} \cup X \qquad \text{for every } i = 0, \dots, d-1, \tag{3.18}$$
$$\operatorname{supp} \eta_d \subset X. \tag{3.19}$$

Let c be Kneser's coloring from Lemma 3.31. If we set $\mu_{\sigma_i} := \eta_{c(\sigma_i)}$, we can define $\psi \colon \widetilde{C}_* \left(\Delta_s^{(k)}; \mathbb{F}\right) \to \widetilde{C}_* \left(\Delta_n^{(k)}; \mathbb{F}\right)$ by

$$\psi(\sigma) := \begin{cases} \sigma & \text{if dimension of } \sigma \text{ is less than } k \text{ and} \\ \sigma_i - \partial(\mu_{\sigma_i} \wedge \sigma_i) & \text{if } \sigma \text{ is the } i\text{th } k\text{-dimensional face } \sigma_i \end{cases}$$

on the generators and extend it linearly to the whole space.

Then $\mathbf{v}(\mu_{\sigma_i}) = \mathbf{v}(\eta_1)$. Since $\mathbf{v}(\mu_{\sigma_i})$ does not depend on i, Observations 3.28 and 3.29 imply that ψ is a well-defined chain map satisfying $\varphi_* \circ \psi_* = 0$.

To show that ψ is an almost embedding, we verify the assumptions of Lemma 3.30. Let σ be a k-face. If[27] $\min \sigma \leq d$, then $\operatorname{supp} \mu_\sigma = \operatorname{supp} \eta_{c(\sigma)} \subseteq \{v_{c(\sigma)}\} \cup X \subseteq \sigma \cup X$, where the last inclusion follows from the fact that $c(\sigma)$ is defined as the minimal number c', for which $v_{c'} \in \sigma$. If $\min \sigma \geq d+1$, $\operatorname{supp} \mu_\sigma = \operatorname{supp} \eta_{d+1} \subseteq X \subseteq \sigma \cup X$. In all cases (3.9) holds true.

We need to check that (3.10) is satisfied as well. If $\sigma \cap \tau = \emptyset$, then $c(\sigma) \neq c(\tau)$ by Lemma 3.31. Hence $\operatorname{supp} \mu_\sigma \cap \operatorname{supp} \mu_\tau = \operatorname{supp} \eta_{c(\sigma)} \cap \operatorname{supp} \eta_{c(\tau)}$. Since the multipoints assigned to distinct colors are disjoint, the intersection is empty and (3.10) is satisfied.

Hence ψ is an almost embedding by Lemma 3.30. As the last step, we use Lemma 3.32 construct an almost-embedding $g \colon \left|\Delta_s^{(k)}\right| \to \left|\Delta_n^{(k)}\right|$ such that $(\iota_n)_* \circ \psi_* = g_* \circ (\iota_s)_*$. Lemma 3.32 also ensures that if φ is an almost embedding, so is g and that the image $g(|\sigma|)$ is a union of faces in $\left|\Delta_n^{(k)}\right|$.

The proof for $s - 2k \geq 0$ is finished.

If $s - 2k < 0$, we may use the similar argumentation. The Kneser's coloring has only one color in such case, so we may use the same multipoint $\mu_{\sigma_i} := \eta_1$ for all faces σ_i (such assignment satisfies $\mathbf{v}(\mu_i) = \mathbf{v}(\mu_1)$ trivially). Since in Theorem 3.26 we assume that $n \geq s + 1$, there exists $x \in V\left(\Delta_n^{(k)}\right) \setminus V\left(\Delta_s^{(k)}\right)$. We set $\eta_1 := x$ and see that the conditions of Lemma 3.30 are satisfied (with $U_{\sigma_i} := \{x\}$ for all σ_i). The rest of the proof goes through as before. $\square$

Now we show that our approach can be regarded as a natural generalization of the case $k = 0$, which is rather trivial: If we have a topological space with $b + 1$ path-connected components and $s(b+1) + 1$ points in it, there are at least $s + 1$ points lying in the same path-connected component.

This can be proven easily using pigeonhole principle, but it also fits into our framework:[28]

[26] We note that if we used Lemma 2.10 instead of Theorem 2.14 and wanted the multipoints satisfy the next equation, we could only allow Y to contain one point. Hence we would be forced to increase the size of X by $d - 1$, which would worsen the bound in Theorem 3.26 by $d - 1 = s - 2k - 1$.

[27] See the discussion before Lemma 3.31, where they symbol $\min \sigma$ is introduced.

[28] Observe that the bound in Theorem 3.25 differs by $+1$. This is caused by the fact the we need to use a multipoint with support outside of $V\left(\Delta_s^{(k)}\right)$ for the faces of the last color. But according to Lemma 3.30 this is only needed if the intersection of all k-faces with that color is empty. Consequently, we can get rid of this $+1$ if we use a coloring, where we color each face by the minimal index of its vertex. This would yield a bound $n \geq \max\{s, \binom{s}{k} b(s-k) + s\}$ for Theorems 3.25 and 3.26. Unless $k = 0$ or $b = 0$, the bounds are worse than the provided ones, however, for $b = 0$ they agree with Volovikov's theorem.

Lemma 3.34. *Let $n, s, b \geq 0$ be integers, p a prime number. Let M be a manifold with 0-th reduced $\mathbb{Z}_p$-Betti number at most b. Let $f\colon \left|\Delta_n^{(k)}\right| \to M$ be a continuous map.*

If

$$n \geq \binom{s}{0} b(s-0) + s = (b+1)s \quad \text{and} \quad n \geq s,$$

then there exists an almost embedding $g\colon \left|\Delta_s^{(0)}\right| \to \left|\Delta_n^{(0)}\right|$ such that the induced homomorphism $(g_* \circ f_*)\colon H_*\left(\left|\Delta_s^{(0)}\right|; \mathbb{Z}_p\right) \to H_*(M; \mathbb{Z}_p)$ is trivial. Moreover, the image $g(|\sigma|)$ of every face $\sigma \in \Delta_s^{(0)}$ has the form $\bigcup_{i \in I_\sigma} |\tau_i|$, where τ_i are some simplices in $\Delta_n^{(0)}$.

The proof of the lemma is an easy exercise. Here we present a line of argumentation that agrees with our reasoning in the proof of Theorem 3.26.

Proof. If the 0th reduced $\mathbb{Z}_p$-Betti number is at most b, the manifold M has at most $b+1$ path-connected components. It follows that for every 0-dimensional face $\sigma \in \Delta_s^{(0)}$ its image $f_\sharp\big(\iota_s(\sigma)\big)$ is uniquely determined by the one element set $f\big(\iota_s(\sigma)(|\Delta_0|)\big)$.

Because there are at most $(b+1)$ path-connected components of M and $\Delta_n^{(0)}$ has $n+1 > (b+1)s$ points, there exist $s+1$ points that are mapped into the same component P.

For every vertex x from $\Delta_s^{(0)}$ with $f(|x|) \in P$ we set $\mu_x := x$.

We take the remaining points, one after the other. For every such point x, we take an unused vertex y of $\Delta_n^{(0)}$ with $f(|y|) \in P$ and set $\mu_x := y$.

If we define $g\colon \left|\Delta_s^{(0)}\right| \to \left|\Delta_n^{(0)}\right|$ by $g(|x|) = |\mu_x|$, then $f \circ g$ maps all vertices of $\left|\Delta_s^{(0)}\right|$ into P. Hence $(f \circ g)_* = 0$ and it is not hard to see that g satisfies all the other conditions as well.

Now we show how this fits into our framework: We have defined μ_x for every 0-face x of $\Delta_s^{(0)}$, hence we may use (3.7) to define ψ.

For every 0-face x with $f(|x|) \in P$, we have $\partial(\mu_x \wedge x) = \partial(x \wedge x) = 0$ and $\psi(x) = x - \partial(\mu_x \wedge x) = x - \partial(x - x) = x$. If $f(|x|) \notin P$, then $\psi(x) = x - \partial(\mu_x \wedge x) = x - \partial(y \wedge x) = x - x + y = y$.

It is then not hard to see that $(\iota_n)_* \circ \psi_* = g_* \circ (\iota_s)_*$. □

Now we finish the proof of Lemma 3.32, hence filling the missing part in the proof of Theorem 3.26.

Proof of Lemma 3.32

We carry out the plan which we have outlined earlier. First we start with two technical lemmas about "addition" of singular simplices, then we finally prove Lemma 3.32.

Lemma 3.35. *Let X be a topological space, $l \geq 0$ an integer and $\mathbb{F}$ a field. If $\tau_0, \tau_1, \ldots, \tau_l$ are ordered l-dimensional singular simplices in X satisfying*[29]

$$\tau_i \circ \delta_l^j = \tau_j \circ \delta_l^{i+1} \quad \text{for all integers } 0 \leq i < j \leq l, \tag{3.20}$$

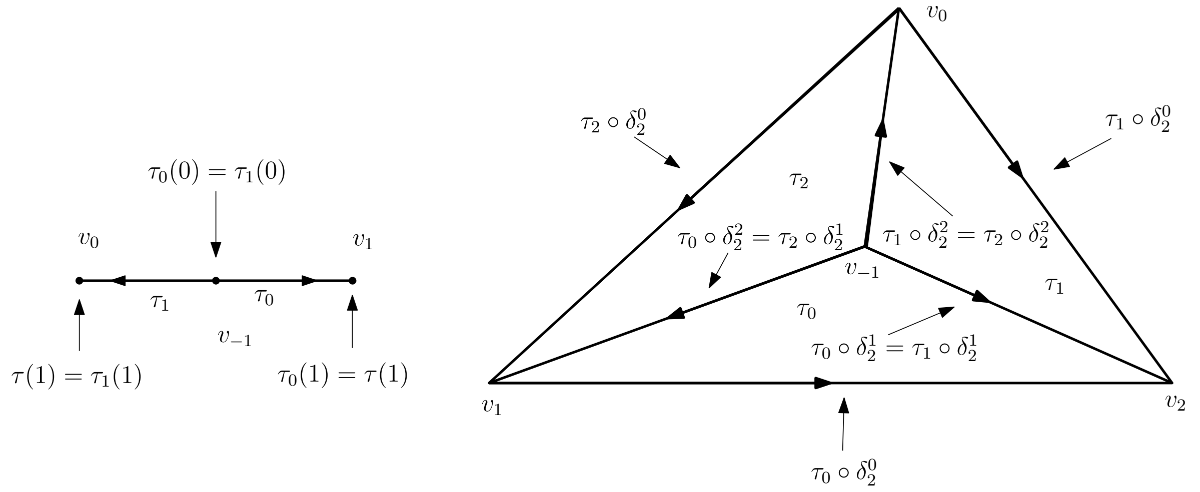
then there is an ordered l-dimensional singular simplex τ such that

$$\tau \equiv \sum_{i=0}^{l} (-1)^i \tau_i \pmod{B_l^O(X; \mathbb{F})},$$

$\tau \circ \delta_l^i = \tau_i \circ \delta_l^0$ for all $i = 0, \ldots, l$ and $\operatorname{Im} \tau = \bigcup_{i=0}^{l} \operatorname{Im} \tau_i$.

The map τ generalizes concatenation of two paths into higher dimensions, see Fig. 3.5.

[29] See Definition 3.12 which introduces the maps δ_l^i.



On the left: the situation of Lemma 3.35 for $l = 1$. Note that we identify $|\Delta_1|$ with the unit interval $[0, 1]$. The resulting map τ "corresponds" to $\tau_0 - \tau_1 = -\tau_1 + \tau_0$.
On the right: the situation for $l = 2$, with the singular 2-simplices τ_i mapping $|\Delta_2|$ onto the corresponding triangles linearly and vertex order preserving ($v_{-1} < v_0 < v_1 < v_2$). The resulting map τ "corresponds" to $\tau_0 - \tau_1 + \tau_2$. The situation in higher dimensions is completely analogous, only worse to draw.

Figure 3.5: Illustration of Lemma 3.35

Proof. Fig. 3.5 describes the idea behind the proof. It is useful to keep it in mind while carrying out the technical details. Let Δ_l be an l-dimensional simplex with the standard geometric realization $|\Delta_l|$ and vertex set $\{v_0 < v_1 < \ldots < v_l\}$. We recall that $|v_i| = \mathbf{e}_{i+1}$, where $\mathbf{e}_{i+1}$ is the $(i+1)$th vector in the standard basis of $\mathbb{R}^{l+1}$. Let $\mathbf{a} = v_{-1} \notin V(\Delta_l)$ be a point, we define its geometric realization $|\mathbf{a}|$ to be the barycenter of $|\Delta_l|$. Let $L := \mathrm{sd}\,(\Delta_l, \mathbf{a})$ be the stellar subdivision of Δ_l with respect to $\mathbf{a}$ and let F_i, $i = 0, \ldots, l$, be the unique l-face of L, that does not contain vertex v_i. Let $\gamma_i \colon |F_i| \to |\Delta_l|$ be the vertex order preserving linear isomorphism of $|F_i|$ onto $|\Delta_l|$.

The resulting map $\tau \colon |\Delta_l| \to X$ can now be defined by:

$$\tau(x) := \tau_i \circ \gamma_i(x) \quad \text{if } x \in F_i. \tag{3.21}$$

If x belongs to two different l-faces F_i and F_j, $0 \le i < j \le l$, the condition $\tau_i \circ \delta_l^j = \tau_j \circ \delta_l^{i+1}$ implies $\tau_i \circ \gamma_i(x) = \tau_j \circ \gamma_j(x)$, hence τ is well defined and continuous.

The formal proof that $\tau \equiv \sum_{i=0}^{l} (-1)^i \tau_i \pmod{B_l^O(X; \mathbb{F})}$ requires a step into dimension by one higher.

Let L' be an $(l+1)$-dimensional simplex with vertex set $\{v_0 < v_1 < \ldots < v_l < v_{l+1}\}$ and standard geometric realization. Let $\mathbf{b} \notin V(L')$ be a point. We define $|\mathbf{b}|$ to be the barycenter of the face $F := |\{v_1, v_2, \ldots, v_{l+1}\}|$.

Let γ be the vertex order preserving linear isomorphism of F and $|\Delta_l|$, in particular $\gamma(|\mathbf{b}|) = |\mathbf{a}|$. Let π be the projection of $|L'|$ onto F in the direction $|\mathbf{b}| - |v_0|$, in particular $\pi(|v_0|) = |\mathbf{b}|$.

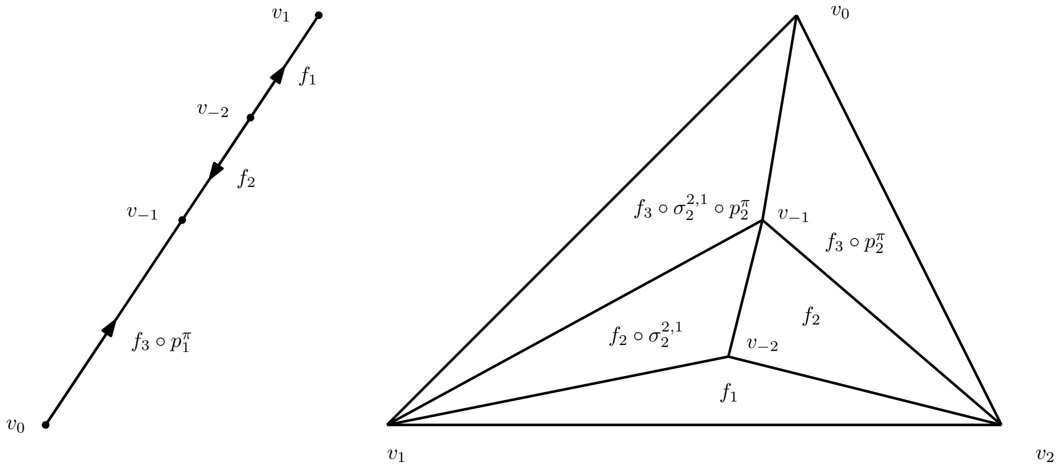
We now define $f \colon |\Delta_{l+1}| \to X$ as $f := \tau \circ \gamma \circ \pi$.

We can check that $f \circ \delta_{l+1}^i = \tau_{i-1}$ for all integers $1 \le i \le l+1$ and $f \circ \delta_{l+1}^0 = \tau$. This implies

$$\tau \equiv \sum_{i=0}^{l} (-1)^i \tau_i \pmod{B_l^O(X; \mathbb{F})},$$

as desired. The condition $\tau \circ \delta_l^i = \tau_i \circ \delta_l^0$ for $i = 0, \ldots, l$ is also easy to verify. □

The next lemma is another variation on "addition" of singular simplices. The basic idea can be described as follows: Let us assume that we have three paths f, g, h with common starting and ending points. Then we can form the concatenation of these paths in a way which preserves boundaries: first



The situation for $l = 1$ (on the left) and $l = 2$ (on the right). In both situations we set $v_{-1} := v_1$ and $v_{-2} := v_0$ (but preserve the ordering $v_{-2} < v_{-1} < v_0 < v_1$). Because $v_{-1} < v_0$, but $v_1 > v_0$, we have to switch v_{-1} and v_0 for all maps that correspond to those vertices. The map p_k^π does exactly that: it is a linear map that switches the first two vertices of Δ_k and leaves other vertices fixed. Because $v_{-1} = v_1$ and $v_{-2} = v_0$, some triangles on the right are degenerated, the degeneracy maps $\sigma_k^{2,1}$, describe the situation (i.e. they fully describe what vertex is mapped where and how does it affect the order of the vertices in the corresponding triangle).
The proof is then finished by two applications of Lemma 3.35. First we apply it to the simplex $v_{-1}, v_1, v_2, v_3, \ldots, v_k$, where we "sum up" the maps appearing in this simplex to a single map τ', then to the simplex $v_0, v_1, v_2, v_3, \ldots, v_k$.

Figure 3.6: Proof of Lemma 3.36

we traverse f then g in the opposite direction and finally h. The lemma generalizes this concept into higher dimensions.

In the following lemma if f is an (ordered) singular simplex in $\widetilde{O}_*(X; \mathbb{F})$, we denote its unordered image in $\widetilde{C}_*(X; \mathbb{F})$ by f^u, i.e., $f^u := f + \widetilde{T}(X; \mathbb{F})$.

Lemma 3.36. *Let X be a topological space, $\mathbb{F}$ a field and $l > 0$ an integer. Let f_1, f_2, f_3 be ordered l-dimensional singular simplices in X satisfying*

$$f_1 \circ \delta_l^i = f_2 \circ \delta_l^i = f_3 \circ \delta_l^i \tag{3.22}$$

for all $i = 0, 1, \ldots, l$. Then there exists an ordered l-dimensional singular simplex τ for which $\tau^u \equiv f_1^u - f_2^u + f_3^u \pmod{B_l(X; \mathbb{F})}$,

$$\tau \circ \delta_l^i = f_1 \circ \delta_l^i = f_2 \circ \delta_l^i = f_3 \circ \delta_l^i \tag{3.23}$$

for all $i = 0, \ldots, l$ and $\operatorname{Im} \tau = \operatorname{Im} f_1 \cup \operatorname{Im} f_2 \cup \operatorname{Im} f_3$.

Proof. The lemma can be proven by two applications of Lemma 3.35, see Fig. 3.6.

Let us now carry out the first step.

Let $\tau_0' := f_1$ and for every $i = 1, 2, 3, \ldots, l$ let[30] $\tau_i' := f_2 \circ \sigma_l^{i,i-1} \circ \sigma_l^{i-1,i-2} \circ \cdots \circ \sigma_l^{2,1}$. Note that $\tau_1' = f_2$ and τ_i' are degenerated for $i > 1$.

Now we verify that such assignment satisfies the hypotheses of Lemma 3.35. We only need to check that

$$\tau_i' \circ \delta_l^j = \tau_j' \circ \delta_l^{i+1} \qquad \text{for all } 0 \le i < j \le l. \tag{3.24}$$

We divide the verification of Equation (3.24) into two cases:

1. If $i = 0$, $1 \le j \le l$ then the left hand side of (3.24) equals $f_1 \circ \delta_l^j$, whereas the right hand-side is equal to $f_2 \circ \sigma_l^{j,j-1} \circ \cdots \circ \sigma_l^{2,1} \circ \delta_l^1$. If we inductively use the relation $\sigma_l^{k+1,k} \circ \delta_l^k = \delta_l^{k+1}$, $k = 1, 2, \ldots, j-1$

[30] The maps $\sigma_l^{j,j-1}$ are introduced in Definition 3.12. Also recall that an l-dimensional singular simplex γ is called degenerated, if $\gamma = \gamma' \circ \sigma_l^{i,j}$ for some $i \neq j$ and an l-dimensional singular simplex γ'.

from Observation 3.13, we see that the right hand-side equals $f_2 \circ \delta_l^j$. From (3.22) we get the equality $f_2 \circ \delta_l^j = f_1 \circ \delta_l^j$, hence both sides of Equation (3.28) are equal.

