

Stabbing simplices by points and flats

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Abstract

The following result was proved by Bárány in 1982: For every $d \geq 1$ there exists $c_d > 0$ such that for every n -point set S in \mathbb{R}^d there is a point $p \in \mathbb{R}^d$ contained in at least $c_d n^{d+1} - O(n^d)$ of the d -dimensional simplices spanned by S .

We investigate the largest possible value of c_d . It was known that $c_d \leq 1/(2^d(d+1)!)$ (this estimate actually holds for *every* point set S). We construct sets showing that $c_d \leq (d+1)^{-(d+1)}$, and we conjecture this estimate to be tight. The best known lower bound, due to Wagner, is $c_d \geq \gamma_d := (d^2 + 1)/((d+1)!(d+1)^{d+1})$; in his method, p can be chosen as any centerpoint of S . We construct n -point sets with a centerpoint that is contained in no more than $\gamma_d n^{d+1} + O(n^d)$ simplices spanned by S , thus showing that the approach using an arbitrary centerpoint cannot be further improved.

We also prove that for every n -point set $S \subset \mathbb{R}^d$ there exists a $(d-2)$ -flat that stabs at least $c_{d,d-2} n^3 - O(n^2)$ of the triangles spanned by S , with $c_{d,d-2} \geq \frac{1}{24}(1 - 1/(2d-1)^2)$. To this end, we establish an equipartition result of independent interest (generalizing planar results of Buck and Buck and of Ceder): Every mass distribution in \mathbb{R}^d can be divided into $4d-2$ equal parts by $2d-1$ hyperplanes intersecting in a common $(d-2)$ -flat.

1 Introduction

Let S be an n -point set in \mathbb{R}^d in general position (no $d+1$ points lying on a common hyperplane). The points of S span $\binom{n}{d+1}$ distinct d -dimensional simplices. The following interesting and useful result in discrete geometry (called the *First Selection Lemma* in [Mat02]), shows that at least a fixed fraction of these simplices have a point in common:

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Theorem 1.1 (Bárány [Bár82]). *For every n -point set in \mathbb{R}^d in general position there exists a point $p \in \mathbb{R}^d$ that is contained in at least $c_d n^{d+1} - O(n^d)$ simplices spanned by S , where c_d is a positive constant depending only on d (and the implicit constant in the $O(\cdot)$ notation may also depend on d).*

In this paper we investigate the value of c_d . More precisely, from now on, let c_d denote the supremum of the numbers such that the statement of Theorem 1.1 holds for all finite sets S in \mathbb{R}^d .

Lower bounds. Bárány’s proof yields

$$c_d \geq \frac{1}{d!(d+1)^{d+1}}.$$

Wagner [Wag03] improved this bound by roughly a factor of d , to

$$c_d \geq \frac{d^2 + 1}{(d+1)!(d+1)^{d+1}}. \tag{1}$$

For the special case $d = 2$, Boros and Füredi [BF84] achieved the better lower bound of $c_2 = 1/27$ (also see Bukh [Buk06] for a simpler proof of this planar bound).

Upper bounds. The following result was proved by Kárteszi [Kár55] for $d = 2$ (also see Moon [Moo68, p. 7] and Boros and Füredi [BF77, BF84]) and by Bárány [Bár82] for general d :

Theorem 1.2. *If S is any n -point set in general position in \mathbb{R}^d , then no point $p \in \mathbb{R}^d$ is contained in more than*

$$\frac{1}{2^d(d+1)!} n^{d+1} + O(n^d)$$

d -simplices spanned by S .

It follows without difficulty from a result of Wendel (reproduced as Lemma 3.1 below) that this bound is asymptotically attained with high probability by points chosen uniformly at random from the unit sphere. Alternatively, as was kindly pointed out to us by Uli Wagner, the tightness also follows by considering the Gale transform of the polar of a cyclic polytope; see, e.g., Welzl [?] for the relevant background.

Bárány’s bound implies that

$$c_d \leq \frac{1}{2^d(d+1)!},$$

which, to our knowledge, was the best known upper bound on c_d for all $d \geq 3$.

For $d = 2$, Boros and Füredi [BF84] claimed the upper bound $c_2 \leq 1/27$ (which would be tight), but it turns out that the construction in their paper gives only $c_2 \leq 1/27 + 1/729$ (see Appendix A of this paper).

Our first result is an improved upper bound for c_d for every d (and the first “non-trivial” one, in the sense that it refers to a specific construction):

Theorem 1.3. *For every fixed $d \geq 2$ and every n there exists an n -point set $S \subset \mathbb{R}^d$ such that no point $p \in \mathbb{R}^d$ is contained in more than $(n/(d+1))^{d+1} + O(n^d)$ d -simplices spanned by S . Thus,*

$$c_d \leq (d+1)^{-(d+1)}. \quad (2)$$

Moreover, such an S can be chosen in convex position.

In particular, the planar bound of $c_2 = 1/27$ is tight, after all.

1.1 The First Selection Lemma and centerpoints

If S is an n -point set in \mathbb{R}^d and $p \in \mathbb{R}^d$, we say that p lies at *depth m with respect to S* if every halfspace that contains p contains at least m points of S . A classical result of Rado [Rad47] states that there always exists a point at depth $n/(d+1)$. Such a point is called a *centerpoint*.

Wagner proved the bound (1) by showing the following:

Theorem 1.4 ([Wag03]). *If S is an n -point set in \mathbb{R}^d and $p \in \mathbb{R}^d$ is a point at depth αn with respect to S , then p is contained in at least*

$$((d+1)\alpha^d - 2d\alpha^{d+1}) \frac{n^{d+1}}{(d+1)!} - O(n^d)$$

d -simplices spanned by S .¹

This, together with Rado's Centerpoint Theorem, immediately implies (1).

In this paper we show that Theorem 1.4 cannot be improved:

Theorem 1.5. *For every α , $0 < \alpha \leq 1/2$, and every n , there exists an n -point set S in \mathbb{R}^d such that the origin is at depth αn with respect to S but is contained in only*

$$((d+1)\alpha^d - 2d\alpha^{d+1}) \frac{n^{d+1}}{(d+1)!} + O(n^d)$$

d -simplices spanned by S .

Thus, the approach of taking an arbitrary centerpoint cannot yield any lower bound better than (1) for the First Selection Lemma.

1.2 Stabbing $(d-k)$ -simplices by k -flats

The First Selection Lemma can be generalized as follows: If S is an n -point set in \mathbb{R}^d and k is an integer, $0 \leq k < d$, then there exists a k -flat that intersects at least $c_{d,k} n^{d-k+1} - O(n^{d-k})$ of the $(d-k)$ -simplices spanned by S , for some positive constants $c_{d,k}$ that depend only on d and k . (By a *k -flat* we mean a k -dimensional affine subspace of \mathbb{R}^d .)

¹This is obtained by setting $k = 0$ in the lower bound for $f_k(\mu, \mathbf{o})$ in the proof of Theorem 4.32 in [Wag03].

