Exercise 2

February, 2006

1. Let T_n be the degree-n Chebyshev polynomial.
   - Show that T_n and T_{n+1} interlace, i.e., between any two roots of one there is a root of the other.
   - Where are the zeroes of T_n most and least concentrated in the interval $[-1, 1]$?
2. (The Riemman-Lebesgue lemma.) Show that if $f : \mathbb{T} \to \mathbb{C}$ is continuous, then $\lim_{|n|\to\infty} \hat{f}(n) = 0$. You can use the fact that every such f can be arbitrarily approximated in L^2 norm by trigonometric polynomials.
3. For any Boolean function $f : \{-1, 1\}^n \to \{1, -1\}$, we say that a polynomial P is a sign-representing polynomial for f if $\forall x : \text{sign} f(x) = \text{sign} P(x)$. The degree of f is defined to be the minimal degree of a sign-representing polynomial of f. Show, by induction, that the degree of the parity function is n. Then show it directly using its Fourier expansion. (What are the characters of $\{-1, 1\}^n$?)
4. For $A, B \subseteq \mathbb{R}^n, p \in \mathbb{R}^n$, define:

$$d_2(p, A) = \inf_{q \in A} \|p - q\|_2$$

$$d_2(A, B) = \left(\sum_{p \in A} d_2^2(p, b) \right)^{\frac{1}{2}}$$

   For B an orthonormal basis of $\mathbb{R}^n$, define

$$S(B) = \{tv | t \in \mathbb{R}, v \in B, \|tv\|_2 = 1\}$$

   the intersection of B's vectors with the unit sphere in $\mathbb{R}^n$. Find an orthonormal basis H for $\mathbb{R}^{2^n}$ such that $d_2(S(H), S(e))$ is maximal (e is the natural basis, $e = \{e_1, e_2, ...e_{2^n}\}$.)

   Hint: consider $\mathbb{R}^{2^n}$ as the space of real functions on the n-dimensional discrete cube.

5. Let T = [0,1].

(a) Show that $l_p \subseteq l_q$ for $p \leq q$.

(b) Show that $L^q(T) \subseteq L^p(T)$ for $p \leq q$.

(c) Show that $L^2(T) \subseteq L^2(T)^*$ (H^* is the space of linear functionals on H.)