Making a K_4-free graph bipartite

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Abstract

We show that every K_4-free graph G with n vertices can be made bipartite by deleting at most $n^2/9$ edges. Moreover, the only extremal graph which requires deletion of that many edges is a complete 3-partite graph with parts of size $n/3$. This proves an old conjecture of P. Erdős.

1   Introduction

The well-known Max Cut problem asks for the largest bipartite subgraph of a graph G. This problem has been the subject of extensive research, both from the algorithmic perspective in computer science and the extremal perspective in combinatorics. Let n be the number of vertices and e be the number edges of G and let $b(G)$ denote the size of the largest bipartite subgraph of G. The extremal part of Max Cut problem asks to estimate $b(G)$ as a function of n and e. This question was first raised almost forty years ago by P. Erdős [8] and attracted a lot of attention since then (see, e.g., [3, 2, 4, 1, 16, 11, 10, 5, 7]).

It is well known that every graph G with e edges can be made bipartite by deleting at most $e/2$ edges, i.e., $b(G) \geq e/2$. To see this just consider a random partition of vertices of G into two parts V_1, V_2 and estimate the expected number of edges in the cut (V_1, V_2). A complete graph K_n on n vertices shows that the constant $1/2$ in the above bound is asymptotically tight. Moreover, this constant can not be improved even if we consider restricted families of graphs, e.g., graphs that contain no copy of a fixed *forbidden* subgraph H. We call such graphs H-free. Indeed, using sparse random graphs one can easily construct a graph G with e edges such that it has no short cycles but can not be made bipartite by deleting less than $e/2 - o(e)$ edges. Such G is clearly H-free for every forbidden graph H which is not a forest. It is a natural question to estimate the error term $b(G) - e/2$ as G ranges over all H-free graph with e edges. We refer interested reader to [3, 2, 1, 16], where such results were obtained for various forbidden subgraphs H.

In this paper we restrict our attention to *dense* $(e = \Omega(n^2))$ H-free graphs for which it is possible to prove stronger bounds for Max Cut. According to a long-standing conjecture of Erdős [9], every triangle-free graph on n vertices can be made bipartite by deleting at most $n^2/25$ edges. This bound, if true, is best possible (consider an appropriate blow-up of a 5-cycle). Erdős, Faudree, Pach and

Spencer proved that for triangle-free G of order n it is enough to delete $(1/18 - \epsilon)n^2$ edges to make it bipartite. They also verify the conjecture for all graphs with at least $n^2/5$ edges. Some extensions of their results were further obtained in [11]. Nevertheless this intriguing problem remains open. Erdős also asked similar question for K_4-free graphs. His old conjecture (see e.g., [10]) asserts that it is enough to delete at most $(1 + o(1))n^2/9$ edges to make bipartite any K_4-free graph on n vertices. Here we confirm this in the following strong form.

Theorem 1.1 *Every K_4-free graph G with n vertices can be made bipartite by deleting at most $n^2/9$ edges. Moreover, the only extremal graph which requires deletion of that many edges is a complete 3-partite graph with parts of size $n/3$.*

This result can be used to prove the following asymptotic generalization.

Corollary 1.2 *Let H be a fixed graph with chromatic number $\chi(H) = 4$. If G is a graph on n vertices not containing H as a subgraph, then we can delete at most $(1 + o(1))n^2/9$ edges from G to make it bipartite.*

Another old problem of Erdős, that is similar in spirit, is to determine the best local density in K_r-free graphs for $r \geq 3$ (for more information see, e.g., [12, 13] and their references). One of Erdős' favorite conjectures was that any triangle-free graph G on n vertices should contain a set of $n/2$ vertices that spans at most $n^2/50$ edges. The blow-up of a 5-cycle in which we replace each vertex by an independent set of size $n/5$ and each edge by a complete bipartite graph shows that this estimate can not be improved. On the other hand, for $r > 3$ Chung and Graham [6] conjectured that Turán graph has the best local density for subsets of size $n/2$. In particular, their conjecture implies that every K_4-free graph on n vertices should contain a set of $n/2$ vertices that spans at most $n^2/18$ edges.