2. If $1 \le i < j \le l$ we use the relations $\sigma_l^{m+1,m} \circ \delta_l^k = \delta_l^k \circ \sigma_{l-1}^{m+1,m}$ for $l \ge k > m+1$ and $\sigma_l^{k+1,k} \circ \delta_l^k = \delta_l^{k+1}$; $\sigma_l^{k+1,k} \circ \delta_l^{k+1} = \delta_l^{k+1}$ for all $k = 0, \dots, l$, all from Observation 3.13. Then the left hand side of (3.24) equals $f_2 \circ \sigma_l^{i,i-1} \cdots \circ \sigma_l^{2,1} \circ \delta_l^j = f_2 \circ \delta_l^j \circ \sigma_{l-1}^{i,i-1} \cdots \circ \sigma_{l-1}^{2,1}$, whereas the right hand side is equal to $f_2 \circ \sigma_l^{j,j-1} \cdots \circ \sigma_l^{2,1} \circ \delta_l^{i+1} = f_2 \circ \sigma_l^{j,j-1} \circ \cdots \circ \sigma_l^{i+2,i+1} \circ \sigma_l^{i+1,i} \circ \delta_l^{i+1} \circ \sigma_{l-1}^{i,i-1} \circ \cdots \circ \sigma_{l-1}^{2,1} = f_2 \circ \delta_l^j \circ \sigma_{l-1}^{i,i-1} \circ \cdots \circ \sigma_{l-1}^{2,1}$.

Hence we see that $\tau_i' \circ \delta_l^j = \tau_j' \circ \delta_l^{i+1}$ for all $0 \le i < j \le l$ and the assumptions of Lemma 3.35 are satisfied.

We conclude that there is a map τ' such that

$$\tau' \circ \delta_l^i = \tau_i' \circ \delta_l^0 \quad \text{for all } i = 0, 1, \dots, l. \tag{3.25}$$

and $\tau' \equiv \sum_{i=0}^{l} (-1)^i \tau_i' \pmod{B_l^O(X; \mathbb{F})}$.

Because $\tau_i' \in \widetilde{T}_l(X; \mathbb{F})$ for all $i = 2, 3, \dots, l$ this implies

$$\tau'^u \equiv f_1^u - f_2^u \pmod{B_l(X; \mathbb{F})}. \tag{3.26}$$

Moreover, since $\operatorname{Im} \tau_i' \subseteq \operatorname{Im} f_2$ for every $i \ge 1$ and $\operatorname{Im} \tau_1' = \operatorname{Im} f_2$, $\operatorname{Im} \tau_0' = \operatorname{Im} f_1$, Lemma 3.35 implies that

$$\operatorname{Im} \tau' = \operatorname{Im} f_1 \cup \operatorname{Im} f_2. \tag{3.27}$$

Now we apply Lemma 3.35 for the second time. Let π be the permutation (01) viewed as an element of $S(\{0, 1, \dots, l\})$ and π' be the permutation (01) viewed as an element of $S(\{0, 1, \dots, l-1\})$. Let $\tau_0 := \tau'$ and[31] $\tau_i := f_3 \circ \sigma_l^{i,i-1} \circ \sigma_l^{i-1,i-2} \circ \cdots \circ \sigma_l^{2,1} \circ p_l^{\pi}$ for $i = 1, 2, \dots, l$. Note that $\tau_1 = f_3 \circ p_l^{\pi}$ and τ_i are degenerated for $i > 1$. Once again we verify that the assignment satisfies the assumptions:

$$\tau_i \circ \delta_l^j = \tau_j \circ \delta_l^{i+1} \qquad \text{for all } 0 \le i < j \le l. \tag{3.28}$$

As before, we divide the verification into two cases:

1. If $i = 0, 1 \le j \le l$ then the left hand-side of (3.28) equals $\tau' \circ \delta_l^j = \tau_j' \circ \delta_l^0 = f_2 \circ \sigma_l^{j,j-1} \circ \cdots \circ \sigma_l^{2,1} \circ \delta_l^0$, where the equalities follow from (3.25) and the definition of τ_j'. Using the relation $\sigma_l^{k+1,k} \circ \delta_l^0 = \delta_l^0 \circ \sigma_{l-1}^{k,k-1}$ for $k = 1, 2, \dots, j-1$ from Observation 3.13, the left hand-side can be further rewritten as $f_2 \circ \delta_l^0 \circ \sigma_{l-1}^{j-1,j-2} \circ \cdots \circ \sigma_{l-1}^{1,0}$. The right hand-side of (3.28) is equal to $f_3 \circ \sigma_l^{j,j-1} \circ \cdots \circ \sigma_l^{2,1} \circ p_l^{\pi} \circ \delta_l^1$. By the relation $p_l^{\pi} \circ \delta_l^1 = \delta_l^0$ and $\sigma_l^{k+1,k} \circ \delta_l^0 = \delta_l^0 \circ \sigma_{l-1}^{k,k-1}$ for $k = 1, 2, \dots, j-1$ from Observation (3.13), we see that the right hand-side of (3.28) is equal to $f_3 \circ \delta_l^0 \circ \sigma_{l-1}^{j-1,j-2} \circ \cdots \circ \sigma_{l-1}^{1,0}$. If we now use the assumption (3.22) that $f_2 \circ \delta_l^0 = f_3 \circ \delta_l^0$, we see that both sides of (3.28) are equal.

2. If $1 \le i < j \le l$ then the left hand side of (3.28) equals $f_3 \circ \sigma_l^{i,i-1} \cdots \sigma_l^{2,1} \circ p_l^{\pi} \circ \delta_l^j = f_3 \circ \delta_l^j \circ \sigma_{l-1}^{i,i-1} \circ \cdots \circ \sigma_{l-1}^{2,1} \circ p_{l-1}^{\pi'}$, where the equalities follow from Observation 3.13, namely from the relations $p_l^{\pi} \circ \delta_l^k = \delta_l^k \circ p_{l-1}^{\pi'}$ for $k > 1$ and $\sigma_l^{k+1,k} \circ \delta_l^m = \delta_l^m \circ \sigma_{l-1}^{k+1,k}$ for $m > k+1$. The right hand-side can be rewritten as follows $f_3 \circ \sigma_l^{j,j-1} \circ \cdots \circ \sigma_l^{2,1} \circ p_l^{\pi} \circ \delta_l^{i+1} = f_3 \circ \sigma_l^{j,j-1} \circ \cdots \circ \sigma_l^{i+2,i+1} \circ \sigma_l^{i+1,i} \circ \delta_l^{i+1} \circ \sigma_{l-1}^{i,i-1} \circ \cdots \circ \sigma_{l-1}^{2,1} \circ p_{l-1}^{\pi'}$, where the equalities follows by the same relations as for the left hand-side. If we now use $\sigma_l^{i+1,i} \circ \delta_l^{i+1} = \delta_l^{i+1}$ and $\sigma_l^{k+1,k} \circ \delta_l^k = \delta_l^{k+1}$ for $k = i+1, i+2, \dots, j-1$, we see that the right hand side of (3.28) equals $f_3 \circ \delta_l^j \circ \sigma_{l-1}^{i,i-1} \circ \cdots \circ \sigma_{l-1}^{2,1} \circ p_{l-1}^{\pi'}$ as well.

Hence we see that $\tau_i \circ \delta_l^j = \tau_j \circ \delta_l^{i+1}$ for all $0 \le i < j \le l$ and the assumptions of Lemma 3.35 are satisfied.

We conclude that there is a map τ such that $\tau \equiv \sum_{i=0}^{l} (-1)^i \tau_i \pmod{B_l^O(X; \mathbb{F})}$ and $\tau \circ \delta_l^i = \tau_i \circ \delta_l^0$ for all $i = 0, \dots, l$.

Because $\tau_i = f_3 \circ \sigma_l^{i,i-1} \circ \cdots \circ \sigma_l^{2,1} \circ p_l^{\pi}$ for $i = 2, 3, \dots, l$ and $\sigma_l^{2,1} \circ p_l^{\pi} = p_l^{\pi} \circ \sigma_l^{2,0}$ (Observation 3.13), we see that $\tau_i \in \widetilde{T}_l(X; \mathbb{F})$ for all $i = 2, 3, \dots, l$. Because also $f_3 \circ p_l^{\pi} - \operatorname{sgn}(\pi) f_3 \in \widetilde{T}_l(X; \mathbb{F})$ and $\operatorname{sgn}(\pi) = -1$, we see that $\tau^u \equiv \tau'^u + f_3^u \pmod{B_l(X; \mathbb{F})}$. Together with (3.26) this yields that

$$\tau^u \equiv f_1^u - f_2^u + f_3^u \pmod{B_l(X; \mathbb{F})},$$

[31] See Definition 3.12 for the introduction of maps p_l^{π}.

as desired.

Now we compute how do $\tau \circ \delta_l^i$ look like. If $i = 0$, we have $\tau \circ \delta_l^0 = \tau_0 \circ \delta_l^0 = \tau' \circ \delta_l^0 = \tau_0' \circ \delta_l^0 = f_1 \circ \delta_l^0$, where the equalities follow from Lemma 3.35 and the definitions of τ_0 and τ_0'.

If $i \geq 1$, we have $\tau \circ \delta_l^i = \tau_i \circ \delta_l^0 = f_3 \circ \sigma_l^{i,i-1} \circ \sigma_l^{i-1,i-2} \circ \cdots \circ \sigma_l^{2,1} \circ p_l^\pi \circ \delta_l^0$. If we now use the relations $p_l^\pi \circ \delta_l^0 = \delta_l^1$ and $\sigma_l^{k+1,k} \circ \delta_l^k = \delta_l^{k+1}$ from Observation 3.13, we see that $\tau \circ \delta_l^i = f_3 \circ \delta_l^i$.

Since $\operatorname{Im} \tau_i \subseteq \operatorname{Im} f_3$ for all $i = 1, 2, \ldots, 3$ and $\operatorname{Im} \tau_1 = \operatorname{Im} f_3$, $\operatorname{Im} \tau_0 = \operatorname{Im} f_1 \cup \operatorname{Im} f_2$ (see Equation (3.27)), Lemma 3.35 implies that $\operatorname{Im} \tau = \operatorname{Im} f_1 \cup \operatorname{Im} f_2 \cup \operatorname{Im} f_3$. □

We may now finally prove Lemma 3.32.

Proof of Lemma 3.32. Recall that $X = V\left(\Delta_n^{(k)}\right)$. For brevity, we further set $Y := \left|\Delta_n^{(k)}\right|$. Let $\sigma = \sigma_i$ be a k-dimensional face of $\Delta_s^{(k)}$. Then $\psi(\sigma_i) = \sigma_i - \partial(\mu_{\sigma_i} \wedge \sigma_i)$.

Because μ_{σ_i} is a multipoint in $\mathcal{M}(X; \mathbb{Z}_p)$, it can be expressed as $\sum_{j \in J_i} a_{i,j} x_{i,j}$ for some sets J_i, points $x_{i,j} \in V\left(\Delta_n^{(k)}\right)$ and coefficients $a_{i,j} \in \mathbb{Z}_p$, where $\sum_{j \in J_i} a_{i,j} = 1$ for all i.

By definition then

$$\psi(\sigma_i) = \sigma_i - \partial(\mu_{\sigma_i} \wedge \sigma_i) = \sum_{j \in J_i} a_{i,j} \left(\sigma_i - \partial\big(x_{i,j} \wedge \sigma_i\big)\right).$$

Since we work in $\mathbb{Z}_p$, we may replace the term $a_{i,j}$ with

$$\underbrace{1 + 1 + \ldots + 1}_{a_{i,j}\text{-times}} \quad \text{or} \quad \underbrace{(-1) + (-1) + \ldots + (-1)}_{(p - a_{i,j})\text{-times}}.$$

Hence after rearranging[32], we obtain that

$$\psi(\sigma_i) = \sum_{j=0}^{2m_i} (-1)^j \left(\sigma_i - \partial\big(y_{i,j} \wedge \sigma_i\big)\right),$$

where $m_i \geq 0$ is an integer and $y_{i,j} \in V\left(\Delta_n^{(k)}\right)$.

Let $\sigma_i = (v_{i,0}, v_{i,1}, \ldots, v_{i,k})$. For every $l = 0, \ldots, k$ we set

$$\sigma_i^l := (v_{i,0}, v_{i,1}, \ldots, \widehat{v_{i,l}}, \ldots, v_{i,k}).$$

Then

$$\psi(\sigma_i) = \sum_{j=0}^{2m_i} (-1)^j \left(\sum_{l=0}^{k} (-1)^l \big(y_{i,j}, v_{i,0}, v_{i,1}, \ldots, \widehat{v_{i,l}}, \ldots, v_{i,k}\big)\right).$$

This yields

$$\begin{aligned}
\iota_n\big(\psi(\sigma_i)\big) &= \sum_{j=0}^{2m_i} (-1)^j \left(\sum_{l=0}^{k} (-1)^l \iota_n\big(y_{i,j}, v_{i,0}, v_{i,1}, \ldots, \widehat{v_{i,l}}, \ldots, v_{i,k}\big)\right) \\
&= \sum_{j=0}^{2m_i} (-1)^j \left(\sum_{l=0}^{k} (-1)^l \iota_n(y_{i,j} \wedge \sigma_i) \circ \delta_{k+1}^{l+1}\right).
\end{aligned}$$

Because the maps $\tau_l = \iota_n(y_{i,j} \wedge \sigma_i) \circ \delta_{k+1}^{l+1}$, $l = 0, 1, \ldots, k$ obviously satisfy the assumptions of Lemma 3.35, we can replace $\sum_{l=0}^{k} (-1)^l \iota_n(y_{i,j} \wedge \sigma_i) \circ \delta_{k+1}^{l+1}$ with a single map $\gamma_{\sigma_i, j} \colon |\Delta_k| \to \left|\Delta_n^{(k)}\right|$ such that

$$\iota_n^o\big(\psi(\sigma_i)\big) \equiv \sum_{j=0}^{2m_i} (-1)^j \gamma_{\sigma_i, j} \pmod{B_k^O(Y; \mathbb{F})}, \tag{3.29}$$

where ι_n^o is the natural inclusion of $O_*\left(\Delta_n^{(k)}; \mathbb{Z}_p\right)$ into $O_*\left(\left|\Delta_n^{(k)}\right|; \mathbb{Z}_p\right)$.

[32] For example over $\mathbb{Z}_5$, the multipoint $3x + 3y$ can be rewritten as $x - y + x - y + x$.

Moreover, by Observation 3.13 the maps $\gamma_{\sigma_i,j}$ satisfy $\gamma_{\sigma_i,j} \circ \delta_k^l = \tau_i \circ \delta_k^0 = \iota_n(y_{i,j} \wedge \sigma_i) \circ \delta_{k+1}^{l+1} \circ \delta_k^0 = \iota_n(y_{i,j} \wedge \sigma_i) \circ \delta_{k+1}^0 \circ \delta_k^l = \iota_n(\sigma_i) \circ \delta_k^l$ for all $l = 0, 1, \ldots, k$ and

$$\operatorname{Im} \gamma_{\sigma_i,j} = \bigcup_{l=0}^{k} \left| (y_{i,j}, v_{i,0}, v_{i,1}, \ldots, \widehat{v_{i,l}}, \ldots, v_{i,k}) \right| .$$

We can now apply Lemma 3.36 to $\gamma_{\sigma_i,2m_i-2} - \gamma_{\sigma_i,2m_i-1} + \gamma_{\sigma_i,2m_i}$ and replace them with a single map $\gamma'_{\sigma_i,2m-2}$. Continuing inductively, in each step decreasing m_i in (3.29) by two, we obtain a map $\gamma_{\sigma_i} \colon |\Delta_k| \to \left|\Delta_n^{(k)}\right|$ satisfying

$$\iota_n\big(\psi(\sigma_i)\big) \equiv \gamma_{\sigma_i}^u \pmod{B_k(Y;\mathbb{F})} \tag{3.30}$$

and $\gamma_{\sigma_i} \circ \delta_k^l = \iota_n(\sigma_i) \circ \delta_k^l$ for all $l = 0, 1, \ldots, k$. Moreover, if N_{σ_i} is the set of all non-degenerated simplices ρ appearing in $\psi(\sigma_i)$ with a non-zero coefficient, then

$$\operatorname{Im} \gamma_{\sigma_i} = \bigcup_{\rho \in N_{\sigma_i}} |\rho| . \tag{3.31}$$

Now we define $g \colon \left|\Delta_s^{(k)}\right| \to \left|\Delta_n^{(k)}\right|$ by

$$g(x) := \gamma_{\sigma_i}\Big(\big(\iota_s(\sigma_i)\big)^{-1}(x)\Big) \quad \text{if } x \in |\sigma_i|. \tag{3.32}$$

If x lies in two different σ_i and σ_j it has to lie on the boundary of both of them. Because $\gamma_{\sigma_i} \circ \delta_k^l = \iota_n(\sigma_i) \circ \delta_k^l$, it follows that

$$g(x) = \iota_s(\sigma_i \cap \sigma_j)\Big(\big(\iota_s(\sigma_i \cap \sigma_j)\big)^{-1}(x)\Big) = x \quad \text{for all } x \in |\sigma_i| \cap |\sigma_j|. \tag{3.33}$$

Hence $g(x)$ is well-defined.

Moreover, because σ_i are all k-dimensional faces of $\Delta_s^{(k)}$, the value $g(x)$ is defined for every $x \in \Delta_s^{(k)}$.

From Eq. (3.32) follows that $g \circ (\iota_s \sigma_i) = \gamma_{\sigma_i}$, which together with (3.30), (3.33) and (3.7) yields $g_\sharp(\iota_s(c)) \equiv \iota_n(\psi(c)) \pmod{B_k(Y;\mathbb{Z}_p)}$ for every chain $c \in \widetilde{C}_*\left(\Delta_s^{(k)};\mathbb{Z}_p\right)$.

Since this equation is true for every chain c, it is also true for every cycle z and because the homology groups are formed by factoring the group of cycles by the group of boundaries $B_l(X;\mathbb{F})$, we see that $g_* \circ (\iota_s)_* = (\iota_n)_* \circ \psi_*$, as desired.

By the definition and Eq. (3.31) $g(|\sigma_i|) = \operatorname{Im} \gamma_{\sigma_i} = \bigcup_{\rho \in N_{\sigma_i}} |\rho|$. If ψ is an almost embedding, the sets N_{σ_i} and N_{σ_j} are disjoint for disjoint k-faces σ_i, σ_j. If follows that g is an almost embedding in that case. $\square$

Chapter 4

Van Kampen-Flores type non-embeddability results for manifolds

In this chapter we apply Theorem 3.25 to provide an upper bound for the following conjecture by Kühnel [Küh94].

Conjecture 4.1. *Let $n, k \geq 1$ be integers. If $\Delta_n^{(k)}$ embeds in a compact, $(k-1)$-connected $2k$-manifold with kth $\mathbb{Z}_2$-Betti number $b_k(M)$ then*

$$\binom{n-k-1}{k+1} \leq \binom{2k+1}{k+1} b_k(M). \tag{4.1}$$

The conjecture generalizes the classical Heawood inequality [Hea90, Hef91] and the Van Kampen-Flores Theorem [vK32, Flo33].

The main result of this chapter is the following mild generalization[1] of Theorem 1.1 and its generalization to q-almost embeddings (Theorem 4.4).

Theorem 4.2. *Let k, n be non-negative integers. Let M be a $2k$-dimensional manifold with kth $\mathbb{Z}_2$-Betti number b_k. If $n \geq 2b_k\binom{2k+2}{k} + 2k + 3$, then $\Delta_n^{(k)}$ does not almost embed into M.*

Remark 4.3. *The reader who skipped the colorful algebraic Tverberg theorem may use the weaker version of Theorem 3.25, see Remark 3.27, and obtains that $\Delta_n^{(k)}$ does not almost embed into M for any $n \geq 2b_k\binom{2k+2}{k} + 2k + 4$.*

Since an almost embedding is defined as 1-almost embedding, see Definition 3.23, one may wonder: If we have a d-dimensional manifold, what are the necessary conditions on k and n that ensure that there is no $(q-1)$-almost embedding $f\colon \left|\Delta_k^{(n)}\right| \to M$?

An obvious condition seems to be $k \geq \left(1 - \frac{1}{q}\right) d$. Otherwise one could consider n points $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^d$ in general position, e.g. on the moment curve, map the ith vertex of $\left|\Delta_n^{(k)}\right|$ to $\mathbf{a}_i$ and extend the map linearly. The general position assumption then ensures that this is indeed a $(q-1)$-almost embedding. Since every d-dimensional manifold is locally homeomorphic to $\mathbb{R}^d$, it follows that the condition $k \geq \left(1 - \frac{1}{q}\right) d$ is necessary for every d-dimensional manifold.

If $k \geq \left(1 - \frac{1}{q}\right) d$ we are able to prove the following bound[2]:

[1] Theorem 1.1 only asserts non-existence of embeddings, but in fact we prove non-existence of almost embeddings.

[2] If we used the weaker version of Theorem 3.25 (Remark 3.27), we would obtain a bound $N \geq \binom{N_0}{k} b\,(N_0 - 2k) + 2N_0 - 2k$ instead of $N \geq \binom{N_0}{k} b\,(N_0 - 2k) + N_0 + 1$.

Theorem 4.4. *Let M be a d-dimensional manifold. Let $q = p^n$ be a prime power. Let b be the kth Betti number of M in the homology with $\mathbb{Z}_p$ coefficients. If $k \geq d\left(1 - \frac{1}{q}\right)$, $N_0 = q(k+1) + q - 2$ and $N \geq \binom{N_0}{k} b\,(N_0 - 2k) + N_0 + 1$, then there is no $(q-1)$-almost embedding of $\Delta_N^{(k)}$ into M.*

Theorem 4.4 not only generalizes Theorem 4.2, it also provides a topological variant of the Tverberg theorem for manifolds.