The problem is to determine the maximum values of the constants $c_{d,k}$. Trivially we have $c_{d,k} \geq c_{d-k}$: Simply project S into an arbitrary generic subspace of dimension $d-k$, and then apply the First Selection Lemma.

Here we derive a nontrivial lower bound for the case $k = d - 2$ (this is the only case for which we could obtain good lower bounds):

Theorem 1.6. *If S is a n -point set in \mathbb{R}^d , then there is a $(d - 2)$ -flat ℓ that intersects at least $\frac{1}{24}(1 - 1/(2d - 1)^2)n^3 - O(n^2)$ triangles spanned by S . Thus,*

$$c_{d,d-2} \geq \frac{1}{24} \left(1 - \frac{1}{(2d - 1)^2} \right). \quad (3)$$

For $d = 2$ this is just the planar version of First Selection Lemma with the optimal constant of $1/27$. And as d increases, the right-hand-side of (3) increases strictly with d , approaching $1/24$ as d tends to infinity. Indeed, it is impossible to stab more than $n^3/24$ triangles for any d , since then projecting into a plane perpendicular to ℓ would result in a point stabbing more than $n^3/24$ triangles in the plane, contradicting Theorem 1.2.

Theorem 1.6 is a consequence of the following equipartition result, which is interesting in its own right. Given an integer $m \geq 2$, define an m -fan as a set of m hyperplanes in \mathbb{R}^d that pass through a common $(d - 2)$ -flat. Then:

Theorem 1.7. *For every probability measure μ on \mathbb{R}^d that is absolutely continuous with respect to the Lebesgue measure, there exists a $(2d - 1)$ -fan that divides μ into $4d - 2$ equal parts.*

For $d = 2$ this theorem specializes to a result of Ceder [Ced64] (also see Buck and Buck [BB49] for a special case).

We also show that $2d - 1$ is the largest possible number of hyperplanes in Theorem 1.7:

Theorem 1.8. *For every integer $m \geq 2d$ there exists an absolutely continuous probability measure μ on \mathbb{R}^d which cannot be partitioned into $2m$ equal parts by an m -fan.*

2 The construction for Theorem 1.3

We now prove Theorem 1.3 by constructing a suitable point set S . Given real numbers $a, b > 1$, let $a \ll b$ mean that $f(a) < b$ for some fixed, sufficiently large function f (concretely, we can take $f(x) = (d + 1)!x^{d+1}$). Our point set is $S = \{p_1, \dots, p_n\}$, with

$$p_i = (p_{i1}, p_{i2}, \dots, p_{id}) \in (1, \infty)^d,$$

where the components p_{ij} satisfy

$$p_{ij} \ll p_{i'j'} \quad \text{whenever} \quad j < j', \quad \text{or} \quad j = j' \text{ and } i < i'$$

(so the ordering of the p_{ij} is first by the coordinate index j and then by the point index i). The idea of taking points separated by rapidly-increasing distances is borrowed from Boros and Füredi's planar construction [BF84]. However, their construction is more complicated, with points grouped into three clusters; see Appendix A.

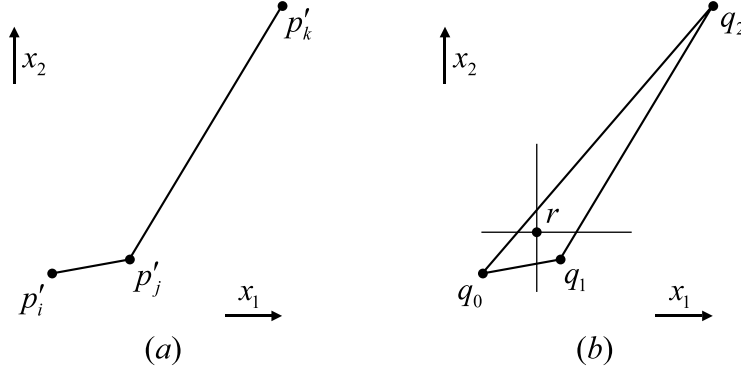


Figure 1: (a) In the x_1x_2 -plane, the segment $p'_i p'_j$ has a smaller slope than the segment $p'_j p'_k$. (b) In the planar case, the point r must lie above-right of q_0 , above-left of q_1 , and below-left of q_2 .

Lemma 2.1. *The set S is in convex position.*

Proof. Let $p'_i = (p_{i1}, p_{i2})$ be the projection of point p_i into the x_1x_2 -plane, for $1 \leq i \leq n$. We claim that the points p'_i lie on an x_1 -monotone convex curve in the x_1x_2 -plane (which implies the lemma).

To this end, we show that for every three points p'_i, p'_j, p'_k , with $i < j < k$, the segment $p'_i p'_j$ has a smaller slope than the segment $p'_j p'_k$; see Figure 1(a). Indeed, this is the case if and only if

$$(p_{k2} - p_{j2})(p_{j1} - p_{i1}) > (p_{j2} - p_{i2})(p_{k1} - p_{j1}). \quad (4)$$

But (4) will hold as long as the function f in the definition of \ll is chosen large enough. Specifically, if $f(x) \geq 4x^2$, then the left-hand side of (4) is at least

$$\frac{1}{2}p_{k2} \cdot \frac{1}{2}p_{j1} \geq \frac{1}{4}p_{k2} \geq p_{j2}^2,$$

which is larger than the right-hand side of (4). \square

Next, we want to show that no point $r = (r_1, \dots, r_d) \in \mathbb{R}^d$ is contained in more than $(n/(d+1))^{d+1} + O(n^d)$ of the d -simplices spanned by S .

We can assume that $p_{1j} \leq r_j \leq p_{nj}$ for each coordinate $1 \leq j \leq d$, since otherwise, r is not contained in any d -simplex spanned by S . For each coordinate $j = 1, \dots, d$, we discard from S the last point p_i with $p_{ij} \leq r_j$ and the first point p_i with $p_{ij} \geq r_j$. Let S' be the resulting set. Since we have discarded at most $2d$ points, the number of d -simplices involving any of the discarded points is only $O(n^d)$. And now, for every $p_i \in S'$ and every j , we have either $r_j \ll p_{ij}$ or $r_j \gg p_{ij}$.

Let $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ be a point; we define the *type* of a with respect to r as $\max\{k : a_j > r_j \text{ for all } j = 1, 2, \dots, k\}$. Note that the type of a is an integer between 0 and d (it is 0 if $a_1 \leq r_1$).

Let $p_{i_0}, \dots, p_{i_d} \in S'$ span a d -simplex containing r , with $i_0 < \dots < i_d$. For convenience, we rename these points and their coordinates as

$$q_\ell = (q_{\ell 1}, \dots, q_{\ell d}), \quad \text{for } \ell = 0, 1, \dots, d.$$

The proof of Theorem 1.3 will be almost finished once we establish the following:

Lemma 2.2. *For each $\ell = 0, 1, \dots, d$, the point q_ℓ has type ℓ with respect to r . (See Figure 1(b) for an illustration of the planar case.)*

Indeed, assuming this lemma, the proof of Theorem 1.3 is concluded as follows. Given r , we partition the points of S' into $d+1$ subsets S'_0, \dots, S'_d according to their type. Then, for a d -simplex spanned by $d+1$ points from S' to contain r , each point must come from a different S'_k . The number of such simplices is thus at most $\prod_{k=0}^d |S'_k| \leq (n/(d+1))^{d+1}$, by the arithmetic-geometric mean inequality.

