Exercise 6

February, 2006

1. Calculate the Fourier coefficients of:
   (a) The parity (XOR) function
   (b) The AND function (AND of all inputs)
   (c) The OR function
   (d) The majority function on $2n+1$ inputs

2. For any $S \subseteq [n]$, show that

$$\sum_{i \in S} \mathrm{Inf}_i(f) \geq \mathrm{Inf}_S(f)$$

3. Are there two monotone nondecreasing Boolean functions f, g on n variables such that $\forall i : \mathrm{Inf}_i(f) = \mathrm{Inf}_i(g)$, yet $f \not\equiv g$, with permutations of variables considered as identical functions?

4. Let f be a monotone Boolean function on $\{0,1\}^n$ with the measure μ_p defined naturally (each input bit is 1 with probability p.) Prove Russo's lemma:

$$\frac{d\mathbb{E}_{\mu_p}(f)}{dp} = \sum_i \mathrm{Inf}_{f,p}(i)$$

   where $\mathrm{Inf}_{f,p}(i)$ is the probability (with the measure μ_p) that the ith bit of f has influence. Hint: generalize to the case where each bit has its own probability p_i.

5. For fixed n, the degree-k Krawtchouk polynomial is given by:

$$K_k(x) = \sum_{j=0}^{k} (-1)^j \binom{x}{j} \binom{n-x}{k-j}$$

   What is the leading coefficient of the degree-k Krawtchouk polynomial?