The original Tverberg theorem can namely be stated as follows:

Theorem 4.5. *There is no affine $(q-1)$-almost embedding of $\Delta^{(d)}_{(q-1)(d+1)}$ into $\mathbb{R}^d$.*

Theorem 4.4 generalizes Theorem 4.5 in three ways. Firstly, we need not restrict our attention to affine maps only, all continuous maps are allowed. Secondly, the theorem holds for maps into arbitrary manifolds. Thirdly, instead of the d-dimensional skeleton, we can use k-dimensional skeleton for any $k \geq d\left(1 - \frac{1}{q}\right)$.

We will strongly use a result by Volovikov [Vol96b] which shows that a continuous map $f\colon \left|\Delta_n^{(k)}\right| \to M$ cannot be a q-almost embedding of $\Delta_n^{(k)}$, provided that f satisfies certain homological triviality condition and n is big enough. (Since the homological triviality condition is satisfied by any continuous map $f\colon \Delta_n^{(k)} \to \mathbb{R}^d$, one can regard the result of Volovikov as a generalization of both Van Kampen-Flores and Tverberg theorems).

Theorem 4.6 (Volovikov [Vol96b])**.** *Let M be a compact d-dimensional manifold with or without boundary and $q = p^n$ a prime power. Consider a map $f\colon \left|\Delta^{(k)}_{q(k+1)+q-2}\right| \to M$ such that $k \geq d\left(1 - \frac{1}{q}\right)$ and the homomorphism $f_*\colon H_k\left(\Delta^{(k)}_{q(k+1)+q-2}; \mathbb{Z}_p\right) \to H_k(M; \mathbb{Z}_p)$ is trivial. Then f is not an $(q-1)$-almost embedding of $\Delta^{(k)}_{q(k+1)+q+2}$ into M.*

We note that we only state a special case of Volovikov's result that we need in our proof. It is obtained by setting $j = 2$ in item 3 of Volovikov's main result. Moreover, the original result is stated in terms of cohomology, i.e., it assumes that $f^*\colon H_*(M; \mathbb{Z}_2) \to H^*\left(\Delta^{(k)}_{2k+2}; \mathbb{Z}_2\right)$ is trivial, however, by the Universal Coefficient Theorem [Mun84, 53.5], $H_k(\,\cdot\,; \mathbb{Z}_2)$ and $H^k(\,\cdot\,; \mathbb{Z}_2)$ are dual vector spaces, and f^* is the adjoint of f_*, hence triviality of f_* implies that of[3] f^*.

Proof of Theorem 4.4. Assume $N \geq \binom{N_0}{k} b\,(N_0 - 2k) + N_0 + 1$. Let $f\colon \left|\Delta_N^{(k)}\right| \to M$ be a continuous map.

By Theorem 3.25 there exists an almost embedding $g\colon \left|\Delta_{N_0}^{(k)}\right| \to \left|\Delta_N^{(k)}\right|$, such that $(f \circ g)_* = 0$ and the g image of every face is a union of faces in $\Delta_N^{(k)}$. In particular, if f is a $(q-1)$-almost embedding of $\Delta_N^{(k)}$ into M, $f \circ g$ is an almost embedding of $\Delta_{N_0}^{(k)}$ into M, we refer to Observation 3.24.

Because $(f \circ g)_* = 0$, $(f \circ g)(|\partial\sigma|)$ is a boundary of some chain γ_σ for every $\sigma \in \Delta_{N_0}^{(k+1)}$.

Let

$$M' := \operatorname{Im} f \cup \bigcup_{\sigma \in \Delta_{N_0}^{(k+1)}} \operatorname{supp} \gamma_\sigma.$$

Then M' is compact and without loss of generality, we may assume that M' is path-connected. Furthermore M' is contained in some compact submanifold M'' (possible with boundary) of M and the map $(f \circ g)$ viewed as a map into M'' still satisfies $(f \circ g)_* = 0$.

But since $(f \circ g)_* = 0$, Volovikov's theorem implies that $f \circ g$ cannot be a $(q-1)$-almost embedding. Therefore f could not be a $(q-1)$-almost embedding either. □

[3]Moreover, if the homology group $H_k(X; \mathbb{Z}_2)$ of a space X is finitely generated, then it is (non-canonically) isomorphic to its dual vector space $H^k(X; \mathbb{Z}_2)$. Therefore, f_* is trivial if and only if f^* is.

Chapter 5

Homological Almost-Embeddings

One may wonder whether Theorem 3.26 can be used iteratively to show that certain continuous maps f cannot exist. More precisely, given a field $\mathbb{F}$, a manifold M and an almost embedding $\theta\colon C_*\left(\Delta_N^{(k)};\mathbb{F}\right) \to C_*\left(M;\mathbb{F}\right)$, where[1] $\theta = f_\sharp \circ \iota_N$ and N is big enough, the proof of Theorem 3.26 ensures an existence of an almost embedding $\psi\colon C_*\left(\Delta_n^{(k)};\mathbb{F}\right) \to C_*\left(\Delta_N^{(k)};\mathbb{F}\right)$ such that $\theta_* \circ \psi_* = 0$. That means that for every $\sigma \in \Delta_n^{(k+1)}$, $\theta\Big(\psi\big(\partial(\sigma)\big)\Big)$ is a boundary of some chain γ_σ. We can then extend $\theta \circ \psi$ to a chain map $\theta'\colon C_*\left(\Delta_n^{(k+1)};\mathbb{F}\right) \to C_*\left(M;\mathbb{F}\right)$ by setting

$$\theta'(\sigma) := \begin{cases} \theta\big(\psi(\sigma)\big) & \text{if } \sigma \text{ is a } \leq k\text{-dimensional face,} \\ \gamma_\sigma & \text{if } \sigma \text{ is a } (k+1)\text{-dimensional face.} \end{cases}$$

Moreover, it is possible to continue this process by induction.

If we reach a contradiction with some non-embeddability result after several steps, we conclude that the initial map f could not exist. However, there are two issues to be addressed: Firstly, non-embeddability result only speak about maps, but we are constructing chain maps. Secondly, we need some mechanism guaranteeing the constructed chain maps are almost embeddings (otherwise we cannot obtain a contradiction with non-embeddability result).

We deal with the first issue in this chapter. We provide several generalizations of non-embeddability results for chain maps, which we will need later in Chapter 6. The second issue is addressed in Chapter 6 and requires a non-trivial adjustment[2] of Theorem 3.26.

We note that we do not need the distinction between ordered and oriented homology from Chapters 3 anymore. Hence the usual singular chain group, that was called $O_*(X)$ there, will be denoted $C_*(X)$, as usual, from now on.

We define *homological almost-embedding*, an analogue of topological embeddings on the level of chain maps, and show that certain simplicial complexes do not admit homological almost-embeddings in $\mathbb{R}^d$, in analogy to classical non-embeddability results due to Van Kampen and Flores. In fact, when this comes at no additional cost we phrase the auxiliary results in a slightly more general setting, replacing $\mathbb{R}^d$ by a general topological space $\mathbf{R}$. Readers that focus on the proof of Theorem 1.8 can safely replace every occurrence of $\mathbf{R}$ with $\mathbb{R}^d$.

We assume that the reader is familiar with basic topological notions and facts concerning simplicial complexes and singular and simplicial homology, as described in textbooks like [Hat02, Mun84]. Throughout the following two chapters we will work with homology with $\mathbb{Z}_2$-coefficients unless explicitly stated otherwise. Moreover, while we will consider singular homology groups for topological spaces in

[1] The symbol ι_N is used for the standard inclusion of $C_*\left(\Delta_N^{(k)};\mathbb{F}\right)$ into $C_*\left(\left|\Delta_N^{(k)}\right|;\mathbb{F}\right)$.

[2] Since the adjustment needs hypergraph Ramsey theorem [Ram29], the obtained bounds are enormous, regardless whether we use Theorem 3.26 or some cruder tool. Since the proof in Chapter 6 is itself relatively complicated, we have decided to replace Theorem 3.26 with a version which is conceptually easier, but provides worse bounds and only works over $\mathbb{Z}_2$.

general, for simplicial complexes we will work with simplicial homology groups. In particular, if X is a topological space then $C_*(X)$ will denote the singular chain complex of X, while if K is a simplicial complex, then $C_*(K)$ will denote the simplicial chain complex of K (both with $\mathbb{Z}_2$-coefficients).

We use the following notation. Let K be a (finite, abstract) simplicial complex. The *underlying topological space* of K is denoted by $|K|$. Moreover, we denote by $K^{(i)}$ the *i-dimensional skeleton* of K, i.e., the set of simplices of K of dimension at most i; in particular $K^{(0)}$ is the set of vertices of K. For an integer $n \geq 0$, let Δ_n denote the n-dimensional simplex.

Given a set X we let 2^X and $\binom{X}{k}$ denote, respectively, the set of all subsets of X (including the empty set) and the set of all k-element subsets of X. If $f\colon X \to Y$ is an arbitrary map between sets then we abuse the notation by writing $f(S)$ for $\{f(s) \mid s \in S\}$ for any $S \subseteq X$; that is, we implicitly extend f to a map from 2^X to 2^Y whenever convenient.

We will consider singular homology groups for topological spaces in general, however, for simplicial complexes we will work with simplicial homology groups. In particular, if X is a topological space then $C_*(X)$ will denote the singular chain complex of X, while if K is a simplicial complex, then $C_*(K)$ will denote the simplicial chain complex of K (both with $\mathbb{Z}_2$-coefficients).

5.1 Non-Embeddable Complexes

We recall that an *embedding* of a finite simplicial complex K into $\mathbb{R}^d$ is simply an injective continuous map $|K| \to \mathbb{R}^d$. As noted before, the fact that the complete graph on five vertices cannot be embedded in the plane has the following generalization.

Proposition 5.1 (Van Kampen [vK32], Flores [Flo33])**.** *For $k \geq 0$, $\left|\Delta_{2k+2}^{(k)}\right|$ cannot be embedded in $\mathbb{R}^{2k}$.*

A basic tool for proving the non-embeddability of a simplicial complex is the so-called *Van Kampen obstruction.* To be more precise, we emphasize that in keeping with our general convention regarding coefficients, we work with the $\mathbb{Z}_2$-coefficient version[3] of the Van Kampen obstruction, which will be reviewed in some detail in Section 5.3 below. Here, for the benefit of readers who are willing to accept certain topological facts as given, we simply collect those statements necessary to motivate the definition of homological almost-embeddings and to follow the logic of the proof of Theorem 1.8.

Given a simplicial complex K, one can define, for each $d \geq 0$, a certain cohomology class $\mathfrak{o}^d(K)$ that resides in the cohomology group $H^d(\overline{K})$ of a certain auxiliary complex $\overline{K}$ (the quotient of the combinatorial deleted product by the natural $\mathbb{Z}_2$-action, see below); this cohomology class $\mathfrak{o}^d(K)$ is called the Van Kampen obstruction to embeddability into $\mathbb{R}^d$ because of the following fact:

Proposition 5.2. *Suppose that K is a finite simplicial complex with $\mathfrak{o}^d(K) \neq 0$. Then K is not embeddable into $\mathbb{R}^d$. In fact, a slightly stronger conclusion holds: there is no* almost-embedding *$f\colon |K| \to \mathbb{R}^d$, i.e., no continuous map such that the images of disjoint simplices of K are disjoint.*

Another basic fact is the following result (for a short proof see, for instance, [Mel09, Example 3.5]).

Proposition 5.3 ([vK32, Flo33])**.** *For every $k \geq 0$, $\mathfrak{o}^{2k}\left(\Delta_{2k+2}^{(k)}\right) \neq 0$.*

As a consequence, one obtains Proposition 5.1, and in fact the slightly stronger statement that $\Delta_{2k+2}^{(k)}$ does not admit an almost-embedding into $\mathbb{R}^{2k}$.

5.2 Van Kampen–Flores Type Result for Homological Almost-Embeddings

For the proof of Theorem 1.8, we wish to replace homotopy-theoretic notions (like k-connectedness) by homological assumptions (bounded Betti numbers). The simple but useful observation that allows us to

[3] There is also a version of the Van Kampen obstruction with integer coefficients, which in general yields more precise information regarding embeddability than the $\mathbb{Z}_2$-version, but we will not need this here. We refer to [Mel09] for further background.

do this is that in the standard proof of Proposition 5.2, which is based on (co)homological arguments, maps can be replaced by suitable chain maps at every step.[4] The appropriate analogue of an almost-embedding is the following.

Definition 5.4. *Let* $\mathbf{R}$ *be a (nonempty) topological space,* K *be a simplicial complex, and consider a chain map*[5] $\gamma\colon C_*(K) \to C_*(\mathbf{R})$ *from the simplicial chains in* K *to singular chains in* $\mathbf{R}$*.*

(i) *The chain map* γ *is called* nontrivial[6] *if the image of every vertex of* K *is a finite set of points in* $\mathbf{R}$ *(a 0-chain) of* odd *cardinality.*

(ii) The chain map γ *is called a* homological almost-embedding *of a simplicial complex* K *in* $\mathbf{R}$ *if it is nontrivial and if, additionally, the following holds: whenever* σ *and* τ *are disjoint simplices of* K*, their image chains* $\gamma(\sigma)$ *and* $\gamma(\tau)$ *have disjoint supports, where the support of a chain is the union of (the images of) the singular simplices with nonzero coefficient in that chain.*

Remark 5.5. *Suppose that* $f\colon |K| \to \mathbb{R}^d$ *is a continuous map.*

(i) The induced chain map[7] $f_\sharp\colon C_*(K) \to C_*(\mathbb{R}^d)$ *is nontrivial.*

(ii) If f *is an almost-embedding then the induced chain map is a homological almost-embedding.*

Moreover, note that without the requirement of being nontrivial, we could simply take the constant zero chain map, for which the second requirement is trivially satisfied.

We have the following analogue of Proposition 5.2 for homological almost-embeddings.

Proposition 5.6. *Suppose that* K *is a finite simplicial complex with* $\mathfrak{o}^d(K) \neq 0$*. Then* K *does not admit a homological almost-embedding in* $\mathbb{R}^d$*.*

As a corollary, we get the following result, which underlies our proof of Theorem 1.8.

Corollary 5.7. *For any* $k \geq 0$*, the* k*-skeleton* $\Delta_{2k+2}^{(k)}$ *of the* $(2k+2)$*-dimensional simplex has no homological almost-embedding in* $\mathbb{R}^{2k}$*.*

We conclude this subsection by two facts that are not needed for the proof of the main result but are useful for the presentation of our method in Section 6.2.

If the ambient dimension $d = 2k+1$ is odd, we can immediately see that $\Delta_{2k+4}^{(k+1)}$ has no homological almost-embedding in $\mathbb{R}^{2k+1}$ since it has no homological almost-embedding in $\mathbb{R}^{2k+2}$; this result can be slightly improved:

Corollary 5.8. *For any* $d \geq 0$*, the* $\lceil d/2 \rceil$*-skeleton* $\Delta_{d+2}^{(\lceil d/2 \rceil)}$ *of the* $(d+2)$*-dimensional simplex has no homological almost-embedding in* $\mathbb{R}^d$*.*

Proof. The statement for even d is already covered by the case $k = d/2$ of Corollary 5.7, so assume that d is odd and write $d = 2k+1$. If K is a finite simplicial complex with $\mathfrak{o}^d(K) \neq 0$ and if CK is the cone over K then $\mathfrak{o}^{d+1}(CK) \neq 0$ (for a proof, see, for instance, [BKK02, Lemma 8]). Since we know that $\mathfrak{o}^{2k}(\Delta_{2k+2}^{(k)}) \neq 0$ it follows that $\mathfrak{o}^{2k+1}(C\Delta_{2k+2}^{(k)}) \neq 0$. Consequently, $\mathfrak{o}^{2k+1}(\Delta_{2k+3}^{(k+1)}) \neq 0$ since $C\Delta_{2k+2}^{(k)}$ is a subcomplex of $\Delta_{2k+3}^{(k+1)}$. Proposition 5.6 then implies that $\Delta_{2k+3}^{(k+1)}$ admits no homological almost-embedding in $\mathbb{R}^{2k+1}$. □

The next fact is the following analogue of Radon's lemma, proved in the next subsection along the proof of Proposition 5.6.

Lemma 5.9 (Homological Radon's lemma)**.** *For any* $d \geq 0$*,* $\mathfrak{o}^d(\partial\Delta_{d+1}) \neq 0$*. Consequently, the boundary of* $(d+1)$*-simplex* $\partial\Delta_{d+1}$ *admits no homological almost-embedding in* $\mathbb{R}^d$*.*

[4] This observation was already used in [Wag11] to study the (non-)embeddability of certain simplicial complexes. What we call a *homological almost-embedding* corresponds to the notion of a *homological minor* used in [Wag11].

[5] We recall that a chain map $\gamma\colon C_* \to D_*$ between chain complexes is simply a sequence of homomorphisms $\gamma_n\colon C_n \to D_n$ that commute with the respective boundary operators, $\gamma_{n-1} \circ \partial_C = \partial_D \circ \gamma_n$.

[6] If we consider augmented chain complexes with chain groups also in dimension -1, then being nontrivial is equivalent to requiring that the generator of $C_{-1}(K) \cong \mathbb{Z}_2$ (this generator corresponds to the empty simplex in K) is mapped to the generator of $C_{-1}(\mathbf{R}) \cong \mathbb{Z}_2$.

[7] See Definition 3.18 for the introduction of the induced chain map $f_\sharp$, which goes as follows: We assume that we have fixed a total ordering of the vertices of K. For a p-simplex σ of K, the ordering of the vertices induces a homeomorphism $h_\sigma\colon |\Delta_p| \to |\sigma| \subseteq |K|$. The image $f_\sharp(\sigma)$ is defined as the singular p-simplex $f \circ h_\sigma$.

5.3 Deleted Products and Obstructions

Here, we review the standard proof of Proposition 5.2 and explain how to adapt it to prove Proposition 5.6, which will follow from Lemma 5.13 and Lemma 5.14 (b) below. The reader unfamiliar with cohomology and willing to accept Proposition 5.6 can safely proceed to Chapter 6.

$\mathbb{Z}_2$-spaces and equivariant maps. We begin by recalling some basic notions of equivariant topology: An *action* of the group $\mathbb{Z}_2$ on a space X is given by an automorphism $\nu\colon X \to X$ such that $\nu \circ \nu = 1_X$; the action is *free* if ν does not have any fixed points. If X is a simplicial complex (or a cell complex), then the action is called simplicial (or cellular) if it is given by a simplicial (or cellular) map. A space with a given (free) $\mathbb{Z}_2$-action is also called a (free) $\mathbb{Z}_2$-space.

A map $f\colon X \to Y$ between $\mathbb{Z}_2$-spaces (X, ν) and (Y, μ) is called *equivariant* if it commutes with the respective $\mathbb{Z}_2$-actions, i.e., $f \circ \nu = \mu \circ f$. Two equivariant maps $f_0, f_1\colon X \to Y$ are *equivariantly homotopic* if there exists a homotopy $F\colon X \times [0,1] \to Y$ such that all intermediate maps $f_t := F(\cdot, t)$, $0 \leq t \leq 1$, are equivariant.

A $\mathbb{Z}_2$-action ν on a space X also yields a $\mathbb{Z}_2$-action on the chain complex $C_*(X)$, given by the induced chain map $\nu_\sharp\colon C_*(X) \to C_*(X)$ (if ν is simplicial or cellular, respectively, then this remains true if we consider the simplicial or cellular chain complex of X instead of the singular chain complex), and if $f\colon X \to Y$ is an equivariant map between $\mathbb{Z}_2$-spaces then the induced chain map is also equivariant (i.e., it commutes with the $\mathbb{Z}_2$-actions on the chain complexes).

Spheres. Important examples of free $\mathbb{Z}_2$-spaces are the standard spheres $\mathbb{S}^d$, $d \geq 0$, with the action given by antipodality, $x \mapsto -x$. There are natural inclusion maps $\mathbb{S}^{d-1} \hookrightarrow \mathbb{S}^d$, which are equivariant. Antipodality also gives a free $\mathbb{Z}_2$-action on the union $\mathbb{S}^\infty = \bigcup_{d \geq 0} \mathbb{S}^d$, the infinite-dimensional sphere. Moreover, one can show that $\mathbb{S}^\infty$ is contractible, and from this it is not hard to deduce that $\mathbb{S}^\infty$ is a universal $\mathbb{Z}_2$-space, in the following sense (see, for instance, [Koz08, Prop. 8.16 and Thm. 8.17]).

Proposition 5.10. *If X is any cell complex with a free cellular $\mathbb{Z}_2$-action, then there exists an equivariant map $f\colon X \to \mathbb{S}^\infty$. Moreover, any two equivariant maps $f_0, f_1\colon X \to \mathbb{S}^\infty$ are equivariantly homotopic.*

Any equivariant map $f\colon X \to \mathbb{S}^\infty$ induces a nontrivial equivariant chain map

$$f_\sharp\colon C_*(X) \to C_*(\mathbb{S}^\infty).$$

A simple fact that will be crucial in what follows is that Proposition 5.10 has an analogue on the level of chain maps.