Proof of Lemma 2.2. We are going to derive the following relations:

$$q_{(j-1)j} \ll r_j \ll q_{jj} \quad \text{for every } j = 1, 2, \dots, d. \quad (5)$$

Let us first check that they imply the lemma. To see that q_ℓ has type ℓ , we need that $q_{\ell j} > r_j$ for $j \leq \ell$ and, if $\ell < d$, also that $q_{\ell(\ell+1)} \leq r_{\ell+1}$. The last inequality follows from (5) with $j = \ell + 1$. To derive $q_{\ell j} > r_j$, we use that the coordinates of q_ℓ are increasing since $q_\ell \in S$, and thus $q_{\ell j} \geq q_{jj} > r_j$.

Now we start working on (5). First we express the condition that r lie in the simplex spanned by q_0, \dots, q_d using determinants. For each ℓ , the points r and q_ℓ must lie on the *same* side of the hyperplane spanned by the points q_m , $m \neq \ell$. Thus, let M be the $(d+1) \times (d+1)$ matrix consisting of rows $(1, q_0), (1, q_1), \dots, (1, q_d)$. For $k = 0, 1, \dots, d$, let M_k be the matrix obtained from M by replacing the row $(1, q_k)$ by $(1, r)$. Then, for each k , $\det M_k$ must have the same sign as $\det M$.

Next, we show that $\det M$ and each $\det M_k$ are “dominated” by a single product of entries. Let A be one of the matrices M, M_0, M_1, \dots, M_d , and denote by $a_{\ell j}$ the entry in row ℓ and column j of A , for $0 \leq \ell, j \leq d$. We claim that if the function f in the definition of \ll is chosen sufficiently large, then there is a single product of the form $\text{sign}(\sigma) \prod_\ell a_{\ell \sigma(\ell)}$, for some permutation σ , which is larger in absolute value than the sum of absolute values of all the other products in $\det A$.

Indeed, let $a_{\ell_d d}$ be the largest entry in the last column of A . This is also the largest entry in the entire matrix. Then, if we take $f(x) \geq (d+1)!x^{d+1}$, any permutation product involving $a_{\ell_d d}$ is larger than $(d+1)!$ times any permutation product *not* involving this entry. Thus, we choose $a_{\ell_d d}$ as the first term in our product, we remove row ℓ_d and column d from A , and we continue in this fashion leftwards. We obtain a product $\prod_\ell a_{\ell \sigma(\ell)}$ which is larger than $(d+1)!$ times any other permutation product in $\det A$. Therefore, this product “dominates” $\det A$ in the above sense, and so $\text{sign}(\det A) = \text{sign}(\sigma)$.

In particular, these considerations for $A = M$ show that $\det M$ is dominated by the product

$$q_{dd}q_{(d-1)(d-1)} \cdots q_{11} \cdot 1$$

corresponding to the identity permutation. Therefore, $\det M > 0$, and so we must have $\det M_k > 0$ for all k .

Now we are ready to prove (5). First we suppose for contradiction that $r_j \gg q_{jj}$ for some $j = 1, 2, \dots, d$. We take the largest such j ; thus, $q_{kk} \gg r_k$ for $k > j$. Then $\det M_{j-1}$ is dominated by the product

$$q_{dd} \cdots q_{(j+1)(j+1)} r_j q_{j(j-1)} q_{(j-2)(j-2)} \cdots q_{11} \cdot 1,$$

so the sign of $\det M_{j-1}$ is the sign of the permutation associated with this product. This is a permutation with exactly one inversion, so $\det M_{j-1} < 0$, which is a contradiction.

Next, we suppose for contradiction that $r_j \ll q_{(j-1)j}$ for some $j = 1, 2, \dots, d$. Now we take the *smallest* such j . We have already shown that $r_k \ll q_{kk}$ for all k . Therefore, $\det M_j$ is dominated by the product

$$q_{dd} \cdots q_{(j+1)(j+1)} q_{(j-1)j} r_{j-1} q_{(j-2)(j-2)} \cdots q_{11} \cdot 1.$$

Again, this product corresponds to a permutation with exactly one inversion, so we have $\det M_j < 0$, which is again a contradiction. \square

3 The construction for Theorem 1.5

We now present the construction that proves Theorem 1.5. Let us call a set $Y \subseteq \mathbb{R}^d$ *antiantipodal* if $Y \cap (-Y) = \emptyset$.

We make use of the following result of Wendel:

Lemma 3.1 ([Wen62]). *Let $X = \{x_1, \dots, x_{d+1}\}$ be a set of $d+1$ points in general position on the unit sphere \mathbb{S}^{d-1} in \mathbb{R}^d . Then there are exactly two antiantipodal $(d+1)$ -point subsets of $X \cup (-X)$ whose convex hull contains the origin; if one of them is Y , then the other one is $-Y$.*

Let $\alpha \in (0, 1/2]$ be a parameter and let n be given. For the moment assume for simplicity that αn is an integer. Let A be a set of αn points on \mathbb{S}^{d-1} and let p be another point on \mathbb{S}^{d-1} such that the set $A \cup (-A) \cup \{p\}$ has $2|A| + 1$ points and no hyperplane containing p and a $(d-1)$ -point antiantipodal subset of $A \cup (-A)$ passes through the origin. Let P be a very small cluster of $(1 - 2\alpha)n$ points around p .

Our set is $S = A \cup (-A) \cup P$. Note that $|S| = n$ as required. The origin clearly lies at depth αn with respect to S . Thus, Theorem 1.5 reduces to the following lemma:

Lemma 3.2. *There number of $(d+1)$ -point subsets B of S such that $\text{conv } B$ contains the origin is*

$$\left((d+1)\alpha^d - 2d\alpha^{d+1} \right) \binom{n}{d+1} + O(n^d).$$

Proof. The number of $(d+1)$ -point subsets of S that contain a pair of antipodal points (one in A and one in $-A$) is $O(n^d)$, and so it suffices to count the number of B that are antiantipodal.

The choice of A and P guarantees that if B is antiantipodal and $|B \cap P| \geq 2$, then $0 \notin \text{conv } B$. So we need to consider the cases $B \cap P = \emptyset$ and $|B \cap P| = 1$.

Let us set $\tilde{B} = \{x \in A \cup P : x \in B \text{ or } -x \in B\}$. For $B \cap P = \emptyset$ there are $\binom{\alpha n}{d+1}$ ways of choosing $\tilde{B} \subseteq A$, and for each of them we have two choices for B by Lemma 3.1.