Krivelevich [15] noticed that for regular graphs a bound in the local density problem implies a bound for the problem of making the graph bipartite. Indeed, suppose n is even, G is a d-regular K_4-free graph on n vertices and S is a set of $n/2$ vertices. Then $dn/2 = \sum_{s \in S} d(s) = 2e(S) + e(S, \bar{S})$ and $dn/2 = \sum_{s \notin S} d(s) = 2e(\bar{S}) + e(S, \bar{S})$, i.e. $e(S) = e(\bar{S})$. Deleting the $2e(S)$ edges within S or $\bar{S}$ makes the graph bipartite, so if we could find S spanning at most $n^2/18$ edges we would delete at most $n^2/9$ in making G bipartite. Unfortunately, the converse reasoning does not work. Nevertheless, we believe that the result of Theorem 1.1 provides some supporting evidence for conjecture of Chung and Graham.

The rest of this short paper is organized as follows. The proof of our main theorem appears in the beginning of next section. Next we show how to obtain Corollary 1.2 using this theorem together with well known Szemerédi's Regularity Lemma [17] (see also [14]). The last section of the paper contains some concluding remarks and open questions.

Notation. We usually write $G = (V, E)$ for a graph G with vertex set $V = V(G)$ and edge set $E = E(G)$, setting $n = |V|$ and $e = e(G) = |E(G)|$. If $X \subset V$ is a subset of the vertex set then $G[X]$ denotes the restriction of G to X, i.e. the graph on X whose edges are those edges of G with

both endpoints in X. We will write $e(X) = e(G[X])$ and similarly, we write $e(X, Y)$ for the number of edges with one endpoint in X and the other in Y. $N(v)$ is the set of vertices adjacent to a vertex v and $d(v) = |N(v)|$ is the degree of v. For any two vertices u, v we denote by $N(u, v)$ the set of common neighbors of u and v, i.e., all the vertices adjacent to both of them. We will also write $d(u, v) = |N(u, v)|$. Finally if three vertices u, v, and w are all adjacent then they form a triangle in G and we denote this by $\triangle = \{u, v, w\}$.

2 Proofs

2.1 Main result

In this subsection we present the proof of our main theorem. We start with the following well known fact (see, e.g., [1]), whose short proof we include here for the sake of completeness.

Lemma 2.1 *Let G be a 4-partite graph with e edges. Then G contains a bipartite subgraph with at least $2e/3$ edges.*

Proof. Let $V_1, \ldots, V_4$ be a partition of vertices of G into four independent sets. Partition these sets randomly into two classes, where each class contains exactly two of the sets V_i. Consider a bipartite subgraph H of G with these color classes. For each fixed edge (u, v) of G the probability that u and v will lie in the different classes is precisely $(2 \cdot 2)/\binom{4}{2} = 2/3$. Therefore, by linearity of expectation, the expected number of edges in H is $2e/3$, completing the proof. $\qquad\square$

Next we need two simple lemmas.

Lemma 2.2 *Let G be a graph with e edges and m triangles. Then it contains a triangle $\{u, v, w\}$ such that*
$$d(u, v) + d(u, w) + d(v, w) \geq \frac{9m}{e}.$$

Proof. A simple averaging argument, using that $\sum_{(x,y)\in E(G)} d(x, y) = 3m$ and Cauchy-Schwartz inequality, shows that there is a triangle $\{u, v, w\}$ in G with

$$
\begin{aligned}
d(u, v) + d(u, w) + d(v, w) &\geq \frac{1}{m} \sum_{\{x,y,z\}=\triangle} \Big(d(x, y) + d(x, z) + d(y, z) \Big) = \frac{1}{m} \sum_{(x,y)\in E(G)} d^2(x, y) \\
&\geq \frac{e}{m} \left(\frac{\sum_{(x,y)\in E(G)} d(x, y)}{e} \right)^2 = \frac{(3m)^2}{me} = \frac{9m}{e}. \qquad\square
\end{aligned}
$$

Lemma 2.3 *Let G be a graph on n vertices with e edges and m triangles. Then G contains a bipartite subgraph of size at least $4e^2/n^2 - 6m/n$.*

Proof. Let v be a vertex of G and let e_v denotes the number of edges spanned by the neighborhood $N(v)$. Consider the bipartite subgraph of G whose parts are $N(v)$ and its complement $V(G) \setminus N(v)$.