We first recall the relevant notion of homotopy between chain maps: Let $C_*(X)$ and $C_*(Y)$ be (singular or simplicial, say) chain complexes, and let $\varphi, \psi\colon C_*(X) \to C_*(Y)$ be chain maps. A *chain homotopy* η between φ and ψ is a family of homomorphisms $\eta_j\colon C_j(X) \to C_{j+1}(Y)$ such that

$$\varphi_j - \psi_j = \partial^Y_{j+1} \circ \eta_j + \eta_{j-1} \circ \partial^X_j$$

for all j.[8] If X and Y are $\mathbb{Z}_2$-spaces then a chain homotopy is called equivariant if it commutes with the (chain maps induced by) the $\mathbb{Z}_2$-actions.[9]

Lemma 5.11. *If X is a cell complex with a free cellular $\mathbb{Z}_2$-action then any two nontrivial equivariant chain maps $\varphi, \psi\colon C_*(X) \to C_*(\mathbb{S}^\infty)$ are equivariantly chain homotopic.*[10]

Proof of Lemma 5.11. Let the $\mathbb{Z}_2$-action on X be given by the automorphism $\nu\colon X \to X$. For each dimension $i \geq 0$, the action partitions the i-dimensional cells of X (the basis elements of $C_i(X)$) into pairs $\sigma, \nu(\sigma)$. For each such pair, we arbitrarily pick one of the cells and call it the representative of the pair.

[8] Here, we use subscripts and superscripts on the boundary operators to emphasize which dimension and which chain complex they belong to; often, these indices are dropped and one simply writes $\varphi - \psi = \partial\eta + \eta\partial$.

[9] We also recall that if $f, g\, X \to Y$ are (equivariantly) homotopic then the induced chain maps are (equivariantly) chain homotopic. Moreover, chain homotopic maps induce *identical* maps in homology and cohomology.

[10] We stress that we work with the cellular chain complex for X.

We define the desired equivariant chain homotopy η between φ and ψ by induction on the dimension, using the fact that all reduced homology groups of $\mathbb{S}^\infty$ are zero.[11]

We start the induction in dimension at $j = -1$ (and for convenience, we also use the convention that all chain groups, chain maps, and η_i are understood to be zero in dimensions $i < -1$). Since we assume that both φ and ψ are nontrivial, we have that $\varphi_{-1}, \psi_{-1} : C_{-1}(X) \to C_{-1}(\mathbb{S}^\infty)$ are identical, and we set $\eta_{-1} : C_{-1}(X) \to C_0(\mathbb{S}^\infty)$ to be zero.

Next, assume inductively that equivariant homomorphisms $\eta_i : C_i(X) \to C_i(\mathbb{S}^\infty)$ have already been defined for $i < j$ and satisfy

$$\varphi_i - \psi_i = \eta_{i-1} \circ \partial + \partial \circ \eta_i \tag{5.1}$$

for all $i < j$ (note that initially, this holds true for $j = 0$).

Suppose that σ is a j-dimensional cell of X representing a pair $\sigma, \nu(\sigma)$. Then $\partial\sigma \in C_{j-1}(X)$, and so $\eta_{j-1}(\partial\sigma) \in C_j(\mathbb{S}^\infty)$ is already defined. We are looking for a suitable chain $c \in C_{j+1}(\mathbb{S}^\infty)$ which we can take to be $\eta_j(\sigma)$ in order to satisfy the chain homotopy relation (5.1) also for $i = j$, such a chain c has to satisfy $\partial c = b$, where

$$b := \varphi_j(\sigma) - \psi_j(\sigma) - \eta_{j-1}(\partial(\sigma)).$$

To see that we can find such a c, we compute

$$\begin{aligned}
\partial b &= \partial\varphi_j(\sigma) - \partial\psi_j(\sigma) - \partial\eta_{j-1}(\partial(\sigma)) \\
&= \varphi_{j-1}(\partial\sigma) - \psi_{j-1}(\partial\sigma) - \Big(\varphi_{j-1}(\partial\sigma) - \psi_{j-1}(\partial\sigma) - \eta_{j-2}(\partial\partial\sigma)\Big) = 0
\end{aligned}$$

Thus, b is a cycle, and since $H_j(\mathbb{S}^\infty) = 0$, b is also a boundary. Pick an arbitrary chain $c \in C_{j+1}(\mathbb{S}^\infty)$ with $\partial c = b$ and set $\eta_j(\sigma) := c$ and $\eta_j(\nu(\sigma)) := \nu_\sharp(c)$. We do this for all representative j-cells σ and then extend η_j by linearity. By definition, η_j is equivariant and (5.1) is now satisfied also for $i = j$. This completes the induction step and hence the proof. □

Deleted products and Gauss maps. Let K be a simplicial complex. Then the Cartesian product $K \times K$ is a cell complex whose cells are the Cartesian products of pairs of simplices of K. The (*combinatorial*) *deleted product* $\widetilde{K}$ of K is defined as the polyhedral subcomplex of $K \times K$ whose cells are the products of vertex-disjoint pairs of simplices of K, i.e., $\widetilde{K} := \{\sigma \times \tau : \sigma, \tau \in K, \sigma \cap \tau = \emptyset\}$. The deleted product is equipped with a natural free $\mathbb{Z}_2$-action that simply exchanges coordinates, $(x, y) \mapsto (y, x)$. Note that this action is cellular since each cell $\sigma \times \tau$ is mapped to $\tau \times \sigma$.

Lemma 5.12. *If $f : |K| \hookrightarrow \mathbb{R}^d$ is an embedding or an almost embedding, then*[12] *there exists an equivariant map $\tilde{f} : \widetilde{K} \to S^{d-1}$.*

Proof. Define $\tilde{f}(x, y) := \frac{f(x) - f(y)}{\|f(x) - f(y)\|}$. This map, called the *Gauss map*, is clearly equivariant. □

For the proof of Proposition 5.6, we use the following analogue of Lemma 5.12.

Lemma 5.13. *Let K be a finite simplicial complex. If $\gamma : C_*(K) \to C_*(\mathbb{R}^d)$ is a homological almost-embedding then there is a nontrivial equivariant chain map (called the* Gauss chain map*) $\tilde{\gamma} : C_*(\widetilde{K}) \to C_*(\mathbb{S}^{d-1})$.*

The proof of this lemma is not difficult but a bit technical, so we postpone it until the end of this section.

Obstructions. Here, we recall a standard method for proving the non-existence of equivariant maps between $\mathbb{Z}_2$-spaces. The arguments are formulated in the language of cohomology, and, as we will see, what they actually establish is the non-existence of nontrivial equivariant chain maps.

Let K be a finite simplicial complex and let $\widetilde{K}$ be its (combinatorial) deleted product. By Proposition 5.10, there exists an equivariant map $G_K : \widetilde{K} \to \mathbb{S}^\infty$, which is unique up to equivariant homotopy.

[11] This just mimics the argument for the existence of an equivariant homotopy, which uses the contractibility of $\mathbb{S}^\infty$.

[12] We remark that a classical result due to Haefliger and Weber [Hae63, Web67] asserts that if $\dim K \le (2d-3)/3$ (the so-called *metastable range*) then the existence of an equivariant map from $\widetilde{K}$ to $\mathbb{S}^{d-1}$ is also *sufficient* for the existence of an embedding $K \hookrightarrow \mathbb{R}^d$ (outside the metastable range, this fails); see [Sko08] for further background.

By factoring out the action of $\mathbb{Z}_2$, this induces a map $\overline{G}_K \colon \overline{K} \to \mathbb{RP}^\infty$ between the quotient spaces $\overline{K} = \widetilde{K}/\mathbb{Z}_2$ and $\mathbb{RP}^\infty = \mathbb{S}^\infty/\mathbb{Z}_2$ (the infinite-dimensional real projective space), and the homotopy class of the map $\overline{G}_K$ depends only[13] on K. Passing to cohomology, there is a uniquely defined induced homomorphism

$$\overline{G}_K^* \colon H^*(\mathbb{RP}^\infty) \to H^*(\overline{K}).$$

It is known that $H^d(\mathbb{RP}^\infty) \cong \mathbb{Z}_2$ for every $d \geq 0$. Letting ξ^d denote the unique generator of $H^d(\mathbb{RP}^\infty)$, there is a uniquely defined cohomology class

$$\mathfrak{o}^d(K) := \overline{G}_K^*(\xi^d),$$

called the *van Kampen obstruction* (with $\mathbb{Z}_2$-coefficients) to embedding K into $\mathbb{R}^d$. For more details and background regarding the van Kampen obstruction, we refer the reader to [Mel09].

The basic fact about the van Kampen obstruction (and the reason for its name) is that K does not embed (not even almost-embed) into $\mathbb{R}^d$ if $\mathfrak{o}^d(K) \neq 0$ (Proposition 5.2). This follows from Lemma 5.12 and Part (a) of the following lemma:

Lemma 5.14. *Let K be a simplicial complex and suppose that $\mathfrak{o}^d(K) \neq 0$.*

(a) *Then there is no equivariant map $\widetilde{K} \to \mathbb{S}^{d-1}$.*

(b) *In fact, there is no nontrivial equivariant chain map $C_*(\widetilde{K}) \to C_*(\mathbb{S}^{d-1})$.*

Together with Lemma 5.13, Part (b) of the lemma also implies Proposition 5.6, as desired. The simple observation underlying the proof of Lemma 5.14 is the following

Observation 5.15. *Suppose $\varphi \colon C_*(\widetilde{K}) \to C_*(\mathbb{S}^\infty)$ is a nontrivial equivariant chain map (not necessarily induced by a continuous map). By factoring out the action of $\mathbb{Z}_2$, φ induces a chain map $\overline{\varphi} \colon C_*(\overline{K}) \to C_*(\mathbb{RP}^\infty)$. The induced homomorphism in cohomology*

$$\overline{\varphi}^* \colon H^*(\mathbb{RP}^\infty) \to H^*(\overline{K})$$

is equal *to the homomorphism $\overline{G}_K^*$ used in the definition of the Van Kampen obstruction, hence in particular*

$$\mathfrak{o}^d(K) = \overline{\varphi}^*(\xi^d).$$

Proof. By Lemma 5.11, φ is equivariantly chain homotopic to the nontrivial equivariant chain map $(G_K)_\sharp$ induced by the map G_K. Thus, after factoring out the $\mathbb{Z}_2$-action, the chain maps $\overline{\varphi}$ and $(\overline{G}_K)_\sharp$ from $C_*(\overline{K})$ to $C_*(\mathbb{RP}^\infty)$ are chain homotopic, and so induce identical homomorphisms in cohomology. ☐

Proof of Lemma 5.14. If there exists an equivariant map $f \colon \widetilde{K} \to \mathbb{S}^{d-1}$, then the induced chain map $f_\sharp \colon C_*(\widetilde{K}) \to C_*(\mathbb{S}^{d-1})$ is equivariant and nontrivial, so (b) implies (a), and it suffices to prove the former.

Next, suppose for a contradiction that $\psi \colon C_*(\widetilde{K}) \to C_*(\mathbb{S}^{d-1})$ is a nontrivial equivariant chain map. Let $i : \mathbb{S}^{d-1} \to \mathbb{S}^\infty$ denote the inclusion map, and let $i_\sharp \colon C_*(\mathbb{S}^{d-1}) \to C_*(\mathbb{S}^\infty)$ denote the induced equivariant, nontrivial chain map. Then the composition $\varphi = (i_\sharp \circ \psi) \colon C_*(\widetilde{K}) \to C_*(\mathbb{S}^\infty)$ is also nontrivial and equivariant, and so, by the preceding observation, for the induced homomorphism in cohomology, we get

$$\mathfrak{o}^d(K) = \overline{(i_\sharp \circ \psi)}^*(\xi^d) = \overline{\psi}^*\left(\overline{i}^*(\xi^d)\right).$$

However, $\overline{i}^*(\xi^d) \in H^d(\mathbb{RP}^{d-1}) = 0$ (for reasons of dimension), hence $\mathfrak{o}^d(K) = 0$, contradicting our assumption. ☐

[13] We stress that this does not mean that there is only one homotopy class of continuous maps $\overline{K} \to \mathbb{RP}^\infty$; indeed, there exist such maps that do not come from equivariant maps $\widetilde{K} \to \mathbb{S}^\infty$, for instance the constant map that maps all of $\overline{K}$ to a single point.

Remark 5.16. *The same kind of reasoning also yields the well-known* Borsuk–Ulam Theorem*, which asserts that there is no equivariant map $\mathbb{S}^d \to \mathbb{S}^{d-1}$, using the fact that the inclusion $\bar{i}\colon \mathbb{RP}^d \to \mathbb{RP}^\infty$ (induced by the equivariant inclusion $i\colon \mathbb{S}^d \to \mathbb{S}^\infty$) has the property that $\bar{i}^*(\xi^d)$, the pullback of the generator $\xi^d \in H^d(\mathbb{RP}^\infty)$, is* nonzero.[14] *In fact, once again one gets a homological version of the Borsuk–Ulam theorem for free: there is no nontrivial equivariant chain map $C_*(\mathbb{S}^d) \to C_*(\mathbb{S}^{d-1})$.*

Proof of Lemma 5.9. It is not hard to see that the deleted product $\widetilde{\partial\Delta_{d+1}} = \widetilde{\Delta_{d+1}}$ of the boundary of $(d+1)$-simplex is combinatorially isomorphic to the boundary of a certain convex polytope and hence homeomorphic to $\mathbb{S}^d$(respecting the antipodality action), see [Mat03, Exercise 5.4.3]. Thus, the assertion $\mathfrak{o}^d(\partial\Delta_{d+1}) \neq 0$ follows immediately from the preceding remark (the homological proof of the Borsuk–Ulam theorem). Together with Proposition 5.6, this implies that there is no homological almost-embedding of $\partial\Delta_{d+1}$ in $\mathbb{R}^d$. □

The proof of Proposition 5.6 is complete, except for the following:

Proof of Lemma 5.13. Once again, we essentially mimic the definition of the Gauss map on the level of chains. There is one minor technical difficulty due to the fact that the cells of $\widetilde{K}$ are products of simplices, whereas the singular homology of spaces is based on maps whose domains are simplices, not products of simplices (this is the same issue that arises in the proof of Künneth type formulas in homology).

Assume that $\gamma\colon C_*(K) \to C_*(\mathbb{R}^d)$ is a homological almost-embedding. The desired nontrivial equivariant chain map $\widetilde{\gamma}\colon C_*(\widetilde{K}) \to C_*(\mathbb{S}^{d-1})$ will be defined as the composition of three intermediate nontrivial equivariant chain maps

$$C_*(\widetilde{K}) \xrightarrow{\alpha} D_* \xrightarrow{\beta} C_*(\widetilde{\mathbb{R}^d}) \xrightarrow{p_\sharp} C_*(\mathbb{S}^{d-1}), \qquad \widetilde{\gamma} = p_\sharp \circ \beta \circ \alpha.$$

These maps and intermediate chain complexes will be defined presently.

We define D_* as a chain subcomplex of the tensor product $C_*(\mathbb{R}^d) \otimes C_*(\mathbb{R}^d)$. The tensor product chain complex has a basis consisting of all elements of the form $s \otimes t$, where s and t range over the singular simplices of $\mathbb{R}^d$, and we take D_* as the subcomplex spanned by all $s \otimes t$ for which s and t have disjoint supports (note that D_* is indeed a chain subcomplex, i.e., closed under the boundary operator, since if s and t have disjoint supports, then so do any pair of simplices that appear in the boundary of s and of t, respectively). The chain complex $C_*(\widetilde{K})$ has a canonical basis consisting of cells $\sigma \times \tau$, and the chain map α is defined on these basis elements by "tensoring" γ with itself, i.e.,

$$\alpha(\sigma \times \tau) := \gamma(\sigma) \otimes \gamma(\tau).$$

Since γ is nontrivial, so is α, the disjointness properties of γ ensure that the image of α does indeed lie in D_*, and α is clearly $\mathbb{Z}_2$-equivariant.

Next, consider the Cartesian product $\mathbb{R}^d \times \mathbb{R}^d$ with the natural $\mathbb{Z}_2$-action given by flipping coordinates. This action is not free since it has a nonempty set of fixed points, namely the "diagonal" $\Delta = \{(x,x) : x \in \mathbb{R}^d\}$. However, the action on $\mathbb{R}^d \times \mathbb{R}^d$ restricts to a free action on the subspace $\widetilde{\mathbb{R}^d} := (\mathbb{R}^d \times \mathbb{R}^d) \setminus \Delta$ obtained by removing the diagonal (this subspace is sometimes called the topological deleted product of $\mathbb{R}^d$). Moreover, there exists an equivariant map $p\colon \widetilde{\mathbb{R}^d} \to \mathbb{S}^{d-1}$ defined as follows: we identify $\mathbb{S}^{d-1}$ with the unit sphere in the orthogonal complement $\Delta^\perp = \{(w,-w) \in \mathbb{R}^d : w \in \mathbb{R}^d\}$ and take $p\colon \widetilde{\mathbb{R}^d} \to \mathbb{S}^{d-1}$ to be the orthogonal projection onto $\Delta^\perp$ (which sends (x,y) to $\frac{1}{2}(x-y, y-x)$), followed by renormalizing,

$$p(x,y) := \frac{\frac{1}{2}(x-y, y-x)}{\|\frac{1}{2}(x-y,y-x)\|} \in \mathbb{S}^{d-1} \subset \Delta^\perp.$$

The map p is equivariant and so the induced chain map $p_\sharp$ is equivariant and nontrivial.

It remains to define $\beta\colon D_* \to C_*(\widetilde{\mathbb{R}^d})$. For this, we use a standard chain map

$$\mathrm{EML}\colon C_*(\mathbb{R}^d) \otimes C_*(\mathbb{R}^d) \to C_*(\mathbb{R}^d \times \mathbb{R}^d),$$

[14] In fact, it is known that $H^*(\mathbb{RP}^\infty)$ is isomorphic to the polynomial ring $\mathbb{Z}_2[\xi]$, that $H^*(\mathbb{RP}^d) \cong \mathbb{Z}_2[\xi]/(\xi^{d+1})$, and that $\bar{i}^*$ is just the quotient map.

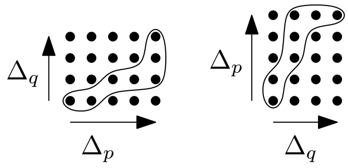


Figure 5.1: A simplex in a triangulation of $\Delta_p \times \Delta_q$ and its twin in $\Delta_q \times \Delta_p$.

sometimes called the Eilenberg–Mac Lane chain map, and then take β to be the restriction to D_*.

Given a basis element $s \otimes t$ of $C_*(\mathbb{R}^d) \otimes C_*(\mathbb{R}^t)$, where $s\colon \Delta_p \to \mathbb{R}^d$ and $t\colon \Delta_q \to \mathbb{R}^d$ are singular simplices, we can view $s \otimes t$ as the map $s \otimes t\colon \Delta_p \times \Delta_q \to \mathbb{R}^d \times \mathbb{R}^d$ with $(x, y) \mapsto (s(x), t(y))$. This is almost like a singular simplex in $\mathbb{R}^d \times \mathbb{R}^d$, except that the domain is not a simplex but a prism (product of simplices). The Eilenberg–Mac Lane chain map is defined by prescribing a systematic and coherent way of triangulating products of simplices $\Delta_p \times \Delta_q$ that is consistent with taking boundaries; then $\mathrm{EML}(s \otimes t) \in C_{p+q}(\mathbb{R}^d \times \mathbb{R}^d)$ is defined as the singular chain whose summands are the restrictions of the map $\sigma \otimes \tau\colon \Delta_p \times \Delta_q$ to the $(p+q)$-simplices that appear in the triangulation of $\Delta_p \times \Delta_q$. We refer to [GDR05] for explicit formulas for the chain map EML. What is important for us is that the chain map EML is equivariant and nontrivial. Both properties follow more or less directly from the construction of the triangulation of the prisms $\Delta_p \times \Delta_q$, which can be explained as follows: Implicitly, we assume that the vertex sets $\{0, 1, \ldots, p\}$ and $\{0, 1, \ldots, q\}$ are totally ordered in the standard way. The vertex set of $\Delta_p \times \Delta_q$ is the grid $\{0, 1, \ldots, p\} \times \{0, 1, \ldots, q\}$, on which we consider the coordinatewise partial order defined by $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$. Then the simplices of the triangulation are all totally ordered subsets of this partial order. Thus, if $\sigma = \{(x_0, y_0), (x_1, y_1), \ldots, (x_r, y_r)\}$ is a simplex that appears in the triangulation of $\Delta_p \times \Delta_q$ then the simplex $\sigma = \{(y_0, x_0), (y_1, x_1), \ldots, (y_r, x_r)\}$ obtained by flipping all coordinates appears in the triangulation of $\Delta_q \times \Delta_p$; see Figure 5.1. This implies equivariance of EML (and it is nontrivial since it maps a single vertex to a single vertex). □

Chapter 6

A general Helly type theorem

In this chapter we finally prove the general Helly type theorem 1.8. Let us recall its statement: There exists a function $h(b,d)$ such that the following holds. If $\mathcal{F}$ is a finite family of sets in $\mathbb{R}^d$ such that $\tilde{\beta}_i\left(\bigcap\mathcal{G};\mathbb{Z}_2\right) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and every $0 \leq i \leq \lceil d/2\rceil - 1$, then $\mathcal{F}$ has Helly number at most $h(b,d)$. If we are only interested whether the Helly numbers are bounded or not, this theorem subsumes a broad class of Helly type theorems for sets in $\mathbb{R}^d$.

Before we prove the theorem, we show that it is qualitatively sharp (Example 6.1) and provide a lower bound for the function $h(b,d)$ (Example 6.2).