For $|B \cap P| = 1$, we have $(1 - 2\alpha)n \binom{\alpha n}{d}$ choices for \tilde{B} , and each of them yields exactly one B (Lemma 3.1 with $X = \tilde{B}$ shows that there are two $B \subset \tilde{B} \cup (-\tilde{B})$ with $0 \in \text{conv } B$, and exactly one of these contains the point $p \in P \cap \tilde{B}$, while the other contains $-p$).

Altogether the number of B 's is $2 \binom{\alpha n}{d+1} + (1 - 2\alpha)n \binom{\alpha n}{d} + O(n^d)$, and the lemma follows by algebraic manipulation. \square

If αn is not an integer, then apply the above argument using $\alpha' = \lceil \alpha n \rceil / n$, and use the fact that $\alpha' - \alpha < 1/n$.

4 Partitioning measures by fans of hyperplanes

In this section we prove Theorem 1.7, which is the main ingredient in the proof of Theorem 1.6. We then prove Theorem 1.8, showing that Theorem 1.7 is optimal. Recall that an m -fan is a set of m hyperplanes in \mathbb{R}^d sharing a common $(d - 2)$ -flat.

Let \mathbb{S}^{d-1} denote the unit sphere in \mathbb{R}^d , and let $\mathbb{V}_{d,2} = \{(v, w) \in (\mathbb{S}^{d-1})^2 : v \perp w\}$ denote the set of ordered pairs of orthonormal vectors in \mathbb{R}^d (called the *Stiefel manifold* of orthogonal 2-frames). The proof of Theorem 1.7 is based on the following topological result:

Lemma 4.1. *There exists no continuous function $g: \mathbb{V}_{d,2} \rightarrow \mathbb{S}^{2d-4}$ with the following property: For every $(v, w) \in \mathbb{V}_{d,2}$, if $g(v, w) = (a_1, \dots, a_{2d-3})$ (with $a_1^2 + \dots + a_{2d-3}^2 = 1$), then*

- $g(-v, w) = (-a_1, \dots, -a_{d-1}, a_d, \dots, a_{2d-3})$, and
- $g(v, -w) = (a_1, \dots, a_{d-1}, -a_d, \dots, -a_{2d-3})$.

Proof. The lemma is a result on nonexistence of an *equivariant map*. Let us briefly recall the basic setting; for more background we refer to [Živ04], [Mat08].

Let G be a finite group. A G -space is a topological space X together with an *action* of G on X , which is a collection $(\varphi_g)_{g \in G}$ of homeomorphisms $\varphi_g: X \rightarrow X$ whose composition agrees with the group operation in G ; that is, $\varphi_e = \text{id}_X$ for the unit element $e \in G$ and $\varphi_g \circ \varphi_h = \varphi_{gh}$ for all $g, h \in G$.

In our case, the relevant group is $G := \mathbb{Z}_2 \times \mathbb{Z}_2$ (the direct product of two cyclic groups of order 2). We can write $G = \{e, g_1, g_2, g_1 g_2\}$, where g_1 and g_2 are two generators of G ; in order to specify an action of G , it is enough to give the homeomorphisms corresponding to g_1 and g_2 . The lemma deals with two G -spaces:

- The Stiefel manifold $\mathbb{V}_{d,2}$ with the action $(\varphi_g)_{g \in G}$ of G given by $\varphi_{g_1}(v, w) = (-v, w)$, $\varphi_{g_2}(v, w) = (v, -w)$.

- The sphere \mathbb{S}^{2d-4} with the action $(\psi_g)_{g \in G}$, where ψ_{g_1} flips the signs of the first $d-1$ coordinates and ψ_{g_2} flips the signs of the remaining $d-2$ coordinates.

We want to prove that there is no *equivariant map* $f: \mathbb{V}_{d,2} \rightarrow \mathbb{S}^{2d-4}$, where an equivariant map is a continuous map that commutes with the actions of G , i.e., such that $f \circ \varphi_g = \psi_g \circ f$ for all $g \in G$. The “usual” elementary methods for showing nonexistence of equivariant maps, explained in [Živ04], [Mat08] and based on the Borsuk–Ulam theorem and its generalizations, cannot be applied here. We use the *ideal-valued cohomological index* of Fadell and Husseini [FH88] (also see [Živ98]).

This method assigns to every G -space X the G -index of X , denoted by $\text{Ind}_G(X)$, which is an ideal in a certain ring R_G (depending only on G). A key property is that whenever there is an equivariant map $f: X \rightarrow Y$, where X and Y are G -spaces, we have $\text{Ind}_G(Y) \subseteq \text{Ind}_G(X)$. For the considered $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, R_G is the ring $\mathbb{Z}_2[t_1, t_2]$ of polynomials in two variables with \mathbb{Z}_2 coefficients. The general definition of $\text{Ind}_G(X)$, as well as its computation, are rather complicated, but fortunately, in our case we can use ready-made results from the literature.

For the G -space \mathbb{S}^{2d-4} with the G -action as above, the G -index is the principal ideal in $\mathbb{Z}_2[t_1, t_2]$ generated by $t_1^{d-1}t_2^{d-2}$ according to Corollary 2.12 in [Živ98]. On the other hand, Fadell [Fad89] proved that the G -index of the G -space $\mathbb{V}_{d,2}$ with the described G -action does not contain the monomial $t_1^{d-1}t_2^{d-2}$ (also see [Ino06] for a statement of this result and some applications of it). This shows that an equivariant map as in the lemma is indeed impossible. \square

Proof of Theorem 1.7. We follow the “configuration space/test map” paradigm (see, e.g., [Živ04]). We encode each “candidate” for the desired equipartition, which in our case is going to be a certain special fan of $4d-2$ half-hyperplanes sharing the boundary $(d-2)$ -flat, by a point of $\mathbb{V}_{d,2}$. Then we define a continuous *test map* that assigns to each candidate fan of half-hyperplanes a $(2d-3)$ -tuple of real numbers, which measures how far the given candidate is from being a $(2d-1)$ -fan of hyperplanes. Finally we will check that if there were no equipartition, the test map would yield an equivariant map $\mathbb{V}_{d,2} \rightarrow \mathbb{S}^{2d-4}$, which would contradict Lemma 4.1. The details follow.

For the proof we may assume that every nonempty open set has a positive μ -measure. (Given an arbitrary μ , we can consider the convolution $\mu * \gamma_\varepsilon$ of μ with a suitable probability measure γ_ε whose density function is everywhere nonzero but for which all but at most ε of the mass lies in a ball of radius ε around 0. The convolution has the required property and then, given an equipartition for each $\mu * \gamma_\varepsilon$ a limit argument, letting $\varepsilon \rightarrow 0$, yields an equipartition for the original μ .)

Let $m = 2d-1$. Suppose we are given two orthonormal vectors $v, w \in \mathbb{S}^{d-1}$. Let h be the unique hyperplane orthogonal to v that splits \mathbb{R}^d into two halfspaces of equal measure with respect to μ . We say that the halfspace in the direction of v is “above” h , and the other halfspace is “below” h .

Let ℓ be a $(d-2)$ -flat orthogonal to w contained in h . Note that ℓ splits h into two half-hyperplanes. We say that the half of h in the direction of w lies “left” of ℓ , and the other half of h lies “right” of ℓ .

