Exercise 3

February, 2006

1. Prove the edge isoperimetric inequality on the discrete cube, as follows: Let $A \subseteq \{0,1\}^n$ be a set; show that the number of edges between A and $\bar{A}$ is at least $-2^n \mu(A) \log_2 \mu(A)$, where μ is the uniform measure on the cube. Show that this bound is tight (at least in the case $\mu(A) = 1/2$.)

2. (a) Recall that $e^{i\theta} = \cos\theta + i\sin\theta$. Use this to show that $\cos n\theta = T_n(\cos\theta)$ for some polynomial T_n of degree at most n, and that $\sin n\theta = \sin\theta U_{n-1}(\cos\theta)$ for some polynomial U_{n-1} of degree at most $n-1$.

   (b) Show that $T_n'(x) = nU_{n-1}(x)$ for all $x \in \mathbb{R}$.

   (c) Show that the degree of T_n is exactly n (and thus that the degree of U_n is exactly n.) What is the leading coefficient of T_n?

3. Build a sequence of nonnegative continuous functions $\{f_i : [0,1] \to \mathbb{R}\}$ such that $\sum_i f_i$ converges uniformly but $\sum_i \sup f_i$ diverges.

4. Let $f(x) = \exp(-x^{\frac{1}{4}}) \sin x^{\frac{1}{4}} : [0,\infty) \to \mathbb{R}$. Show that f is continuous, $f \not\equiv 0$, and yet $\forall r \in \mathbb{N} : \int x^r f(x) \mathrm{d}x = 0$. (Hint: write $\omega = \exp(\pi i/4)$ and integrate by parts.)