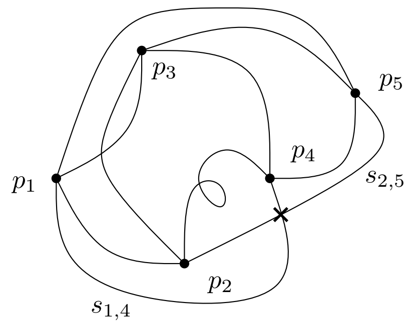
Example 6.1. *Fix some k with $0 \leq k \leq \lceil d/2\rceil - 1$. For n arbitrarily large, consider a geometric realization in $\mathbb{R}^d$ of the k-skeleton of the $(n-1)$-dimensional simplex (see [Mat03, Section 1.6]); more specifically, let $V = \{v_1,\dots,v_n\}$ be a set of points in general position in $\mathbb{R}^d$ (for instance, n points on the moment curve) and consider all geometric simplices $\sigma_A := \mathrm{conv}(A)$ spanned by subsets $A \subseteq V$ of cardinality $|A| \leq k+1$. By general position, $\sigma_A \cap \sigma_B = \sigma_{A\cap B}$, so this yields indeed a geometric realization.*

For $1 \leq j \leq n$, let U_j be the union of all the simplices not containing the vertex v_j. We set $\mathcal{F} = \{U_1,\dots,U_n\}$. Then, $\bigcap\mathcal{F} = \emptyset$, and for any proper sub-family $\mathcal{G} \subsetneq \mathcal{F}$, the intersection $\bigcap\mathcal{G}$ is either $\mathbb{R}^d$ (if $\mathcal{G} = \emptyset$) or (homeomorphic to) the k-dimensional skeleton of a $(n-1-|\mathcal{G}|)$-dimensional simplex. Thus, the Helly number of $\mathcal{F}$ equals n. Moreover, the k-skeleton $\Delta^{(k)}_{m-1}$ of an $(m-1)$-dimensional simplex has reduced Betti numbers $\tilde{\beta}_i = 0$ for $i \neq k$ and $\tilde{\beta}_k = \binom{m-1}{k+1}$. Thus, we can indeed obtain arbitrarily large Helly number as soon as at least one $\tilde{\beta}_k$ is unbounded.

Example 6.2. *First, we observe that for every $d \geq 2$ there is a geometric simplicial complex K_d with $d+2$ vertices, embedded in $\mathbb{R}^d$, such that every nonempty induced subcomplex L of K_d is connected and $\beta_i(L) = \tilde{\beta}_i(L) \leq 1$ for any $i \geq 1$.*

Indeed, it is sufficient to consider K_d as the stellar subdivision of the d-simplex (a. k. a. the cone over the boundary of the d-simplex): Among the vertices of K_d, $d+1$ of them, say $v_1,\dots,v_{d+1}$, form a simplex, and the last one, say w, is situated in the barycenter of that simplex. The maximal simplices of K_d contain w and d of the v-vertices. Given an induced subcomplex L, either L misses one of the v-vertices, and then L is a subcomplex of a simplex; or L contains all the v-vertices, and then $L = K$ or L is the boundary of the simplex formed by the v-vertices.

Now, with the knowledge of K, we can construct a set-system $\mathcal{F}$ with $b(d+2)$ sets such that $\bigcap\mathcal{F} = \emptyset$, the intersection of any proper subsystem of $\mathcal{F}$ is nonempty, and the reduced Betti numbers of the intersection of any proper subsystem are bounded by b: We consider a complex $K_{b,d}$ which consists of b disjoint copies of K_d, embedded in $\mathbb{R}^d$. For any vertex v of $K_{b,d}$ we let F_v be the induced subcomplex of $K_{b,d}$ on all vertices but v. We set $\mathcal{F}$ to be the collection of F_vs for all possible v. It follows that $\bigcap\mathcal{F} = \emptyset$ and also that $\bigcap\mathcal{G}$ is a nonempty induced subcomplex of $K_{b,d}$ for any nonempty $\mathcal{G} \subsetneq \mathcal{F}$. Therefore, $\bigcap\mathcal{G}$ is nonempty and its reduced Betti numbers are bounded by b from the construction of K_d.



Two edges (arcs) with no common vertices intersect (in this case $s_{1,4}$ and $s_{2,5}$). The point in the intersection then belongs to all sets in $\mathcal{F}$.

Figure 6.1: Illustration of the planar case

6.1 Proof outline

Using the machinery of Chapter 5, we prove Theorem 1.8 in two steps. First we present, in Section 6.2, variations of the technique that derives Helly type theorems from non-embeddability. We finally introduce our refinement of this technique and the proof of Theorem 1.8 in Section 6.5.

We derive Theorem 1.8 from obstructions to embeddability using a technique we learned from the work of Matoušek [Mat97]. First, we illustrate this technique, which in fact already appears in the classical proof of Helly's convex theorem from Radon's lemma, on a few examples, then formalize its ingredients.

6.2 Helly type theorems from homotopic assumptions

Let $\mathcal{F} = \{U_1, U_2, \dots, U_n\}$ denote a family of subsets of $\mathbb{R}^d$. We assume that $\mathcal{F}$ has empty intersection and that any proper subfamily of $\mathcal{F}$ has nonempty intersection. Our goal is to show how various conditions on the topology of the intersections of the subfamilies of $\mathcal{F}$ imply bounds on the cardinality of $\mathcal{F}$. For any (possibly empty) proper subset I of $[n] = \{1, 2, \dots, n\}$ we write $U_{\overline{I}}$ for $\bigcap_{i \in [n] \setminus I} U_i$. We also put $U_{\overline{[n]}} = \mathbb{R}^d$.

Path-connected intersections in the plane. Consider the case where $d = 2$ and the intersections $\bigcap \mathcal{G}$ are path-connected for all subfamilies $\mathcal{G} \subsetneq \mathcal{F}$. Since every intersection of $n-1$ members of $\mathcal{F}$ is nonempty, we can pick, for every $i \in [n]$, a point p_i in $U_{\overline{\{i\}}}$. Moreover, as every intersection of $n-2$ members of $\mathcal{F}$ is connected, we can connect any pair of points p_i and p_j by an arc $s_{i,j}$ inside $U_{\overline{\{i,j\}}}$. We thus obtain a drawing of the complete graph on $[n]$ in the plane in a way that the edge between i and j is contained in $U_{\overline{\{i,j\}}}$ (see Figure 6.1). If $n \geq 5$ then the stronger form of non-planarity of K_5 implies that there exist two edges $\{i, j\}$ and $\{k, \ell\}$ with no vertex in common and whose images intersect (see Proposition 5.2 and Lemma 5.3). Since $U_{\overline{\{i,j\}}} \cap U_{\overline{\{k,\ell\}}} = \bigcap \mathcal{F} = \emptyset$, this cannot happen and $\mathcal{F}$ has cardinality at most 4.

$\lceil d/2 \rceil$-connected intersections in $\mathbb{R}^d$. The previous argument generalizes to higher dimension as follows. Assume that the intersections $\bigcap \mathcal{G}$ are $\lceil d/2 \rceil$-connected[1] for all subfamilies $\mathcal{G} \subsetneq \mathcal{F}$. Then we can build by induction a function f from the $\lceil d/2 \rceil$-skeleton of Δ_{n-1} to $\mathbb{R}^d$ in a way that for any simplex σ, the image $f(\sigma)$ is contained in $U_{\overline{\sigma}}$. The previous case shows how to build such a function from the 1-skeleton of Δ_{n-1}. Assume that a function f from the ℓ-skeleton of Δ_{n-1} is built. For every $(\ell+1)$-simplex σ of Δ_{n-1}, for every facet τ of σ, we have $f(\tau) \subset U_{\overline{\tau}} \subseteq U_{\overline{\sigma}}$. Thus, the set

$$\bigcup_{\tau \text{ facet of } \sigma} f(\tau)$$

[1] Recall that a set is k-connected if it is connected and has vanishing homotopy in dimension 1 to k.

is the image of an ℓ-dimensional sphere contained in $U_{\overline{\sigma}}$, which has vanishing homotopy of dimension ℓ. We can extend f from this sphere to an $(\ell+1)$-dimensional ball so that the image is still contained in $U_{\overline{\sigma}}$. This way we extend f to the $(\ell+1)$-skeleton of Δ_{n-1}.

The Van Kampen-Flores theorem asserts that for any continuous function from $\Delta_{2k+2}^{(k)}$ to $\mathbb{R}^{2k}$ there exist two disjoint faces of $\Delta_{2k+2}^{(k)}$ whose images intersect (see Proposition 5.2 and Lemma 5.3). So, if $n \geq 2\lceil d/2 \rceil + 3$, then there exist two disjoint simplices σ and τ of $\Delta_{2\lceil d/2\rceil+2}^{(\lceil d/2 \rceil)}$ such that $f(\sigma) \cap f(\tau)$ is nonempty. Since $f(\sigma) \cap f(\tau)$ is contained in $U_{\overline{\sigma}} \cap U_{\overline{\tau}} = \bigcap \mathcal{F} = \emptyset$, this is a contradiction and $\mathcal{F}$ has cardinality at most $2\lceil d/2 \rceil + 2$.

By a more careful inspection of odd dimensions, the bound $2\lceil d/2 \rceil + 2$ can be improved to $d+2$. We skip this in the homotopic setting, but we will do so in the homological setting (which is stronger anyway); see Corollary 6.3 below.

Contractible intersections. Of course, the previous argument works with other non-embeddability results. For instance, if the intersections $\bigcap \mathcal{G}$ are contractible for all subfamilies then the induction yields a map f from the d-skeleton of Δ_{n-1} to $\mathbb{R}^d$ with the property that for any simplex σ, the image $f(\sigma)$ is contained in $U_{\overline{\sigma}}$. The topological Radon theorem [BB79] (see also [Mat03, Theorem 5.1.2]) states that for any continuous function from Δ_{d+1} to $\mathbb{R}^d$ there exist two disjoint faces of Δ_{d+1} whose images intersect. So, if $n \geq d+2$ we again obtain a contradiction (the existence of two disjoint simplices σ and τ such that $f(\sigma) \cap f(\tau) \neq \emptyset$ whereas $U_{\overline{\sigma}} \cap U_{\overline{\tau}} = \bigcap \mathcal{F} = \emptyset$), and the cardinality of $\mathcal{F}$ must be at most $d+1$.

6.3 From homotopy to homology

The previous reasoning can be transposed to homology as follows. Assume that for $i = 0, 1, \ldots, k-1$ and all subfamilies $\mathcal{G} \subsetneq \mathcal{F}$ we have $\tilde{\beta}_i(\bigcap \mathcal{G}) = 0$. We construct a nontrivial[2] chain map f from the simplicial chains of $\Delta_{n-1}^{(k)}$ to the singular chains of $\mathbb{R}^d$ by increasing dimension:

- For every $\{i\} \subset [n]$ we let $p_i \in U_{\overline{\{i\}}}$. This is possible since every intersection of $n-1$ members of $\mathcal{F}$ is nonempty. We then put $f(\{i\}) = p_i$ and extend it by linearity into a chain map from $\Delta_{n-1}^{(0)}$ to $\mathbb{R}^d$. Notice that f is nontrivial and that for any 0-simplex $\sigma \subseteq [n]$, the support of $f(\sigma)$ is contained in $U_{\overline{\sigma}}$.

- Now, assume, as an induction hypothesis, that there exists a nontrivial chain map f from the simplicial chains of $\Delta_{n-1}^{(\ell)}$ to the singular chains of $\mathbb{R}^d$ with the property that for any $(\leq \ell)$-simplex $\sigma \subseteq [n], \ell < k$, the support of $f(\sigma)$ is contained in $U_{\overline{\sigma}}$. Let σ be a $(\ell+1)$-simplex in $\Delta_{n-1}^{(\ell+1)}$. For every ℓ-dimensional face τ of σ, the support of $f(\tau)$ is contained in $U_{\overline{\tau}} \subseteq U_{\overline{\sigma}}$. It follows that the support of $f(\partial\sigma)$ is contained in $U_{\overline{\sigma}}$, which has trivial homology in dimension $\ell+1$. As a consequence, $f(\partial\sigma)$ is a boundary in $U_{\overline{\sigma}}$. We can therefore extend f to every simplex of dimension $\ell+1$ and then, by linearity, to a chain map from the simplicial chains of $\Delta_{n-1}^{(\ell+1)}$ to the singular chains of $\mathbb{R}^d$. This chain map remains nontrivial and, by construction, for any $(\leq \ell+1)$-simplex $\sigma \subseteq [n]$, the support of $f(\sigma)$ is contained in $U_{\overline{\sigma}}$.

If σ and τ are disjoint simplices of $\Delta_{n-1}^{(k)}$ then the intersection of the supports of $f(\sigma)$ and $f(\tau)$ is contained in $U_{\overline{\sigma}} \cap U_{\overline{\tau}} = \bigcap \mathcal{F} = \emptyset$ and these supports are disjoint. It follows that f is not only a nontrivial chain map, but also a homological almost-embedding in $\mathbb{R}^d$. We can then use obstructions to the existence of homological almost-embeddings to bound the cardinality of $\mathcal{F}$. Specifically, since we assumed that $\mathcal{F}$ has empty intersection and any proper subfamily of $\mathcal{F}$ has nonempty intersection, Corollary 5.8 implies:

Corollary 6.3. *Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^d$ such that $\tilde{\beta}_i(\bigcap \mathcal{G}) = 0$ for every $\mathcal{G} \subsetneq \mathcal{F}$ and $i = 0, 1, \ldots, \lceil d/2 \rceil - 1$. Then the Helly number of $\mathcal{F}$ is at most $d+2$.*

The homological Radon's lemma (Lemma 5.9) yields (noting $\partial\Delta_{d+1} = \Delta_{d+1}^{(d)}$):

[2] See Definition 5.4.

Corollary 6.4. *Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^d$ such that $\tilde{\beta}_i(\bigcap \mathcal{G}) = 0$ for every $\mathcal{G} \subsetneq \mathcal{F}$ and $i = 0, 1, \ldots, d-1$. Then the Helly number of $\mathcal{F}$ is at most $d+1$.*

Remark 6.5. *The following modification of Example 6.1 shows that the two previous statements are sharp in various ways. First assume that for some values k, n there exists some embedding f of $\Delta_{n-1}^{(k)}$ into $\mathbb{R}^d$. Let K_i be the simplicial complex obtained by deleting the ith vertex of $\Delta_{n-1}^{(k)}$ (as well as all simplices using that vertex) and put $U_i := f(K_i)$. The family $\mathcal{F} = \{U_1, \ldots, U_n\}$ has Helly number exactly n, since it has empty intersection and all its proper subfamilies have nonempty intersection. Moreover, for every $\mathcal{G} \subseteq \mathcal{F}$, $\bigcap \mathcal{G}$ is the image through f of the k-skeleton of a simplex on $|\mathcal{F} \setminus \mathcal{G}|$ vertices, and therefore $\tilde{\beta}_i(\bigcap \mathcal{G}) = 0$ for every $\mathcal{G} \subseteq \mathcal{F}$ and $i = 0, \ldots, k-1$. Now, such an embedding exists for:*

$k = d$ **and** $n = d+1$, *as the d-dimensional simplex easily embeds into $\mathbb{R}^d$. Consequently, the bound of $d+1$ is best possible under the assumptions of Corollary 6.4.*

$k = d-1$ **and** $n = d+2$, *as we can first embed the $(d-1)$-skeleton of the d-simplex linearly, then add an extra vertex at the barycenter of the vertices of that simplex and embed the remaining faces linearly. This implies that if we relax the condition of Corollary 6.4 by only controlling the first $d-2$ Betti numbers then the bound of $d+1$ becomes false. It also implies that the bound of $d+2$ is best possible under (a strengthening of) the assumptions of Corollary 6.3.*

(Recall that, as explained in Example 6.1, the $\lceil d/2 \rceil - 1$ in the assumptions of Corollary 6.3 cannot be reduced without allowing unbounded Helly numbers.)

Constrained chain map. Let us formalize the technique illustrated by the previous example. We focus on the homological setting, as this is what we use to prove Theorem 1.8, but this can be easily transposed to homotopy.

Considering a slightly more general situation, we let $\mathcal{F} = \{U_1, U_2, \ldots, U_n\}$ denote a family of subsets of some topological space $\mathbf{R}$. As before for any (possibly empty) proper subset I of $[n] = \{1, 2, \ldots, n\}$ we write $U_{\overline{I}}$ for $\bigcap_{i \in [n] \setminus I} U_i$ and we put $U_{\overline{[n]}} = \mathbf{R}$.

Let K be a simplicial complex and let $\gamma : C_*(K) \to C_*(\mathbf{R})$ be a chain map from the simplicial chains of K to the singular chains of $\mathbf{R}$. We say that γ is *constrained by* $(\mathcal{F}, \Phi)$ if:

(i) Φ is a map from K to $2^{[n]}$ such that $\Phi(\sigma \cap \tau) = \Phi(\sigma) \cap \Phi(\tau)$ for all $\sigma, \tau \in K$ and $\Phi(\emptyset) = \emptyset$.

(ii) For any simplex $\sigma \in K$, the support of $\gamma(\sigma)$ is contained in $U_{\overline{\Phi(\sigma)}}$.

See Figure 6.2. We also say that a chain map γ from K is *constrained by* $\mathcal{F}$ if there exists a map Φ such that γ is constrained by $(\mathcal{F}, \Phi)$. In the above constructions, we simply set Φ to be the identity. As we already saw, constrained chain maps relate Helly numbers to homological almost-embeddings (see Definition 5.4) via the following observation:

Lemma 6.6. *Let $\gamma : C_*(K) \to C_*(\mathbf{R})$ be a nontrivial chain map constrained by $\mathcal{F}$. If $\bigcap \mathcal{F} = \emptyset$ then γ is a homological almost-embedding of K.*

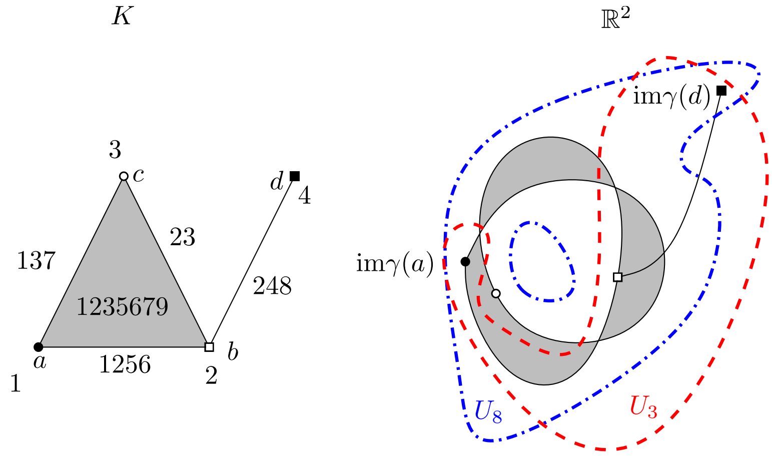
Proof. Let $\Phi : K \to 2^{[n]}$ be such that γ is constrained by $(\mathcal{F}, \Phi)$. Since γ is nontrivial, it remains to check that disjoint simplices are mapped to chains with disjoint support. Let σ and τ be two disjoint simplices of K. The supports of $\gamma(\sigma)$ and $\gamma(\tau)$ are contained, respectively, in $U_{\overline{\Phi(\sigma)}}$ and $U_{\overline{\Phi(\tau)}}$, and

$$U_{\overline{\Phi(\sigma)}} \cap U_{\overline{\Phi(\tau)}} = U_{\overline{\Phi(\sigma) \cap \Phi(\tau)}} = U_{\overline{\Phi(\sigma \cap \tau)}} = U_{\overline{\Phi(\emptyset)}} = U_{\overline{\emptyset}} = \bigcap \mathcal{F}.$$

Therefore, if $\bigcap \mathcal{F} = \emptyset$ then γ is a homological almost-embedding of K. □

6.4 Relaxing the connectivity assumption

In all the examples listed so far, the intersections $\bigcap \mathcal{G}$ must be connected. A relaxation of this condition was given by Matoušek [Mat97] who allowed "having a bounded number of connected components", the assumptions then being on the topology of the components, by using Ramsey's theorem. The gist of our proof is to extend his idea to allow a bounded number of homology classes not only in the first dimension but in *any* dimension. Let us illustrate how Matoušek's idea works in two dimension:



An example of a constrained map $\gamma\colon K \to \mathbb{R}^2$. A label at a face σ of K denotes $\Phi(\sigma)$. Note, for example, that the support of $\gamma(\{a,b,c\})$ needn't be a triangle since we work with chain maps. Constrains by Φ mean that a set U_i must contain cover images of all faces without label i. It is demonstrated by U_3 and U_8 for example.

Figure 6.2: An example of a constrained map

Theorem 6.7 ([Mat97, Theorem 2 with $d = 2$])**.** *For every positive integer b there is an integer $h(b)$ with the following property. If $\mathcal{F}$ is a finite family of subsets of $\mathbb{R}^2$ such that the intersection of any subfamily has at most b path-connected components, then the Helly number of $\mathcal{F}$ is at most $h(b)$.*

Let us fix b from above and assume that for any subfamily $\mathcal{G} \subsetneq \mathcal{F}$ the intersection $\bigcap \mathcal{G}$ consists of at most b path-connected components and that $\bigcap \mathcal{F} = \emptyset$. We start, as before, by picking for every $i \in [n]$, a point p_i in $U_{\overline{\{i\}}}$. This is possible as every intersection of $n-1$ members of $\mathcal{F}$ is nonempty. Now, if we consider some pair of indices $i, j \in [n]$, the points p_i and p_j are still in $U_{\overline{\{i,j\}}}$ but may lie in different connected components. It may thus not be possible to connect p_i to p_j *inside* $U_{\overline{\{i,j\}}}$. If we, however, consider $b+1$ indices $i_1, i_2, \ldots, i_{b+1}$ then all the points $p_{i_1}, p_{i_2}, \ldots, p_{i_{b+1}}$ are in $U_{\overline{\{i_1,i_2,\ldots,i_{b+1}\}}}$ which has at most b connected components, so at least one pair among of these points can be connected by a path inside $U_{\overline{\{i_1,i_2,\ldots,i_{b+1}\}}}$. Thus, while we may not get a drawing of the complete graph on n vertices we can still draw many edges.