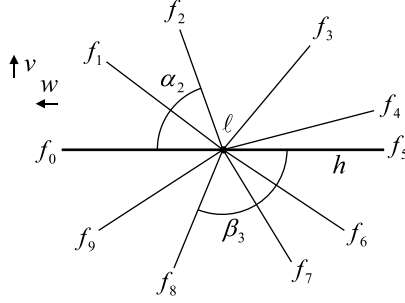


Figure 2: $2m$ half-hyperplanes coming out of ℓ that partition the measure μ into $2m$ equal parts. (Here $m = 5$.)

Every half-hyperplane with boundary ℓ is uniquely determined by the angle it makes with the left half of h . Let $f_0, f_1, \dots, f_{2m-1}$ be $2m$ half-hyperplanes coming out of ℓ , listed in circular order, that split the measure μ into $2m$ equal parts, as follows:

- f_0 is the left half of h ;
- f_1, \dots, f_{m-1} lie above h ;
- f_m is the right half of h ; and
- f_{m+1}, \dots, f_{2m-1} lie below h .

See Figure 2.

For $i = 1, \dots, m - 1$, let α_i be the angle between f_0 and f_i , and let β_i be the angle between f_m and f_{m+i} . Let $\gamma_i = \alpha_i - \beta_i$. Note that $\gamma_i = 0$ means that f_i and f_{m+i} are aligned into a hyperplane.

Translating ℓ within h to the left causes the α_i 's to increase and the β_i 's to decrease, while translating it to the right has the opposite effect. Therefore, there exists a unique position of ℓ for which $\sum \alpha_i = \sum \beta_i$, or equivalently, $\sum \gamma_i = 0$, and we fix ℓ there. In this way, we have defined each α_i and β_i as a function of the given vectors v, w . Using the assumption that μ is absolutely continuous with respect to the Lebesgue measure and each open set has a positive μ -measure, it is routine to verify the continuity of the α_i and β_i as functions of v and w .

Let us examine what happens when we change the sign of v or w . We have:

$$\begin{aligned} \alpha_i(-v, w) &= \pi - \beta_{m-i}(v, w), & \beta_i(-v, w) &= \pi - \alpha_{m-i}(v, w), \\ \alpha_i(v, -w) &= \pi - \alpha_{m-i}(v, w), & \beta_i(v, -w) &= \pi - \beta_{m-i}(v, w). \end{aligned}$$

See Figure 3. Therefore,

$$\begin{aligned} \gamma_i(-v, w) &= \gamma_{m-i}(v, w), \\ \gamma_i(v, -w) &= -\gamma_{m-i}(v, w). \end{aligned}$$

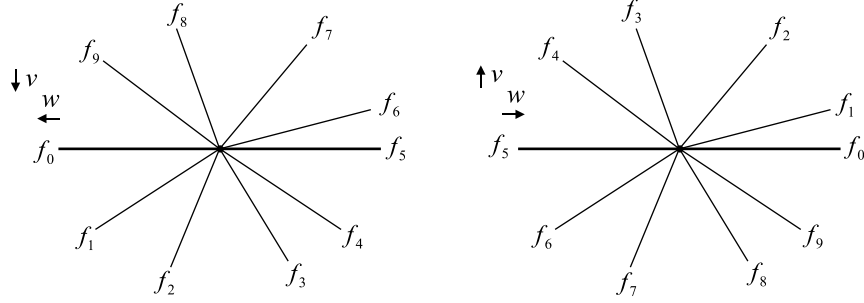


Figure 3: The effect of changing the sign of v (left) or w (right).

Now we introduce a suitable change of coordinates in the target space so that the resulting map behaves as the map g considered in Lemma 4.1. Namely, we set

$$\begin{aligned}\lambda_i &= \gamma_i - \gamma_{m-i}, & \text{for } i = 1, \dots, (m-1)/2; \\ \mu_i &= \gamma_i + \gamma_{m-i}, & \text{for } i = 2, \dots, (m-1)/2.\end{aligned}$$

Note that

$$\begin{aligned}\lambda_i(-v, w) &= -\lambda_i(v, w), & \mu_i(-v, w) &= \mu_i(v, w), \\ \lambda_i(v, -w) &= \lambda_i(v, w), & \mu_i(v, -w) &= -\mu_i(v, w).\end{aligned}$$

We have $\gamma_i = 0$ for all i if and only if $\lambda_i, \mu_i = 0$ for all i . (Recall that $\sum \gamma_i = 0$.) Now we define the “test map” $G: \mathbb{V}_{d,2} \rightarrow \mathbb{R}^{m-2}$ by

$$G(v, w) = (\lambda_1, \dots, \lambda_{(m-1)/2}, \mu_2, \dots, \mu_{(m-1)/2}).$$

Then, our desired equipartition of μ exists if and only if $G(v, w) = (0, \dots, 0)$ for some (v, w) .

But G is a continuous map such that flipping v flips the first $(m-1)/2 = d-1$ coordinates of the image, while flipping w flips the last $(m-3)/2 = d-2$ coordinates of the image. If we had $G \neq (0, \dots, 0)$ for all (v, w) , the map $g: \mathbb{V}_{d,2} \rightarrow \mathbb{S}^{2d-4}$ given by $g(v, w) = G(v, w)/\|G(v, w)\|$ would contradict Lemma 4.1. Therefore, the desired equipartition exists. \square

We conclude this section by proving Theorem 1.8, which shows that Theorem 1.7 is best possible, in the sense that an equipartition of a measure μ in \mathbb{R}^d by a fan of $2d$ or more hyperplanes does not necessarily exist. The proof is based on the following lemma:

Lemma 4.2. *Let $m > 0$ be an integer and let $t \geq 2d + m - 1$. Then there exists a t -point set $T \subset \mathbb{R}^d$ which cannot be covered by any m -fan in \mathbb{R}^d .*

The basic idea, roughly speaking, is that an m -fan in \mathbb{R}^d has $2d + m - 2$ degrees of freedom, while each point in T takes away one degree of freedom. Therefore, T can be completely covered by an m -fan only if it is degenerate an appropriate sense.

Proof of Lemma 4.2. For convenience we first prove the result in \mathbb{RP}^d , the d -dimensional projective space, and then we show that the result also applies to \mathbb{R}^d .

A set of m hyperplanes in \mathbb{RP}^d share a common $(d-2)$ -flat if and only if their dual points, when considered as vectors in \mathbb{R}^{d+1} , span a vector space of dimension at most 2. Thus, define the projective variety

$$V = \{(p_1, \dots, p_m) \in (\mathbb{RP}^d)^m : \text{rank}(p_1, \dots, p_m) \leq 2\},$$

where $\text{rank}(p_1, \dots, p_m)$ denotes the dimension of the vector space spanned by p_1, \dots, p_m as vectors in \mathbb{R}^{d+1} . The variety V has dimension $\dim V = 2d + m - 2$.