To find many vertices among which every pair can be connected we will use the hypergraph version of the classical theorem of Ramsey:

Theorem 6.8 (Ramsey [Ram29])**.** *For any x, y and z there is an integer $R_x(y,z)$ such that any x-uniform hypergraph on at least $R_x(y,z)$ vertices colored with at most y colors contains a subset of z vertices inducing a monochromatic sub-hypergraph.*

From the discussion above, for any $b+1$ indices $i_1 < i_2 < \ldots < i_{b+1}$ there exists a pair $\{k, \ell\} \in \binom{[b+1]}{2}$ such that p_{i_k} and p_{i_ℓ} can be connected inside $U_{\overline{\{i_1,i_2,\ldots,i_{b+1}\}}}$. Let us consider the $(b+1)$-uniform hypergraph on $[n]$ and color every set of indices $i_1 < i_2 < \ldots < i_{b+1}$ by one of the pairs in $\binom{[b+1]}{2}$ that can be connected inside $U_{\overline{\{i_1,i_2,\ldots,i_{b+1}\}}}$ (if more than one pair can be connected, we pick one arbitrarily). Let t be some integer to be fixed later. By Ramsey's theorem, if $n \geq R_{b+1}\left(\binom{b+1}{2}, t\right)$ then there exist a pair $\{k, \ell\} \in \binom{[b+1]}{2}$ and a subset $T \subseteq [n]$ of size t with the following property: for any $(b+1)$-element subset $S \subset T$, the points whose indices are the kth and ℓth indices of S can be connected inside $U_{\overline{S}}$.

Now, let us set $t = 5 + \binom{5}{2}(b-1) = 10b - 5$. We claim that we can find five indices in T, denoted $i_1, i_2, \ldots, i_5$, and, for each pair $\{i_u, i_v\}$ among these five indices, some $(b+1)$-element subset $Q_{u,v} \subset T$ with the following properties:

(i) i_u and i_v are precisely in the kth and ℓth position in $Q_{u,v}$, and

(ii) for any $1 \leq u, v, u', v' \leq 5$, $\quad Q_{u,v} \cap Q_{u',v'} = \{i_u, i_v\} \cap \{i_{u'}, i_{v'}\}$.

We first conclude the argument, assuming that we can obtain such indices and sets. Observe that from the construction of T, the i_u's and the $Q_{u,v}$'s we have the following property: for any $u, v \in [5]$, we can connect p_{i_u} and p_{i_v} inside $U_{\overline{Q_{u,v}}}$. This gives a drawing of K_5 in the plane. Since K_5 is not planar, there exist two edges with no vertex in common, say $\{u, v\}$ and $\{u', v'\}$, that cross. This intersection point must lie in

$$U_{\overline{Q_{u,v}}} \cap U_{\overline{Q_{u',v'}}} = U_{\overline{Q_{u,v} \cap Q_{u',v'}}} = U_{\overline{\{i_u,i_v\} \cap \{i_{u'},i_{v'}\}}} = U_{\overline{\emptyset}} = \bigcap \mathcal{F} = \emptyset,$$

a contradiction. Hence the assumption that $n \geq R_{b+1}\left(\binom{b+1}{2}, t\right)$ is false and $\mathcal{F}$ has cardinality at most $R_{b+1}\left(\binom{b+1}{2}, 10b-5\right) - 1$, which is our $h(b)$.

The selection trick. It remains to derive the existence of the i_u's and the $Q_{u,v}$'s. It is perhaps better to demonstrate the method by a simple example to develop some intuition before we formalize it.

Example. Let us fix $b = 4$ and $\{k, \ell\} = \{2, 3\} \in \binom{[4+1]}{2}$. We first make a 'blueprint' for the construction inside the rational numbers. For any two indices $u, v \in [5]$ we form a totally ordered set $Q'_{u,v} \subseteq \mathbb{Q}$ of size $b + 1 = 5$ by adding three rational numbers (different from $1, \ldots, 5$) to the set $\{u, v\}$ in such a way that u appears at the second and v at the third position of $Q'_{u,v}$. For example, we can set $Q'_{1,4}$ to be $\{0.5; 1; 4; 4.7; 5.13\}$. Apart from this we require that we add a different set of rational numbers for each $\{u, v\}$. Thus $Q'_{u,v} \cap Q'_{u',v'} = \{u, v\} \cap \{u', v'\}$. Our blueprint now appears inside the set $T' := \bigcup_{1 \leq u < v \leq 5} Q'_{u,v}$; note that both this set T' and the set T in which we search for the sets $Q_{u,v}$ have 35 elements. To obtain the required indices i_u and sets $Q_{u,v}$ it remains to consider the unique strictly increasing bijection $\pi_0 \colon T' \to T$ and set $i_u := \pi_0(u)$ and $Q_{u,v} := \pi_0(Q'_{u,v})$.

The general case. Let us now formalize the generalization of this trick that we will use to prove Theorem 1.8. Let Q be a subset of $[w]$. If $e_1 < e_2 < \ldots < e_w$ are the elements of a totally ordered set W then we call $\{e_i : i \in Q\}$ the *subset selected by Q in W*.

Lemma 6.9. *Let $1 \leq q \leq w$ be integers and let Q be a subset of $[w]$ of size q. Let Y and Z be two finite totally ordered sets and let $A_1, A_2, \ldots, A_r$ be q-element subsets of Y. If $|Z| \geq |Y| + r(w - q)$, then there exist an injection $\pi : Y \to Z$ and r subsets $W_1, W_2, \ldots, W_r \in \binom{Z}{w}$ such that for every $i \in [r]$, Q selects $\pi(A_i)$ in W_i. We can further require that $W_i \cap W_j = \pi(A_i \cap A_j)$ for any two $i, j \in [r]$, $i \neq j$.*

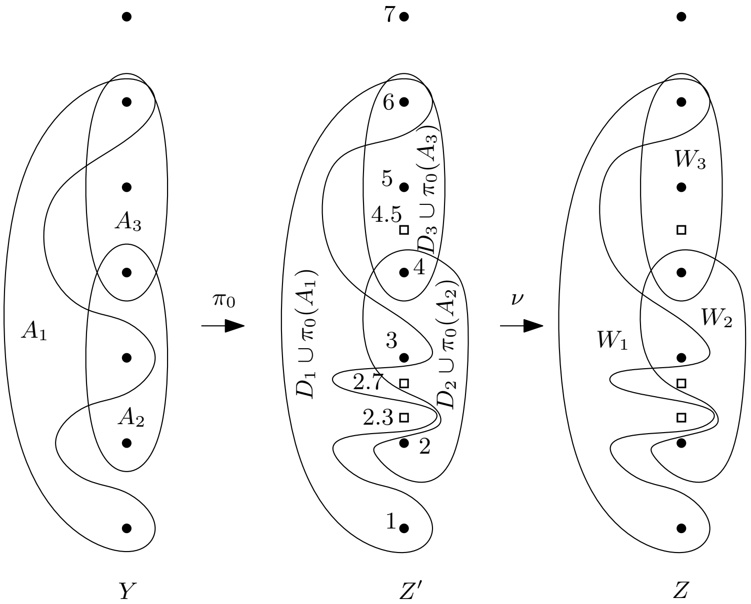
Proof. Let π_0 denote the monotone bijection between Y and $[|Y|]$. For $i \in [r]$ we let D_i denote a set of $w - q$ rationals, disjoint from $[|Y|]$, such that Q selects $\pi_0(A_i)$ in $D_i \cup \pi_0(A_i)$. We further require that the D_i are pairwise disjoint, and put $Z' = [|Y|] \cup \left(\bigcup_{i \in [r]} D_i\right)$. Since $|Z| \geq |Y| + r(w - q) = |Z'|$ there exists a strictly increasing map $\nu : Z' \to Z$. We set $\pi := \nu \circ \pi_0$ and $W_i := \nu(D_i \cup \pi_0(A_i)) \in \binom{Z}{w}$. The desired condition is satisfied by this choice. See Figure 6.3. □

6.5 Constrained chain maps and Helly number

We now generalize the technique presented in Section 6.2 to obtain Helly type theorems from non-embeddability results. We will construct constrained chain maps for arbitrary complexes. As above, $\mathcal{F} = \{U_1, U_2, \ldots, U_n\}$ denotes a family of subsets of some topological space $\mathbf{R}$ and for $I \subseteq [n]$ we keep the notation $U_{\overline{I}}$ as used in the previous section. Note that although so far we only used the *reduced* Betti numbers $\tilde{\beta}$, in this section it will be convenient to work with *standard* (non-reduced) Betti numbers β, starting with the following proposition.

Proposition 6.10. *For any finite simplicial complex K and non-negative integer b there exists a constant $h_K(b)$ such that the following holds. For any finite family $\mathcal{F}$ of at least $h_K(b)$ subsets of a topological space $\mathbf{R}$ such that $\bigcap \mathcal{G} \neq \emptyset$ and $\beta_i(\cap \mathcal{G}) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and any $0 \leq i < \dim K$, there exists a nontrivial chain map $\gamma : C_*(K) \to C_*(\mathbf{R})$ that is constrained by $\mathcal{F}$.*

The case $K = \Delta_{2k+2}^{(k)}$, with $k = \lceil d/2 \rceil$ and $\mathbf{R} = \mathbb{R}^d$, of Proposition 6.10 implies Theorem 1.8.



The situation for $w = 4$ and $Q = \{1, 3, 4\}$.

Figure 6.3: Illustration for the proof of Lemma 6.9

Proof of Theorem 1.8. Let b and d be fixed integers, let $k = \lceil d/2 \rceil$ and let $K = \Delta_{2k+2}^{(k)}$. Let $h_K(b+1)$ denote the constant from Proposition 6.10 (we plug in $b+1$ because we need to switch between reduced and non-reduced Betti numbers). Let $\mathcal{F}$ be a finite family of subsets of $\mathbb{R}^d$ such that $\tilde{\beta}_i\left(\bigcap \mathcal{G}\right) \leq b$ for any $\mathcal{G} \subsetneq \mathcal{F}$ and every $0 \leq i \leq \dim K = \lceil d/2 \rceil - 1$, in particular $\beta_i\left(\bigcap \mathcal{G}\right) \leq b+1$ for such $\mathcal{G}$. Let $\mathcal{F}^*$ denote an inclusion-minimal sub-family of $\mathcal{F}$ with empty intersection: $\bigcap \mathcal{F}^* = \emptyset$ and $\bigcap(\mathcal{F}^* \setminus \{U\}) \neq \emptyset$ for any $U \in \mathcal{F}^*$. If $\mathcal{F}^*$ has size at least $h_K(b+1)$, it satisfies the assumptions of Proposition 6.10 and there exists a nontrivial chain map from K that is constrained by $\mathcal{F}^*$. Since $\mathcal{F}^*$ has empty intersection, this chain map is a homological almost-embedding by Lemma 6.6. However, no such homological almost-embedding exists by Corollary 5.7, so $\mathcal{F}^*$ must have size at most $h_K(b+1) - 1$. As a consequence, the Helly number of $\mathcal{F}$ is bounded and the statement of Theorem 1.8 holds with $h(b, d) = h_K(b+1) - 1$. □

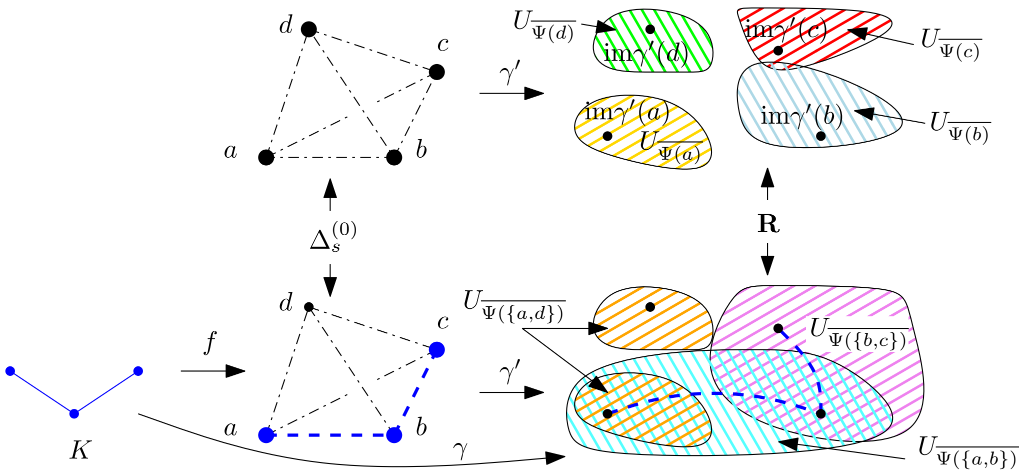
The rest of this section is devoted to proving Proposition 6.10. We proceed by induction on the dimension of K, Section 6.5.1 settling the case of 0-dimensional complexes and Section 6.5.3 showing that if Proposition 6.10 holds for all simplicial complexes of dimension i then it also holds for all simplicial complexes of dimension $i+1$. As the proof of the induction step is quite technical, as a warm-up, we provide the reader with a simplified argument for the induction step from $i = 0$ to $i = 1$ in Section 6.5.2. We let $V(K)$ and $v(K)$ denote, respectively, the set of vertices and the number of vertices of K.

6.5.1 Initialization $(\dim K = 0)$

If K is a 0-dimensional simplicial complex then Proposition 6.10 holds with $h_K(b) = v(K)$. Indeed, consider a family $\mathcal{F}$ of at least $v(K)$ subsets of $\mathbf{R}$ such that all proper subfamilies have nonempty intersection. We enumerate the vertices of K as $\{v_1, v_2, \ldots, v_{v(K)}\}$ and define $\Phi(\{v_i\}) = \{i\}$; in plain English, Φ is a bijection between the set of vertices of K and $\{1, 2, \ldots, v(K)\}$. We first define γ on K by mapping every vertex $v \in K$ to a point $p(v) \in U_{\overline{\Phi(v)}}$, then extend it linearly into a chain map $\gamma : C_0(K) \to C_0(\mathbf{R})$. It is clear that γ is nontrivial and constrained by $(\mathcal{F}, \Phi)$, so Proposition 6.10 holds when $\dim K = 0$.

6.5.2 Principle of the induction mechanism $(\dim K = 1)$

As a warm-up, we now prove Proposition 6.10 for 1-dimensional simplicial complexes. While this merely amounts to reformulating Matoušek's proof for embeddings [Mat97] in the language of chain maps, it



Injecting $V(K)$ into $V(\Delta_s)$ by f in a way that the constrained chain map γ' from $V(\Delta_s)$ (top) can give rise to a constrained chain map from $V(K)$ (bottom); for the sake of illustration we use maps instead of chain maps. The situation considered here is simple, for instance $\gamma'(a+b)$ is a boundary in $U_{\overline{\Psi(\{a,b\})}}$ so $\gamma' \circ f_\sharp$ can be extended to the edge $\{f^{-1}(a), f^{-1}(b)\}$ of K. Note that if we wanted to use the edge ad, since $\gamma'(a+d)$ is not a boundary in $U_{\overline{\Psi(\{a,d\})}}$ we would need to add "dummy" elements to $\Psi(\{a,d\})$.

Figure 6.4: Injecting $V(K)$ into $V(\Delta_s)$

still introduces several key ingredients of the induction while avoiding some of its complications. To avoid further technicalities, we use the non-reduced version of Betti numbers here.

Let K be a 1-dimensional simplicial complex with vertices $\{v_1, v_2, \ldots, v_{v(K)}\}$ and assume that $\mathcal{F}$ is a finite family of subsets of a topological space $\mathbf{R}$ such that for any $\mathcal{G} \subsetneq \mathcal{F}$, $\bigcap \mathcal{G} \neq \emptyset$ and $\beta_0 (\cap \mathcal{G}) \leq b$. Let $s \in \mathbb{N}$ denote some parameter, to be fixed later. We assume that the cardinality of $\mathcal{F}$ is large enough (as a function of s) so that, as argued in Subsection 6.5.1, there exist a bijection $\Psi : \Delta_s^{(0)} \to [s+1]$ and a nontrivial chain map $\gamma' : C_*(\Delta_s^{(0)}) \to C_*(\mathbf{R})$ constrained by $(\mathcal{F}, \Psi)$. We extend Ψ to Δ_s by putting $\Psi(\sigma) = \cup_{v \in \sigma} \Psi(v)$ for any $\sigma \in \Delta_s$ and $\Psi(\emptyset) = \emptyset$. Remark that for any $\sigma, \tau \in \Delta_s$ we have $\Psi(\sigma \cap \tau) = \Psi(\sigma) \cap \Psi(\tau)$.

We now look for an injection f of $V(K)$ into $V(\Delta_s)$ such that the chain map $\gamma' \circ f_\sharp : C_*(K^{(0)}) \to C_*(\mathbf{R})$ can be extended into a chain map $\gamma : C_*(K) \to C_*(\mathbf{R})$ constrained by $\mathcal{F}$. Let $e = \{u, v\}$ be an edge in K. If we could arrange that $\gamma'(f(u) + f(v))$ is a boundary in $U_{\overline{\Psi(\{f(u),f(v)\})}}$ then we could simply define $\gamma(e)$ to be a chain in $U_{\overline{\Psi(\{f(u),f(v)\})}}$ bounded by $\gamma'(f(u) + f(v))$ (see Figure 6.4). Unfortunately this is too much to ask for but we can still follow the Ramsey-based approach of Subsection 6.4: we add "dummy" vertices to $\{\Psi(\{f(u), f(v)\})\}$ to obtain a set W_e such that $\gamma'(f(u) + f(v))$ is a boundary in $U_{\overline{W_e}}$. If we use different dummy vertices for distinct edges then setting $\gamma(e)$ to be a chain in $U_{\overline{W_e}}$ bounded by $\gamma'(f(u) + f(v))$ still yields a chain map constrained by $\mathcal{F}$. We spell out the details in four steps.

Step 1. Any set S of $2^b + 1$ vertices of Δ_s contains two vertices $u_S, v_S \in S$ such that $\gamma'(u_S + v_S)$ is a boundary in $U_{\overline{\Psi(S)}}$.[3] Indeed, notice first that for any $u \in S$, the support of $\gamma'(u)$ is contained in $U_{\overline{\Psi(S)}}$. The assumption on $\mathcal{F}$ about bounded Betti numbers of intersections of subfamilies of $\mathcal{F}$ then ensures that there are at most 2^b distinct elements[4] in $H_0(U_{\overline{\Psi(S)}})$. Thus, there are two vertices

[3] We could require that γ' sends every vertex to a point in $U_{\overline{\Psi(S)}}$, i.e. is a chain map induced by a map, and simply argue that since $U_{\overline{\Psi(S)}}$ has at most b connected components, any $b+1$ vertices of Δ_s contains some pair that can be connected inside $U_{\overline{\Psi(S)}}$. This argument does not, however, work in higher dimension and its higher dimensional analogue, Theorem 3.26 would cause unnecessary technicalities later on. Since Section 6.5.2 is meant as an illustration of the general case, we choose to follow the general simple argument.

[4] $H_0(U_{\overline{\Psi(S)}}) \simeq \mathbb{Z}_2^m$ for some $m \leq b$.

$u_S, v_S \in S$ such that $\gamma'(u_S)$ and $\gamma'(v_S)$ are in the same homology class in $H_0(U_{\overline{\Psi(S)}})$. Since we consider homology with coefficients over $\mathbb{Z}_2$, the sum of two chains that are in the same homology class is always a boundary. In particular, $\gamma'(u_S + v_S) = \gamma'(u_S) + \gamma'(v_S)$ is a boundary in $U_{\overline{\Psi(S)}}$.

Step 2. We use Ramsey's theorem (Theorem 6.8) to ensure a uniform "2-in-(2^b+1)" selection. Let t be some parameter to be fixed in Step 3 and let H denote the (2^b+1)-uniform hypergraph with vertex set $V(\Delta_s)$. For every hyperedge $S \in H$ there exists (by Step 1) a pair $Q_S \in \binom{[2^b+1]}{2}$ that selects a pair whose sum is mapped by γ' to a boundary in $U_{\overline{\Psi(S)}}$. We color H by assigning to every hyperedge S the "color" Q_S. Ramsey's theorem thus ensures that if $s \geq R_{2^b+1}\left(\binom{2^b+1}{2}, t\right)$ then there exist a set T of t vertices of Δ_s and a pair $Q^* \in \binom{[2^b+1]}{2}$ so that Q^* selects in *any* $S \in \binom{T}{2^b+1}$ a pair $\{u_S, v_S\}$ such that $\gamma'(u_S + v_S)$ is a boundary in $U_{\overline{\Psi(S)}}$.