Given a point $p = p_0 : p_1 : \dots : p_d \in \mathbb{RP}^d$, let $p^* = \{x_0 : \dots : x_d \in \mathbb{RP}^d : \sum_i x_i p_i = 0\}$ denote the hyperplane dual to p . For each $v = (p_1, \dots, p_m) \in V$, let $v^* = \bigcup_i p_i^* \subset \mathbb{RP}^d$ be the variety which consists of the union of the hyperplanes dual to the points in v . Finally, define the moduli space

$$C = \{(v, q_1, \dots, q_t) \in V \times (\mathbb{RP}^d)^t : q_1, \dots, q_t \in v^*\}$$

of all t -tuples of points lying on a fan v^* , for all $v \in V$. The dimension of C is $\dim C = \dim V + t(d-1) = 2d + m - 2 + t(d-1)$.

Consider the projection map $\pi : V \times (\mathbb{RP}^d)^t \rightarrow (\mathbb{RP}^d)^t$. Then the projection $\pi(C)$ is the set of t -tuples of points in \mathbb{RP}^d that can be covered by an m -fan. By the Tarski–Seidenberg Theorem [?] $\pi(C)$ is a semialgebraic subset of $(\mathbb{RP}^d)^t$. Since projection does not increase dimension, $\pi(C)$ is of dimension at most $2d + m - 2 + t(d-1)$, which by our choice of t is smaller than $td = \dim(\mathbb{RP}^d)^t$.

Thus, there exists a t -point set T in \mathbb{RP}^d that cannot be covered by an m -fan. Finally, a generic \mathbb{R}^d inside \mathbb{RP}^d completely contains T , and the lemma follows. \square

Proof of Theorem 1.8. Given an integer $m \geq 2d$, let $t = 2m - 1$. We have $t \geq 2d + m - 1$, so by the preceding lemma there exists a t -point set $T \subset \mathbb{R}^d$ that cannot be covered by any m -fan in \mathbb{R}^d .

There must exist a positive radius r such that, for every m -fan F in \mathbb{R}^d , some point of T lies at distance at least r from the closest hyperplane in F . (Otherwise a limit argument would yield an m -fan that covers T .)

Let $C_r(p)$ denote the ball of radius r centered at p . Let μ be the uniform measure on $\bigcup_{p \in T} C_r(p)$. Then, in every partition of \mathbb{R}^d into $2m$ parts by an m -fan, there exists a part that completely contains one of the balls $C_r(p)$. This part has measure at least $\frac{1}{t} > 1/(2m)$, and so the partition is not an equipartition. \square

5 Stabbing many triangles in \mathbb{R}^d

In this section we prove Theorem 1.6 by means of Theorem 1.7. The proof is an extension of the technique in [Buk06] for the case $d = 2$.

By a standard approach (see e.g. Theorem 3.1.2 in [Mat08]), Theorem 1.7 implies the following discrete version, which is what we actually use:

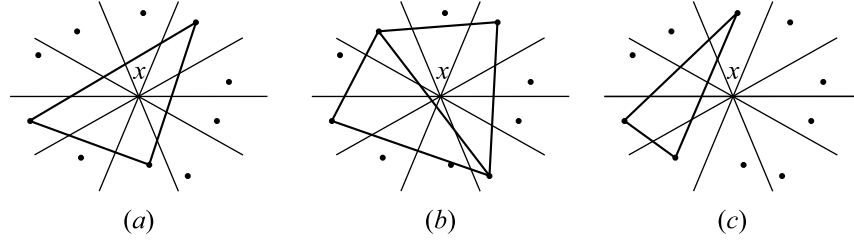


Figure 4: (a) A triangle with three short sides always contains x . (b) The triangles with one medium side can be partitioned into pairs, such that at least one triangle from each pair contains x . (c) A triangle with one long side never contains x .

Corollary 5.1. *Let S be a set of n points in \mathbb{R}^d . Then there exist $2d - 1$ hyperplanes passing through a common $(d - 2)$ -flat that divide the space into $4d - 2$ parts, each containing at most $n/(4d - 2) + O(1)$ points of S .*

We start with the following lemma:

Lemma 5.2. *Let $\ell_0, \dots, \ell_{m-1}$ be m lines in the plane passing through a common point x , dividing the plane into $2m$ sectors. Let $P = \{p_0, \dots, p_{2m-1}\}$ be $2m$ points, one from each sector, listed in circular order around x . Then, out of the $\binom{2m}{3}$ triangles defined by P , at least $(m + 1)m(m - 1)/3$ contain x . (This minimum is achieved if $P \cup \{x\}$ is in general position, in particular if no two points of P are collinear with x .)*

Proof. Let $p_i p_j$ be a directed segment joining two points of P , and let $d = (j - i) \bmod 2m$. If $0 \leq d \leq m - 1$, we call the segment $p_i p_j$ *short*; if $d = m$, we call it *medium*; and if $m + 1 \leq d \leq 2m - 1$, we call it *long*.

A triangle $p_i p_j p_k$, with $i < j < k$, can have either three short sides, or two short sides and one medium side, or two short sides and one long side.

It is easy to see that all the triangles with three short sides contain x , and none of the triangles with one long side contain x . Furthermore, the triangles with one medium side can be grouped into pairs, such that from each pair, at least one triangle contains x (exactly one triangle if $P \cup \{x\}$ is in general position). See Figure 4.

The number of triangles with three short sides is

$$\frac{2m}{3}(1 + 2 + \dots + (m - 2)) = \frac{m(m - 1)(m - 2)}{3};$$

and the number of triangles with one medium side is $2m(m - 1)$. Thus, P defines at least

$$\frac{m(m - 1)(m - 2)}{3} + m(m - 1) = \frac{(m + 1)m(m - 1)}{3}$$

triangles that contain x (exactly these many if $P \cup \{x\}$ is in general position). \square

Corollary 5.3. *Let h_0, \dots, h_{m-1} be m hyperplanes in \mathbb{R}^d that pass through a common $(d - 2)$ -flat ℓ and divide space into $2m$ parts. Let $P = \{p_0, \dots, p_{2m-1}\}$ be $2m$ points, one from each part. Then P defines at least $(m + 1)m(m - 1)/3$ triangles that intersect ℓ .*

Proof of Theorem 1.6. Let S be an n -point set in \mathbb{R}^d . By Corollary 5.1 there exist $2d - 1$ hyperplanes that pass through a common $(d - 2)$ -flat ℓ and partition S into parts of size at most $n/(4d - 2) + O(1)$ each. We show that ℓ is our desired $(d - 2)$ -flat.

Each part has at least $n/(4d - 2) - O(1)$ points, so there are at least $\left(\frac{n}{4d - 2} - O(1)\right)^{4d - 2}$ ways to choose $4d - 2$ points, one from each part.