Step 3. Now, let r be the number of edges of K and let $\sigma_1, \sigma_2, \ldots, \sigma_r$ denote the edges of K. We define

$$h_K(b) = R_{2^b+1}\left(\binom{2^b+1}{2}, r(2^b-1) + v(K)\right) + 1$$

and assume that $s \geq h_K(b) - 1$. We set the parameter t introduced in Step 2 to $t = r(2^b-1) + v(K)$. We can now apply Lemma 6.9 with $Y = V(K)$, $Z = T$, $q = 2$, $w = 2^b+1$, and $A_i = \sigma_i$ for $i \in [r]$. As a consequence, there exist an injection $f : V(K) \to T$ and $W_1, W_2, \ldots, W_r$ in $\binom{T}{2^b+1}$ such that (i) for each i, Q^* selects $f(\sigma_i)$ in W_i, and (ii) $W_i \cap W_j = f(\sigma_i \cap \sigma_j)$ for $i, j \in [r], i \neq j$.

Step 4. We define Φ by

$$\begin{array}{rcll} \Phi(\emptyset) & = & \emptyset & \\ \Phi(\{v_i\}) & = & \Psi(f(v_i)) & \text{for } i = 1, 2, \ldots, v(K) \\ \Phi(\sigma_i) & = & \Psi(W_i) & \text{for } i = 1, 2, \ldots, r \end{array}$$

We define γ on the vertices of K by putting $\gamma(v) = \gamma'(f(v))$ for any $v \in V(K)$. Now remark that for any edge $\sigma_i = \{u, v\}$ of K, $\gamma'(f(u) + f(v))$ is a boundary in $U_{\overline{\Psi(W_i)}}$; this follows from the definition of T and the fact that Q^* selects $\{f(u), f(v)\}$ in W_i. We can therefore define $\gamma(\{u, v\})$ to be some (arbitrary) chain in $U_{\overline{\Psi(W_i)}}$ with boundary $\gamma'(f(u) + f(v))$. We then extend this map linearly into a chain map $\gamma : C_*(K) \to C_*(\mathbf{R})$.

To conclude the proof of Proposition 6.10 for 1-dimensional complexes it remains to check that the chain map γ and the function Φ defined in Step 4 have the desired properties.

Observation 6.11. *γ is a nontrivial chain map constrained by $(\mathcal{F}, \Phi)$.*

Proof. First, it is clear from the definition that γ is a chain map. Moreover, the definition of γ' ensures that for every vertex $v \in K$ the support of $\gamma(v)$ is a finite set of points with odd cardinality. So γ is indeed a nontrivial chain map.

The map Φ is from K to $2^{[s+1]}$ and $\Phi(\emptyset)$ is by definition the empty set. The next property to check is that the identity $\Phi(\sigma \cap \tau) = \Phi(\sigma) \cap \Phi(\tau)$ holds for all $\sigma, \tau \in K$. When σ and τ are vertices this follows from the injectivity of Ψ and f. When σ and τ are edges this follows from the same identity for Ψ and the fact that Step 4 guaranteed that $W_i \cap W_j = f(\sigma_i \cap \sigma_j)$ for $i, j \in [r], i \neq j$. The remaining case is when $\sigma = \sigma_i$ is an edge and τ a vertex. Then, by construction, $\tau \in \sigma_i$ if and only if $f(\tau) \in W_i$, and

$$\begin{array}{rcl} \Phi(\sigma_i) \cap \Phi(\tau) & = & \Psi(W_i) \cap \Psi(f(\tau)) = \Psi(W_i \cap f(\tau)) \\ & = & \left\{ \begin{array}{cl} \Psi(\emptyset) & \text{if } f(\tau) \notin W_i \\ \Psi(f(\tau)) & \text{if } f(\tau) \in W_i \end{array} \right\} = \Phi(\sigma_i \cap \tau). \end{array}$$

It remains to check that for any simplex $\sigma \in K$, the support of $\gamma(\sigma)$ is contained in $U_{\overline{\Phi(\sigma)}}$. When $\sigma = \{v\}$ is a vertex then $\gamma(\sigma) = \gamma'(f(v))$. Since γ' is constrained by $(\mathcal{F}, \Psi)$, the support of $\gamma'(f(v))$ is contained in $U_{\overline{\Psi(f(v))}} = U_{\overline{\Phi(v)}}$, so the property holds. When $\sigma = \sigma_i$ is an edge, $\gamma(\sigma_i)$ is, by construction, a chain in $U_{\overline{\Psi(W_i)}} = U_{\overline{\Phi(\sigma_i)}}$ and the property also holds. □

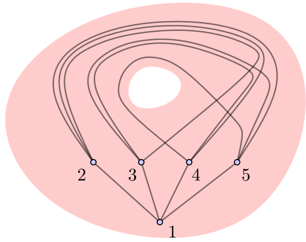


Figure 6.5: No trivial triangles

6.5.3 The induction

Let $k \geq 2$, let K be a simplicial complex of dimension k and assume that Proposition 6.10 holds for all simplicial complexes of dimension $k-1$ or less. Let $\mathcal{F}$ be a finite family of subsets of a topological space $\mathbf{R}$ such that for any $\mathcal{G} \subsetneq \mathcal{F}$ and any $0 \leq i \leq k-1$, $\bigcap \mathcal{G} \neq \emptyset$ and $\beta_i(\cap \mathcal{G}) \leq b$. Assuming that $\mathcal{F}$ contains sufficiently many sets, we want to construct a nontrivial chain map $\gamma : C_*(K) \to C_*(\mathbf{R})$ constrained by $\mathcal{F}$.

Preliminary example. When going from $k=0$ to $k=1$, the first step (as described in Section 6.5.2) is to start with a constrained chain map $\gamma' : C_*(K^{(0)}) \to C_*(\mathbf{R})$ and observe that for some 1-simplices $\{u, v\} \in K$ the chain $\gamma'(\partial\{u, v\})$ must already be a boundary. To see that this is not the case in general, consider the drawing of $\Delta_4^{(1)}$ in an annulus depicted in Fig. 6.5. Observe that for every triangle $\{i, j, k\} \in \Delta_4^{(2)}$ the image, in this drawing, of $\partial\{i, j, k\}$ is a cycle going around the hole of the annulus and is therefore not a boundary. So, if we start with a chain map γ' corresponding to that drawing, we will not be able to extend it by "filling" any triangle directly. This is not a peculiar example, and a similar construction can easily be done with arbitrarily many vertices. Observe, though, that the cycle going from 1 to 2, then 4, then 3 and then back to 1 *is* a boundary; in other words, if we replace, in the triangle $\partial\{1, 2, 3\}$, the edge from 2 to 3 by the concatenation of the edges from 2 to 4 and from 4 to 3, we build, using a chain map of $\Delta_4^{(1)}$ where no 2-face can be filled, a chain map of $\Delta_2^{(2)}$ where the 2-face can be filled. We systematize this observation using the barycentric subdivision of K.

Barycentric subdivision. The idea behind the notion of *barycentric subdivision* is that the geometric realization of a simplicial complex K' can be subdivided by inserting a vertex at the barycenter of every face, resulting in a new, finer, simplicial complex, denoted $\operatorname{sd} K'$, that is still homeomorphic to K'. Formally, the vertices of $\operatorname{sd} K'$ consist of the faces of K', except for the empty face, and the faces of $\operatorname{sd} K'$ are the collections $\{\sigma_1, \ldots, \sigma_\ell\}$ of faces of K' such that

$$\emptyset \neq \sigma_1 \subsetneq \sigma_2 \subsetneq \cdots \subsetneq \sigma_\ell.$$

In other words, the set of vertices of $\operatorname{sd} K'$ is $K' \setminus \{\emptyset\}$ and the faces of $\operatorname{sd} K'$ are the chains of $K' \setminus \{\emptyset\}$. For $\sigma \in K'$ we abuse the notation and let $\operatorname{sd} \sigma$ denote the subdivision of σ regarded as a subcomplex of $\operatorname{sd} K'$, that is,

$$\operatorname{sd} \sigma = \{\{\sigma_1, \ldots, \sigma_\ell\} \subseteq K' : \emptyset \neq \sigma_1 \subsetneq \sigma_2 \subsetneq \cdots \subsetneq \sigma_\ell \subseteq \sigma\}.$$

We will mostly manipulate barycentric subdivisions through the $\operatorname{sd} \sigma$. For further reading on barycentric subdivisions we refer the reader, for example, to [Mat03, Section 1.7].

Overview of the construction of γ. Let $s \in \mathbb{N}$ be some parameter depending on K and to be determined later. To construct γ we will define three auxiliary chain maps

$$C_*\left(K^{(k-1)}\right) \xrightarrow{\alpha} C_*\left((\operatorname{sd} K)^{(k-1)}\right) \xrightarrow{\beta_\sharp} C_*\left(\Delta_s^{(k-1)}\right) \xrightarrow{\gamma'} C_*(\mathbf{R})$$

As before, γ' is a chain map from $C_*(\Delta_s^{(k-1)})$ constrained by $\mathcal{F}$ and is obtained by applying the induction hypothesis. Unlike in Section 6.5.2, we do not inject the vertices of K into those of Δ_s directly but proceed through $\operatorname{sd} K$, the barycentric subdivision of K. We "inject" $K^{(k-1)}$ into $\operatorname{sd} K^{(k-1)}$ by means of a chain map α. We then construct an injection β of the vertices of $\operatorname{sd} K$ into the vertices of Δ_s which we extend linearly into a chain map $\beta_\sharp$. The key idea is the following:

> The boundary of any k-simplex σ of K is mapped, under α, to a sum of $k!$ boundaries of k-simplices of $\operatorname{sd} K$, all of which are mapped through $\beta_\sharp$ to chains with the same homology in some appropriate $U_{\overline{W_\sigma}}$.

Since $k!$ is even and we consider homology with coefficients in $\mathbb{Z}_2$, it follows that $\gamma' \circ \beta_\sharp \circ \alpha(\sigma)$ is a boundary in $U_{\overline{W_\sigma}}$. We therefore construct γ as an extension of $\gamma' \circ \beta_\sharp \circ \alpha$.

Definition of γ'. Since $\Delta_s^{(k-1)}$ has dimension $k-1$, the induction hypothesis ensures that if the cardinality of $\mathcal{F}$ is large enough then there exists a nontrivial chain map $\gamma' : C_*(\Delta_s^{(k-1)}) \to C_*(\mathbf{R})$ constrained by $\mathcal{F}$. We denote by Ψ a map such that γ' is constrained by $(\mathcal{F}, \Psi)$. Remark that Ψ must be monotone over $\Delta_s^{(k-1)}$ as for any $\sigma \subseteq \tau \in \Delta_s^{(k-1)}$ we have $\Psi(\sigma) = \Psi(\sigma \cap \tau) = \Psi(\sigma) \cap \Psi(\tau) \subseteq \Psi(\tau)$. It follows that for any $\sigma \in \Delta_s^{(k-1)}$ we have

$$\Psi(\sigma) = \bigcup_{\tau \in \Delta_s^{(k-1)}, \tau \subseteq \sigma} \Psi(\tau)$$

We use this identity to extend Ψ to Δ_s, that is we define:

$$\forall A \subseteq V(\Delta_s), \quad \Psi(A) = \bigcup_{\tau \in \Delta_s^{(k-1)}, \tau \subseteq A} \Psi(\tau).$$

Remark that the extended map still commutes with the intersection:

Lemma 6.12. *For any $A, B \subseteq V(\Delta_s)$ we have $\Psi(A) \cap \Psi(B) = \Psi(A \cap B)$.*

Proof. For any $A, B \subseteq V(\Delta_s)$ we have

$$\Psi(A) \cap \Psi(B) = \left(\bigcup_{\sigma \in \Delta_s^{(k-1)}, \sigma \subseteq A} \Psi(\sigma) \right) \cap \left(\bigcup_{\tau \in \Delta_s^{(k-1)}, \tau \subseteq B} \Psi(\tau) \right)$$

Distributing the union over the intersections we get

$$\Psi(A) \cap \Psi(B) = \bigcup_{\sigma, \tau \in \Delta_s^{(k-1)}, \sigma \subseteq A, \tau \subseteq B} \Psi(\sigma) \cap \Psi(\tau)$$

and as $\Psi(\sigma \cap \tau) = \Psi(\sigma) \cap \Psi(\tau)$ if σ, τ are simplices of $\Delta_s^{(k-1)}$, this rewrites as

$$\Psi(A) \cap \Psi(B) = \bigcup_{\sigma, \tau \in \Delta_s^{(k-1)}, \sigma \subseteq A, \tau \subseteq B} \Psi(\sigma \cap \tau).$$

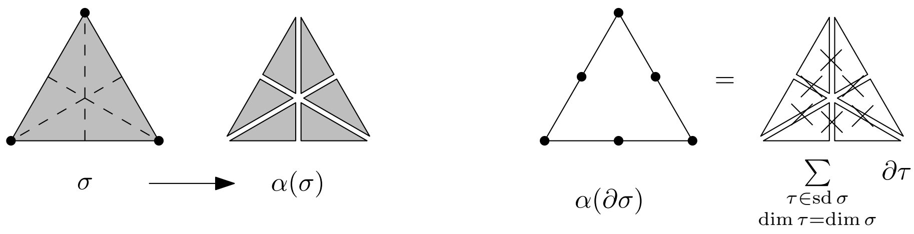
Finally, observing that

$$\{\sigma \cap \tau : \sigma, \tau \in \Delta_s^{(k-1)}, \sigma \subseteq A, \tau \subseteq B\} = \{\vartheta : \vartheta \in \Delta_s^{(k-1)}, \vartheta \subseteq A \cap B\}$$

we get

$$\Psi(A) \cap \Psi(B) = \bigcup_{\vartheta \in \Delta_s^{(k-1)}, \vartheta \subseteq A \cap B} \Psi(\vartheta) = \Psi(A \cap B)$$

which proves the desired identity. □



The map α applied to a simplex σ (left) and to $\partial\sigma$ (right). Significant parts of the boundaries $\partial\tau$ cancel out.

Figure 6.6: Map α

Definition of α. Now we define a chain map $\alpha : C_*\left(K^{(k-1)}\right) \to C_*\left(\operatorname{sd} K^{(k-1)}\right)$ by first putting

$$\alpha : \sigma \in K^{(k-1)} \mapsto \sum_{\substack{\tau \in \operatorname{sd}\sigma \\ \dim\tau = \dim\sigma}} \tau,$$

and then extending that map linearly to $C_*\left(K^{(k-1)}\right)$. See Figure 6.6. Remark that α behaves nicely with respect to the differential:

$$\alpha(\partial\sigma) = \sum_{\substack{\tau \in \operatorname{sd}\sigma \\ \dim\tau = \dim\sigma}} \partial\tau.$$

Note that the formula above makes sense and is valid even if σ is a k-simplex although we define α only up to dimension $k-1$.

Definition of β. We now construct the injection $\beta : V(\operatorname{sd} K) \to V(\Delta_s)$ and, for constraining purposes, an auxiliary function κ associating with every k-dimensional simplex of K some simplex of Δ_s. We want these functions to satisfy:

(P1) For any simplex $\sigma \in K$, $\kappa(\sigma) \cap \operatorname{Im}\beta = \beta(V(\operatorname{sd}\sigma))$.

(P2) For any k-simplices $\sigma, \tau \in K$, $\kappa(\sigma) \cap \kappa(\tau) = \beta(V(\operatorname{sd}\sigma)) \cap \beta(V(\operatorname{sd}\tau))$.

(P3) For any k-simplex $\sigma \in K$, when τ ranges over all k-simplices of $\operatorname{sd}\sigma$, all chains $\gamma' \circ \beta_\sharp(\partial\tau)$ have support in $U_{\overline{\Psi(\kappa(\sigma))}}$ and are in the same homology class in $H_{k-1}(U_{\overline{\Psi(\kappa(\sigma))}})$.

The intuition behind these properties is that $\kappa(\sigma)$ should augment $\beta(V(\operatorname{sd}\sigma))$ by "dummy" vertices (P1) in a way that distinct simplices use disjoint sets of "dummy" vertices (P2). Property (P3), will allow building γ over k-simplices as explained in the preceding overview.

We start the construction of β and κ with a combinatorial lemma. Let $\ell = 2^{k+1} - 1$ stand for the number of vertices of the barycentric subdivision of a k-dimensional simplex, and set $m = R_{k+1}(2^b, \ell)$.

Claim 6.1. *For any integer t, if $s \geq R_m\left(\binom{m}{\ell}, t\right)$ then there exist a set T of t vertices of Δ_s and a set $Q^* \in \binom{[m]}{\ell}$ such that Q^* selects in any $M \in \binom{T}{m}$ a subset L_M with the following property: when σ ranges over all k-simplices of Δ_s with $\sigma \subseteq L_M$, all chains $\gamma'(\partial\sigma)$ are in the same homology class in $H_{k-1}\left(U_{\overline{\Psi(M)}}\right)$.*

Proof. Let M be a subset of m vertices of Δ_s. Since γ' is constrained by $(\mathcal{F}, \Psi)$, for every k-simplex $\sigma \subseteq M$ the support of $\gamma'(\partial\sigma)$ is contained in $U_{\overline{\Psi(\partial\sigma)}} \subseteq U_{\overline{\Psi(\sigma)}} \subseteq U_{\overline{\Psi(M)}}$. We can therefore color the $(k+1)$-uniform hypergraph on M by assigning to every hyperedge σ the homology class of $\gamma'(\partial\sigma)$ in $U_{\overline{\Psi(M)}}$. Since $\beta_{k-1}\left(U_{\overline{\Psi(M)}}\right) \leq b$, there are at most 2^b colors in this coloring. As $m = R_{k+1}(2^b, \ell)$, Ramsey's Theorem implies that there exists a subset $L \subset M$ of ℓ vertices inducing a monochromatic hypergraph. We let Q_M denote an element of $\binom{[m]}{\ell}$ that selects such a subset L.

It remains to find a subset T of vertices of Δ_s so that all m-element subsets $M \subseteq T$ give rise to the same Q_M. This is done by another application of Ramsey's theorem to the m-uniform hypergraph on the vertices of Δ_s where each hyperedge M is colored by the ℓ-element subset Q_M. The subset T can have size t as soon as $s \geq R_m\left(\binom{m}{\ell}, t\right)$, which proves the statement. □

Now, back to the construction of β and κ. We first want a subset of $V(\Delta_s)$ with a "uniform ℓ-in-m selection" property of Claim 6.1 large enough so that we can inject $V(\operatorname{sd} K)$ using Lemma 6.9. We set:

$$t = v(\operatorname{sd} K) + r(m-\ell) \quad \text{and} \quad s^* = R_m\left(\binom{m}{\ell}, t\right),$$

and assume that $s \geq s^*$; since s^* only depends on b and K, this merely requires that $\mathcal{F}$ is large enough, again as a function of b and K, so that γ' still exists. We let T and Q^* denote the subset of $V(\Delta_s)$ and the element of $\binom{[m]}{\ell}$ whose existence follows from applying Claim 6.1. Let $\sigma_1, \sigma_2, \ldots, \sigma_r$ denote the k-dimensional simplices of K. We apply Lemma 6.9 with

$$Y = V(\operatorname{sd} K), \quad Z = T, \quad A_i = V(\operatorname{sd} \sigma_i), \quad q = \ell, \quad \text{and } w = m,$$

and obtain an injection $\pi : Y \to Z$ and $W_1, W_2, \ldots, W_r \in \binom{Z}{m}$ such that (i) for every $i \leq r$, Q^* selects $\pi(A_i)$ in W_i, and (ii) for any $i \neq j \leq r$, $W_i \cap W_j = \pi(A_i \cap A_j)$. This injection π is our map β and we put $\kappa(\sigma_i) = W_i$. It is clear that Property (P1) holds, and since

$$\kappa(\sigma_i) \cap \kappa(\sigma_j) = W_i \cap W_j = \pi(A_i \cap A_j) = \beta(V(\operatorname{sd} \sigma_i) \cap V(\operatorname{sd} \sigma_j)) = \beta(V(\operatorname{sd} \sigma_i)) \cap \beta(V(\operatorname{sd} \sigma_j)),$$

Property (P2) also holds. The set Q^* selects $\pi(A_i)$ in W_i (Lemma 6.9) so Claim 6.1 ensures that when τ ranges over all k-simplices of Δ_s with $\tau \subseteq \pi(A_i)$, all chains $\gamma'(\partial\tau)$ have support in $U_{\overline{\Psi(W_i)}}$ and are in the same homology class in $H_{k-1}\left(U_{\overline{\Psi(W_i)}}\right)$. Substituting $\pi(A_i) = \beta(V(\operatorname{sd} \sigma_i))$ and $W_i = \kappa(\sigma_i)$, we see that (P3) holds.

Construction of γ. Recall that we have the chain maps[5]:

$$C_*\left(K^{(k-1)}\right) \xrightarrow{\ \alpha\ } C_*\left((\operatorname{sd} K)^{(k-1)}\right) \xrightarrow{\ \beta_\sharp\ } C_*\left(\Delta_s^{(k-1)}\right) \xrightarrow{\ \gamma'\ } C_*(\mathbf{R}).$$

We define $\gamma = \gamma' \circ \beta_\sharp \circ \alpha$ as a chain map from $C_*\left(K^{(k-1)}\right)$ to $C_*(\mathbf{R})$. Let σ be a k-dimensional simplex of K. From the definition of α we have

$$\gamma(\partial\sigma) = \sum_{\substack{\tau \in \operatorname{sd} \sigma \\ \dim \tau = \dim \sigma}} \gamma' \circ \beta_\sharp(\partial\tau).$$

By property (P3), all summands in the above chain have support in $U_{\overline{\Psi(\kappa(\sigma))}}$ and belong to the same homology class in $H_{k-1}\left(U_{\overline{\Psi(\kappa(\sigma))}}\right)$. There is an even number of summands, namely $k!$ and we are using homology over $\mathbb{Z}_2$, so $\gamma' \circ \beta_\sharp \circ \alpha(\partial\sigma)$ has support in $U_{\overline{\Psi(\kappa(\sigma))}}$ and is a boundary in $U_{\overline{\Psi(\kappa(\sigma))}}$. We can therefore extend γ into a chain map from $C_*(K)$ to $C_*(\mathbf{R})$ in a way that for any k-simplex σ of K, the support of $\gamma(\sigma)$ is contained in $U_{\overline{\Psi(\kappa(\sigma))}}$.