By Corollary 5.3, each such choice of points defines at least $2d(2d - 1)(2d - 2)/3$ triangles that intersect ℓ . On the other hand, each such triangle is counted at most $\left(\frac{n}{4d - 2} + O(1)\right)^{4d - 5}$ times. Thus the number of triangles intersected by ℓ is at least

$$\begin{aligned} \left(\frac{n}{4d - 2} - O(1)\right)^{4d - 2} \cdot \frac{2d(2d - 1)(2d - 2)}{3} &/ \left(\frac{n}{4d - 2} + O(1)\right)^{4d - 5} \\ &= \frac{d^2 - d}{6(2d - 1)^2} n^3 - O(n^2). \quad \square \end{aligned}$$

6 Discussion

The main open problem is to determine the exact value of the constants c_d of the First Selection Lemma for $d \geq 3$. There remains a multiplicative gap of roughly $(d - 1)!$ between the current lower bound (1) and our upper bound (given by Theorem 1.3). We conjecture that Theorem 1.3 is tight, and that the correct constants are $c_d = (d + 1)^{-(d + 1)}$.

We suspect that the construction in Theorem 1.3 also witnesses sharpness of Theorem 1.6. But, to our embarrassment, we have been unable to find even the line that stabs most triangles in this construction for $d = 3$.

6.1 A generalization of centerpoints

Rado's Centerpoint Theorem [Rad47] implies that for every n -point set S in \mathbb{R}^d there exists a $(d - 2)$ -flat ℓ that lies at depth $n/3$ with respect to S , in the sense that every halfspace that contains ℓ contains at least $n/3$ points of S . (Simply project S into an arbitrary plane.)

But the $(d - 2)$ -flat ℓ of Corollary 5.1 lies at depth $(d - 1)n/(2d - 1) - O(1)$ with respect to S . (Indeed, every halfspace that contains S completely contains $2d - 2$ of the $4d - 2$ parts mentioned in Corollary 5.1.) We do not know whether this bound is tight.

This suggests the following generalization of Rado's Theorem: If S is an n -point set in \mathbb{R}^d and $0 \leq k < d$, then there always exists a k -flat at depth $\delta_{d,k}n$ with respect to S —a “center- k -flat”—for some constants $\delta_{d,k}$. The general question of determining these constants $\delta_{d,k}$ has not been explored, as far as we know. (The formula $\delta_{d,k} = (k + 1)/(d + k + 1)$ seems to fit all the currently known data.)

6.2 From the First Selection Lemma to the Second

The First Selection Lemma has been generalized by Bárány et al. [BFL90], in conjunction with Alon et al. [ABFK92], and Živaljević and Vrećica [ŽV92], to the following result,

called the *Second Selection Lemma* in [Mat02]:

If S is an n -point set in \mathbb{R}^d and \mathcal{F} is a family of $m \leq \binom{n}{d+1}$ d -simplices spanned by S , then there exists a point $p \in \mathbb{R}^d$ contained in at least

$$c'_d \left(\frac{m}{n^{d+1}} \right)^{s_d} n^{d+1} \quad (6)$$

simplices of \mathcal{F} , for some constants c'_d and s_d that depend only on d . (Note that $m/n^{d+1} = O(1)$, so the smaller the constant s_d , the stronger the bound.)

The Second Selection Lemma is an important ingredient in the derivation of non-trivial upper bounds for the number of k -sets in \mathbb{R}^d (see [Mat02, ch. 11] for the definition and details). The derivation proceeds by “lifting” the lemma by one dimension, obtaining that if \mathcal{F} is a family of m d -simplices spanned by n points in \mathbb{R}^{d+1} , then there exists a *line* that stabs $\Omega\left(\left(\frac{m}{n^{d+1}}\right)^{s_d} n^{d+1}\right)$ simplices of \mathcal{F} .

Does this lifting step result in a loss of tightness? If we may make an analogy from the results of this paper, it seems that the answer is yes. (As we showed, $c_{2,0} = 1/27$ by Theorem 1.3, whereas $c_{3,1} \geq 1/25$ by Theorem 1.6.)

The current best bound for the Second Selection Lemma for $d = 2$ is $\Omega(m^3/(n^6 \log^2 n))$, due to Eppstein [Epp93, NS] (so s_2 can be taken arbitrarily close to 3 in (6)). On the other hand, we know that if \mathcal{F} is a set of m triangles in \mathbb{R}^3 spanned by n points, there exists a line (specifically, a line determined by two points of S) that stabs $\Omega(m^3/n^6)$ triangles of \mathcal{F} (see [DE94] and [Smo03] for two different proofs of this fact). It might turn out that this logarithmic gap between the two cases is an artifact of the current proofs, but we believe that the three-dimensional problem *does* have a larger bound than the planar one.

Acknowledgment

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References

- [ABFK92] Noga Alon, Imre Bárány, Zoltán Füredi, and Daniel Kleitman. Point selections and weak ϵ -nets for convex hulls. *Combin., Probab. Comput.*, 1:189–200, 1992.
- [Bár82] Imre Bárány. A generalization of Carathéodory’s theorem. *Discrete Math.*, 40:141–152, 1982.
- [BB49] Robert C. Buck and Ellen F. Buck. Equipartition of convex sets. *Math. Mag.*, 22:195–198, 1948/49.
- [BF77] Endre Boros and Zoltán Füredi. Su un teorema di Kárteszi nella geometria combinatoria. *Archimede*, 2:71–76, 1977.
- [BF84] Endre Boros and Zoltán Füredi. The number of triangles covering the center of an n -set. *Geom. Dedicata*, 17:69–77, 1984.

- [BFL90] Imre Bárány, Zoltán Füredi, and László Lovász. On the number of halving planes. *Combinatorica*, 10(2):175–183, 1990.
- [Buk06] Boris Bukh. A point in many triangles. *Electron. J. Combin.*, 13(1):Note 10, 3 pp. (electronic), 2006.
- [Ced64] Jack G. Ceder. Generalized sixpartite problems. *Bol. Soc. Mat. Mexicana (2)*, 9:28–32, 1964.
- [DE94] Tamal K. Dey and Herbert Edelsbrunner. Counting triangle crossings and halving planes. *Discrete and Computational Geometry*, 12(1):281–289, 1994.
- [Epp93] David Eppstein. Improved bounds for intersecting triangles and halving planes. *J. Comb. Theory Ser. A*, 62(1):176–182, 1993.
- [Fad89] Edward R. Fadell. Ideal-valued generalizations of Ljusternik-Schnierlmann category, with applications. In Edward Fadell et al., editors, *Topics in Equivariant Topology*, volume 108 of *Sém. Math. Sup.*, pages 11–54. Press Univ. Montréal, 1989.
- [FH88] Edward R. Fadell and Sufian Hussein. An ideal-valued cohomological index theory with applications to Borsuk–Ulam and Bourgin–Yang theorems. *Ergod. Th. Dynam. Sys.*, 8:73–85, 1988.
- [Ino06] Akira Inoue. Borsuk-Ulam type theorems on Stiefel manifolds. *Osaka J. Math.*, 43(1):183–191, 2006. <http://projecteuclid.org/euclid.ojm/1146243001>.
- [Kár55] Franz Kárteszi. Extremalaufgaben über endliche Punktsysteme. *Publ. Math. Debrecen*, 4:16–27, 1955.
- [Mat02] Jiří Matoušek. *Lectures on Discrete Geometry*. Springer-Verlag New York, Inc., Secaucus, NJ, USA, 2002.
- [Mat08] Jiří Matoušek. *Using the Borsuk–Ulam theorem*. Springer, Berlin, 2008. Revised 2nd printing.
- [Moo68] John W. Moon. *Topics on Tournaments*. Holt, Rinehart and Winston, New York, 1968.
- [NS] Gabriel Nivasch and Micha Sharir. Correct proof of a result of Eppstein on intersecting triangles. In preparation.
- [Rad47] Richard Rado. A theorem on general measure. *J. London Math. Soc.*, 21:291–300, 1947.
- [Smo03] Shakhar Smorodinsky. *Combinatorial problems in computational geometry*. PhD thesis, Tel Aviv University, June 2003. http://www.cs.bgu.ac.il/~shakhar/my_papers/phd.ps.gz.