Properties of γ. First we verify that γ is nontrivial. If v is a vertex of K then $\operatorname{sd} v$ consists of a single simplex, also a vertex. The chain $\alpha(v)$ is thus a single vertex of $\operatorname{sd} K$, and $\beta_\sharp \circ \alpha(v)$ is still a single vertex $\beta(\operatorname{sd} v)$. Since γ' is nontrivial, the support of $\gamma(v)$ is an odd number of points and therefore γ is also nontrivial. It remains to argue that γ is constrained by $(\mathcal{F}, \Phi)$ where:

$$\Phi : \left\{ \begin{array}{ccl} K & \to & 2^{\mathcal{F}} \\ \sigma & \mapsto & \left\{ \begin{array}{ll} \Psi(\beta(V(\operatorname{sd} \sigma))) & \text{if } \dim \sigma \leq k-1 \\ \Psi(\kappa(\sigma)) & \text{if } \dim \sigma = k \end{array} \right. \end{array} \right.$$

It is clear that $\Phi(\emptyset) = \Psi(\emptyset) = \emptyset$ by definition of Ψ. Also, the construction of γ immediately ensures that for any $\sigma \in K$ the support of $\gamma(\sigma)$ is contained in $U_{\overline{\Phi(\sigma)}}$. To conclude the proof that γ is constrained by $(\mathcal{F}, \Phi)$ and therefore the induction it only remains to check that Φ commutes with the intersection:

Claim 6.2. *For any $\sigma, \tau \in K$, $\Phi(\sigma \cap \tau) = \Phi(\sigma) \cap \Phi(\tau)$.*

[5] $\beta_\sharp$ is the chain map induced by β restricted to chains of dimension at most $(k-1)$.

Proof. The claim is obvious for $\sigma = \tau$, so from now on assume that this is not the case. First assume that σ and τ have dimension at most $k-1$. Then,

$$\Phi(\sigma) \cap \Phi(\tau) = \Psi(\beta(V(\operatorname{sd}\sigma))) \cap \Psi(\beta(V(\operatorname{sd}\tau))) = \Psi(\beta(V(\operatorname{sd}\sigma)) \cap \beta(V(\operatorname{sd}\tau))),$$

the last equality following from Lemma 6.12. Since the map β on subsets of $V(\Delta_s)$ is induced by a map β on vertices of Δ_s we have $\beta(V(\operatorname{sd}\sigma)) \cap \beta(V(\operatorname{sd}\tau)) = \beta(V(\operatorname{sd}\sigma) \cap V(\operatorname{sd}\tau))$. Moreover, by the definition of the barycentric subdivision we have $V(\operatorname{sd}\sigma) \cap V(\operatorname{sd}\tau) = V(\operatorname{sd}(\sigma \cap \tau))$. Thus,

$$\Psi(\beta(V(\operatorname{sd}\sigma)) \cap \beta(V(\operatorname{sd}\tau))) = \Psi(\beta(V(\operatorname{sd}(\sigma \cap \tau)))) = \Phi(\sigma \cap \tau),$$

and the statement holds for simplices of dimension at most $k-1$.

Now assume that σ and τ are both k-dimensional so that

$$\Phi(\sigma) \cap \Phi(\tau) = \Psi(\kappa(\sigma)) \cap \Psi(\kappa(\tau)) = \Psi(\kappa(\sigma) \cap \kappa(\tau)) = \Psi(\beta(V(\operatorname{sd}\sigma)) \cap \beta(V(\operatorname{sd}\tau))),$$

the last identity following from Property (P2) of the map κ. Again, from the definition of β and the barycentric subdivision we have

$$\beta(V(\operatorname{sd}\sigma)) \cap \beta(V(\operatorname{sd}\tau)) = \beta(V(\operatorname{sd}(\sigma \cap \tau))).$$

We thus obtain

$$\Phi(\sigma) \cap \Phi(\tau) = \Psi \circ \beta \circ V(\operatorname{sd}(\sigma \cap \tau)) = \Phi(\sigma \cap \tau),$$

the last identity following from the definition of Φ on simplices of dimension at most $k-1$. The statement also holds for simplices of dimension k.

Finally assume that σ and τ are of dimension k and at most $k-1$ respectively. Then, applying Lemma 6.12 we have:

$$\Phi(\sigma) \cap \Phi(\tau) = \Psi(\kappa(\sigma)) \cap \Psi(\beta(V(\operatorname{sd}\tau))) = \Psi(\kappa(\sigma) \cap \beta(V(\operatorname{sd}\tau))).$$

Note that $\beta(V(\operatorname{sd}\tau)) \subseteq \operatorname{Im}\beta$ and that, by property (P1), $\kappa(\sigma) \cap \operatorname{Im}\beta = \beta(V(\operatorname{sd}\sigma))$. We thus have

$$\kappa(\sigma) \cap \beta(V(\operatorname{sd}\tau)) = \beta(V(\operatorname{sd}\sigma)) \cap \beta(V(\operatorname{sd}\tau)) = \beta(V(\operatorname{sd}(\sigma \cap \tau))),$$

the last equality following, again, from the definition of barycentric subdivision. As $\sigma \cap \tau$ has dimension at most $k-1$ we have

$$\Phi(\sigma) \cap \Phi(\tau) = \Psi(\beta(V(\operatorname{sd}(\sigma \cap \tau)))) = \Phi(\sigma \cap \tau)$$

and the statement holds for the last case. $\square$

List of Symbols

Common symbols

$\bigcap \mathcal{G}$	the intersection of $\mathcal{G}$, i.e., the set of all elements that belong to every single set from the family $\mathcal{G}$
$A \setminus B$	the set difference, i.e., the set of all elements contained in A but not in B
$A \cong B$	symbol used to denote isomorphism of objects A and B
$\bigoplus V_i$, $\bigoplus f_i$	the direct sum of vector spaces or maps between them
$\mathbf{a}, \mathbf{b}$	points in an affine space
$\mathbb{A}, \mathbb{B}$	affine spaces
aff	the affine closure
cl	the closure operator of a matroid, see Definition 2.5
conv	the convex hull of a set
$\lvert C\rvert$	cardinality of set C
d	an integer, usually used to denote dimension
D^i	the d-dimensional disk
$\mathbf{e}_i$	ith standard basis vector of $\mathbb{F}^m$
$\dim \mathbb{A}, \dim V$	dimension of an affine space $\mathbb{A}$ or vector space V, respectively
f	a continuous map
$f \circ g$	a composition of maps
$\mathbb{F}$	a field
$\mathbb{F}^X$	the Cartesian product of $\mathbb{F}$, i.e., all functions from X to $\mathbb{F}$ with addition and multiplication defined coordinate-wise
$\mathcal{F}, \mathcal{G}$	families of sets
i, j, k, l, m, n, r, s	integers
K, L	simplicial complexes
K_n	complete graph on n vertices
M	a topological manifold or a matroid
$\mathbb{N}$	the set of all non-negative integers
$O(f(n))$	the O-notation, a function $g\colon \mathbb{N} \to \mathbb{N}$ satisfies $g = O(f(n))$ if there exists $C_1, C_2 \in \mathbb{N}$ such that $g(n) \leq C_1 f(n)$ for all $n \geq C_2$
p	a prime number
$\mathbb{Q}$	the field of rationals
R	an arbitrary commutative ring
$\mathbb{R}$	the field of reals
S^i	the i-dimensional sphere
U, V	vector spaces
wcl	weak closure operator (satisfying only (CL2) and (CL3) of Definition 2.5)
X, Y	topological spaces or arbitrary sets
$\mathbb{Z}_p$	the unique p-element field
$\mathbb{Z}$	the ring of integers

Chapter 2

a_x	an element of the field $\mathbb{F}$
c	the coloring from Definition 2.11
C	a set split into color classes $C_0, \dots, C_m$
$C_0, \dots, C_m$	color classes
C_I	a shorthand for $\bigcup_{i \in I} C_i$
$F_1, \dots, F_r$	the pairwise disjoint rainbow sets with intersecting affine hulls these are the sets we want to find
G_j	a rainbow set with $\lvert G_j \rvert = j+1$ and $\dim \operatorname{aff} \psi(G_j) = j$
K_k	the set of "allowed" colors
μ	a multipoint (see Definition 2.2)
$\mathcal{M}(X; \mathbb{F})$	the set of all multipoints in X (see Definition 2.2)
ψ	a map from a set C to affine space $\mathbb{A}$
$\operatorname{supp} \mu$	support of a multipoint (see Definition 2.2)
$r(M)$	the rank of matroid M
R_k	subset of G_j
R_k^p	a subset of C containing p and satisfying certain additional conditions
$\mathbf{y}, \mathbf{x}_i$	points

Chapter 3

$X \hookrightarrow Y$	a symbol for the inclusion of X into Y
$\gamma \equiv \tau \pmod{B}$	a shorthand for $\{\gamma + b \mid b \in B\} = \{\tau + b \mid b \in B\}$
$2^{V(K)}$	the abstract simplicial complex of all subsets of $V(K)$; note that $K \subseteq 2^{V(K)}$
$a \wedge \sigma$	for an ordered l-face $\sigma = (v_0, \dots, v_l)$, the ordered $(l+1)$-face $(a, v_0, \dots, v_l)$ (see also Definition 3.8)
b	an upper bound for the kth reduced $\mathbb{F}$-Betti number of the manifold M
B_l	the lth group of boundaries of the chain complex C_* (see Definition 3.6)
$B_l^o(K; \mathbb{F})$	the lth ordered simplicial boundary group (see Definition 3.11)
$B_l^O(X; \mathbb{F})$	the lth ordered singular boundary group (see Definition 3.17)
$B_l(K; \mathbb{F})$	the lth unordered simplicial boundary group (see Definition 3.11)
$B_l(X; \mathbb{F})$	the lth unordered singular boundary group (see Definition 3.17)
c	the Kneser's coloring (see Lemma 3.31)
C_*, D_*	a chain complexes (see Definition 3.5)
C_l	the set of all elements of C_* of degree l (see Definition 3.5)
$\widetilde{C}_l(K; \mathbb{F})$	$\widetilde{O}_l(K; \mathbb{F}) / \widetilde{T}_l(K; \mathbb{F})$ (see Definition 3.9)
$\widetilde{C}_l(X; \mathbb{F})$	$\widetilde{O}_l(X; \mathbb{F}) / \widetilde{T}_l(X; \mathbb{F})$ (see Definition 3.15)
$\widetilde{C}_*(K; \mathbb{F})$	$\widetilde{O}_*(K; \mathbb{F}) / \widetilde{T}_*(K; \mathbb{F})$ (see Definition 3.11)
$\widetilde{C}_*(X; \mathbb{F})$	$\widetilde{O}_*(X; \mathbb{F}) / \widetilde{T}_*(X; \mathbb{F})$ (see Definition 3.17)
∂^C, ∂^D	boundary operators for chain complexes C or D, respectively
∂_l^o	the lth boundary operator for $\widetilde{O}_l(K; \mathbb{F})$ (see Definition 3.10)
∂_l^O	the lth boundary operator for $\widetilde{O}_l(X; \mathbb{F})$ (see Definition 3.16)
∂^o	the boundary operator for $\widetilde{O}_*(K; \mathbb{F})$ (see Definition 3.11)
∂^O	the boundary operator for $\widetilde{O}_*(X; \mathbb{F})$ (see Definition 3.17)

∂_l	the lth boundary operator for $\widetilde{C}_l(K;\mathbb{F})$ or $\widetilde{C}_l(X;\mathbb{F})$ (see Definitions 3.10 and 3.16)
∂	the boundary operator for $\widetilde{C}_*(K;\mathbb{F})$ or $\widetilde{C}_*(X;\mathbb{F})$ (see Definitions 3.10 and 3.16)
$\lvert\Delta_l\rvert$	the standard l-dimensional simplex in $\mathbb{R}^{l+1}$
Δ_d	the abstract d-dimensional simplex, i.e., set of all subsets of $\{0,1,\ldots,d\}$
δ_l^i	the standard ith face map, mapping $\lvert\Delta_{l-1}\rvert$ onto the facet of $\lvert\Delta_l\rvert$ that does not contain the ith vertex (see Definition 3.12)
D_l	the set of all elements of D_* of degree l (see Definition 3.5)
$\operatorname{card} S$	cardinality of a set S
$f_\sharp$	the chain map between $\widetilde{C}_*(X;\mathbb{F})$ and $\widetilde{C}_*(Y;\mathbb{F})$ induced by the continuous map $f\colon X\to Y$ (see Definition 3.18)
$f_\sharp^0$	the chain map between $\widetilde{O}_*(X;\mathbb{F})$ and $\widetilde{O}_*(Y;\mathbb{F})$ induced by the continuous map $f\colon X\to Y$ (see Definition 3.18)
f_*	the chain map between $\widetilde{H}_*(X;\mathbb{F})$ and $\widetilde{H}_*(Y;\mathbb{F})$ induced by the continuous map $f\colon X\to Y$ (see Definition 3.18)
f_*^0	the chain map between $\widetilde{H}_*^O(X;\mathbb{F})$ and $\widetilde{H}_*^O(Y;\mathbb{F})$ induced by the continuous map $f\colon X\to Y$ (see Definition 3.18)
γ,γ'	(ordered) singular simplices (see Definition 3.14)
H_l	the lth homology group of the chain complex C_* (see Definition 3.6)
H_*	the chain complex of homology groups (see Definition 3.6)
$\widetilde{H}_l^o(K;\mathbb{F})$	the lth reduced ordered simplicial homology group (see Definition 3.11)
$\widetilde{H}_l^O(X;\mathbb{F})$	the lth reduced ordered singular homology group (see Definition 3.17)
$\widetilde{H}_l(K;\mathbb{F})$	the lth reduced unordered simplicial homology group (see Definition 3.11)
$\widetilde{H}_l(X;\mathbb{F})$	the lth reduced unordered singular homology group (see Definition 3.17)
ι_K^o	the natural inclusion of $\widetilde{O}_*(K;\mathbb{F})$ into $\widetilde{O}_*(\lvert K\rvert;\mathbb{F})$ (see Definition 3.19)
ι_K	the natural inclusion of $\widetilde{C}_*(K;\mathbb{F})$ into $\widetilde{C}_*(\lvert K\rvert;\mathbb{F})$ (see Definition 3.19)
ι_s	the natural inclusion of $\widetilde{C}_*(\Delta_s^{(k)};\mathbb{F})$ into $\widetilde{C}_*(\left\lvert\Delta_s^{(k)}\right\rvert;\mathbb{F})$ (see Definition 3.19)
ι_n	the natural inclusion of $\widetilde{C}_*(\Delta_n^{(k)};\mathbb{F})$ into $\widetilde{C}_*(\left\lvert\Delta_n^{(k)}\right\rvert;\mathbb{F})$ (see Definition 3.19)
$\lvert K\rvert$	geometric realization of an abstract simplicial complex (see Definition 3.2)
m	the number of all k-faces of $\Delta_s^{(k)}$
$\min\sigma$	if σ is a face of $\Delta_s^{(k)}$ with vertex set $V\left(\Delta_s^{(k)}\right)=\{v_0,v_1,\ldots,v_s\}$, the minimal index of a vertex in σ

φ	the chain map $\theta \circ (\iota_n)$
$\varphi\colon C_* \to D_*$	a chain map (see Definition 3.5)
φ_*	for a chain map φ, the induced homomorphism in homology (see Definition 3.7)
$\varphi_*[z]$	a shorthand for $\varphi_*([z])$
p_l^π	the linear automorphism of $\lvert\Delta_l\rvert$ induced by a permutation π of vertices (see Definition 3.12)
$\pi(\sigma)$	the ordered face σ permuted according to permutation π (see Definition 3.8)
$K^{(k)}$	the k-skeleton of a simplicial complex K, i.e., all faces up to dimension k
$\widetilde{O}_l(K;\mathbb{F})$	the lth augmented chain group of ordered simplices (see Definition 3.9)
$\widetilde{O}_l(X;\mathbb{F})$	the lth augmented chain group of ordered singular simplices (see Definition 3.15)
$\widetilde{O}_*(K;\mathbb{F})$	the augmented simplicial chain complex of a simplicial simplices in K (see Definition 3.11)
$\widetilde{O}_*(X;\mathbb{F})$	the augmented singular chain complex of singular simplices in X (see Definition 3.17)
P	a path-connected component of a topological space
$S(\{0,1,\ldots,l\})$	the group of permutations on $\{0,1,\ldots,s\}$, i.e., set of all bijections $\pi\colon\{0,1,\ldots,s\}\to\{0,1,\ldots,s\}$, endowed with the composition of maps
$\operatorname{sd}(K,\mathbf{a})$	the stellar subdivison of K with respect to $\mathbf{a}$ (see Definition 3.4)
$\sigma_1,\ldots,\sigma_m$	all the k-faces of $\Delta_s^{(k)}$, linearly ordered
σ,τ	(ordered) faces of an abstract simplicial complex
$\sigma_l^{i,j}$	the degeneracy map for $\lvert\Delta_l\rvert$ that maps ith vertex onto jth and leaves others unchanged (see Definition 3.12)
$\lvert\sigma\rvert$	geometric realization of a face σ (see Definition 3.2)
$\operatorname{supp}\gamma$	support of the (ordered or unordered) singular chain γ (see Definition 3.22)
$\widetilde{T}_l(K;\mathbb{F})$	the lth augmented chain group of degenerated chains of simplices (see Definition 3.9)
$\widetilde{T}_l(X;\mathbb{F})$	the lth augmented chain group of degenerated chains of singular simplices (see Definition 3.15)
θ	a chain map from $\widetilde{C}_*\left(\left\lvert\Delta_n^{(k)}\right\rvert;\mathbb{Z}_p\right)$ to $\widetilde{C}_*\left(M;\mathbb{Z}_p\right)$
$v_0,\ldots,v_l$	vertices of an abstract simplicial complex
$(v_0,\ldots,v_l)$	an ordered l-simplex in K (see Definition 3.8)
$(v_0,\ldots,\widehat{v_i},\ldots,v_l)$	a shorthand for $(v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_l)$
$\mathbf{v}(\mu)$	$\mathbf{v}(\mu) := \Big(\varphi_*\big[\partial(\mu\wedge\sigma_1)\big],\varphi_*\big[\partial(\mu\wedge\sigma_2)\big],\ldots,\varphi_*\big[\partial(\mu\wedge\sigma_m)\big]\Big)$ (See Equation (3.6))
$V(K)$	vertex set of an abstract simplicial complex K
Z_l	the lth group of cycles of the chain complex C_* (see Definition 3.6)
$Z_l^o(K;\mathbb{F})$	the lth ordered simplicial cycle group (see Definition 3.11)
$Z_l^O(X;\mathbb{F})$	the lth ordered singular cycle group

	(see Definition 3.17)
$Z_l(K;\mathbb{F})$	the lth unordered simplicial cycle group (see Definition 3.11)
$Z_l(X;\mathbb{F})$	the lth unordered singular cycle group (see Definition 3.17)
$[z]$	the homology class of a cycle Z in the corresponding homology group

Chapter 4

M, M', M''	manifolds, M'' is assumed to be compact
N, N_0	non-negative integers
q	an integer

Chapter 5

$\otimes$	the tensor product
2^X	the set of all subsets of X
CK	the cone over K
$C_*(K)$	the simplicial chain complex of K with $\mathbb{Z}_2$ coefficients
$C_*(X)$	the singular chain complex of X with $\mathbb{Z}_2$ coefficients
	note that its augmented analogues is denoted $\widetilde{O}_*(X;\mathbb{Z}_2)$ in Chapter 3
$H^d(K)$	the dth singular) cohomology group of K with coefficients in $\mathbb{Z}_2$
$\widetilde{K}$	the combinatorial deleted product of K
$\mathfrak{o}^d(K)$	the $\mathbb{Z}_2$-Van Kampen obstruction to embeddability of K into $\mathbb{R}^d$
$\mathbf{R}$	an arbitrary topological space, e.g. $\mathbb{R}^d$
$\mathbb{RP}^d$	the d-dimensional projective space over the reals
$\mathbb{RP}^\infty$	the infinitely dimensional projective space over the reals
$\mathbb{S}^d$	the d-dimensional sphere
$\mathbb{S}^\infty$	the infinitely dimensional sphere
$\binom{X}{k}$	the set of all k-element subsets of X

Chapter 6

$\mathcal{F}$	a fixed family of sets
γ	a chain map
H	a hypergraph
$[m]$	the set $\{1, 2, \ldots, m\}$
n	the number of sets in the family $\mathcal{F}$, i.e., $\mathcal{F} = \{U_1, \ldots, U_n\}$
Φ, Ψ	constraning maps (see the discussion before Lemma 6.6)
$U_{\overline{I}}$	a shorthand for $\bigcap_{i\in[n]\setminus I} U_i$
$U_{\overline{n}}$	defined as $\mathbb{R}^d$
S	an edge of an hypergraph
$\operatorname{sd} K$	the barycentric subdivision of K
$v(K)$	the number of vertices of K

List of Figures

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