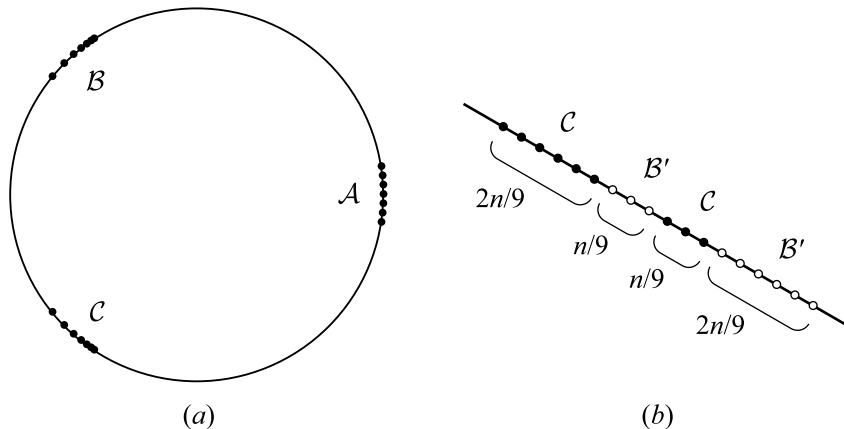


Figure 5: (a) The set \mathcal{P}_n of Boros and Füredi. (b) Relative order of the points of \mathcal{B}' (white circles) and \mathcal{C} (black circles) along the unit circle (which appears here almost as a straight line).

- [Wag03] Ulrich Wagner. *On k -Sets and Applications*. PhD thesis, ETH Zürich, June 2003. <http://www.inf.ethz.ch/~emo/DoctThesisFiles/wagner03.pdf>.
- [Wen62] J. G. Wendel. A problem in geometric probability. *Math. Scand.*, 11:109–111, 1962.
- [Živ98] Rade T. Živaljević. User’s guide to equivariant methods in combinatorics II. *Publ. Inst. Math. (Beograd) (N. S.)*, 64(78):107–132, 1998.
- [Živ04] Rade T. Živaljević. Topological methods. In Jacob E. Goodman and Joseph O’Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 14. CRC Press, Boca Raton, FL, second edition, 2004.
- [ŽV92] Rade T. Živaljević and Sinša T. Vrećica. The colored Tverberg’s problem and complexes of injective functions. *J. Comb. Theory Ser. A*, 61(2):309–318, 1992.

A The planar construction of Boros and Füredi

Boros and Füredi [BF84] constructed a planar n -point set \mathcal{P}_n for which, they claimed, no point $x \in \mathbb{R}^2$ is contained in more than $n^3/27 + O(n^2)$ triangles spanned by \mathcal{P}_n . There is a problem in their construction, however, and, as we show here, there exists a point x contained in $(\frac{1}{27} + \frac{1}{729})n^3$ triangles spanned by \mathcal{P}_n .

Their set \mathcal{P}_n lies on the unit circle, and it consists of three clusters of $n/3$ points, denoted \mathcal{A} , \mathcal{B} , and \mathcal{C} .

Each cluster is very small (sufficiently so), and the clusters are separated by an angular distance of roughly $2\pi/3$ from each other. The construction is not symmetric,

however. The points in cluster \mathcal{A} are uniformly separated, while the points in clusters \mathcal{B} and \mathcal{C} are separated by rapidly increasing distances (sufficiently rapidly so), with the distances increasing *away* from \mathcal{A} ; see Figure 5(a).

Now, let x be a point in the convex hull of $\mathcal{B} \cup \mathcal{C}$. For each point $b \in \mathcal{B}$, trace a line ℓ_b from b through x , until it intersects the unit circle again at b' . Let \mathcal{B}' denote the set of these points b' . If \mathcal{B} is sufficiently small and x is not too close to \mathcal{B} , then the lines ℓ_b will be almost parallel to each other.

The relative order between the points of \mathcal{B}' and the points of \mathcal{C} along the unit circle determines the number of triangles of the form \mathcal{ABC} , \mathcal{BBC} , and \mathcal{BCC} that contain x . (By a triangle of the form \mathcal{ABC} we mean a triangle abc with $a \in \mathcal{A}$, $b \in \mathcal{B}$, $c \in \mathcal{C}$; and so on.)

In fact, each triangle of the form \mathcal{ABC} containing x corresponds to a triple $ab'c$, with $a \in \mathcal{A}$, $b' \in \mathcal{B}'$, $c \in \mathcal{C}$, such that c is farther from a than b' . Similarly, each triangle of the form \mathcal{BBC} containing x corresponds to a triple $b'_1 b'_2 c$, with $b'_1, b'_2 \in \mathcal{B}'$, $c \in \mathcal{C}$, where c lies between b'_1 and b'_2 . And each triangle of the form \mathcal{BCC} containing x corresponds to a triple $b'c_1c_2$, with $b' \in \mathcal{B}'$, $c_1, c_2 \in \mathcal{C}$, where b' lies between c_1 and c_2 .

Note that the distances between the points of \mathcal{B}' increase rapidly *towards* \mathcal{A} . Also note that moving the point x towards or away from \mathcal{B} has the effect of enlarging or shrinking the image \mathcal{B}' , while moving x sideways has the effect of moving \mathcal{B}' sideways.

Therefore, it is not hard to see, we can position the point x such that the order of the points in $\mathcal{B}' \cup \mathcal{C}$, reading *towards* \mathcal{A} , is: $2n/9$ points of \mathcal{C} , followed by $n/9$ points of \mathcal{B}' , followed by $n/9$ points of \mathcal{C} , followed by $2n/9$ points of \mathcal{B}' ; see Figure 5(b).

It follows that x is contained in $\frac{8}{243}n^3$ triangles of the form \mathcal{ABC} , $\frac{2}{729}n^3$ triangles of the form \mathcal{BBC} , and $\frac{2}{729}n^3$ triangles of the form \mathcal{BCC} . Thus, x is contained in a total of $\frac{28}{729}n^3 = (\frac{1}{27} + \frac{1}{729})n^3$ triangles.

On the other hand, it can be checked that this point x is the one that stabs asymptotically the maximum number of triangles. Hence, this construction gives a bound of $c_2 \leq 1/27 + 1/729$.