It is easy to see that number of edges in this subgraph is $\sum_{u \in N(v)} d(u) - 2e_v$. Thus averaging over all vertices v we have that

$$b(G) \;\geq\; \frac{1}{n}\sum_v \left(\sum_{u \in N(v)} d(u) - 2e_v \right) = \frac{1}{n}\sum_v d^2(v) - \frac{2}{n}\sum_v e_v \tag{1}$$

$$\geq\; \left(\frac{\sum_v d(v)}{n} \right)^2 - 6m/n = 4e^2/n^2 - 6m/n.$$

Here we used Cauchy-Schwartz inequality together with identities $\sum_v e_v = 3m$, $\sum_v d(v) = 2e$. $\qquad\square$

Now we can obtain our first estimate on the Max Cut in K_4-free graphs. This result can be used to prove the conjecture for graphs with $\leq n^2/4$ edges.

Lemma 2.4 *Let G be a K_4-free graph on n vertices with e edges. Then it contains a bipartite subgraph of size at least $2e/7 + 8e^2/(7n^2)$.*

Proof. Let v be a vertex of G and denote by e_v the number of edges spanned by the neighborhood of v. Consider a subgraph of G induced by the set $N(v)$. This subgraph $G[N(v)]$ has $d(v)$ vertices, e_v edges and contains no triangles, since G is K_4-free. Therefore by previous lemma (with $m = 0$) it has a bipartite subgraph H of size at least $4e_v^2/d^2(v)$. Let $(A, B), A \cup B = N(v)$ be the bipartition of H. Consider a bipartite subgraph H' of G with parts (A', B'), where $A \subset A'$, $B \subset B'$ and we place each vertex $v \in V(G) \setminus N(v)$ in A' or B' randomly and independently with probability $1/2$. All edges of H are edges of H', and each edge incident to a vertex in $V(G) \setminus N(v)$ appears in H' with probability $1/2$. As the number of edges incident to vertices $V(G) \setminus N(v)$ is $e - e_v$, by linearity of expectation, we have $b(G) \geq \mathbb{E}\big[e(H')\big] \geq (e - e_v)/2 + 4e_v^2/d^2(v)$. By averaging over all vertices v

$$b(G) \geq \frac{1}{2}e + \frac{1}{n}\sum_v \left(4e_v^2/d^2(v) - e_v/2 \right). \tag{2}$$

To finish the proof we take a convex combination of inequalities (1) and (2) with coefficients $3/7$ and $4/7$ respectively. This gives

$$
\begin{aligned}
b(G) \;\geq\;& \frac{3}{7}\left(\frac{1}{n}\sum_v d^2(v) - \frac{2}{n}\sum_v e_v \right) + \frac{4}{7}\left(\frac{1}{2}e + \frac{1}{n}\sum_v \left(4e_v^2/d^2(v) - e_v/2 \right) \right) \\
=\;& \frac{2}{7}e + \frac{1}{7n}\sum_v \left(3d^2(v) - 8e_v + 16e_v^2/d^2(v) \right) \\
=\;& \frac{2}{7}e + \frac{1}{7n}\sum_v d^2(v)\left(3 - 8\left(e_v/d^2(v)\right) + 16\left(e_v/d^2(v)\right)^2 \right) \\
\geq\;& \frac{2}{7}e + \frac{2}{7n}\sum_v d^2(v) \geq \frac{2}{7}e + \frac{2}{7}\left(\frac{\sum_v d(v)}{n} \right)^2 = \frac{2}{7}e + \frac{8}{7}e^2/n^2,
\end{aligned}
$$

where we used that $3 - 8t + 16t^2 = (4t - 1)^2 + 2 \geq 2$ for all t, $\sum_v d(v) = 2e$ and Cauchy-Schwartz inequality. $\qquad\square$

Remark. The above result is enough for our purposes, but one can get a slightly better inequality by taking a convex combination of (1) and (2) with coefficients $1/(1+a)$ and $a/(1+a)$ with $a = 1.38$.

Lemma 2.5 *Let* $f(t) = t/18 + \frac{2}{9}\left(5/2 - t - 1/t\right)^2$. *Then* $f(t) \leq 1/9$ *for all* $t \in [3/2, 2]$ *and equality holds only when* $t = 2$.

Proof. Note that $f(2) = 1/9$ and

$$f(t) - 1/9 = \frac{4t^4 - 19t^3 + 31t^2 - 20t + 4}{18t^2} = \frac{(t-2)(4t^3 - 11t^2 + 9t - 2)}{18t^2}.$$

Consider $g(t) = 4t^3 - 11t^2 + 9t - 2$ in the interval $[3/2, 2]$. The derivative of this function $g'(t) = 12t^2 - 22t + 9$ is zero when $t = \frac{22 \pm \sqrt{52}}{24}$, so the largest root of $g'(t)$ is less than $3/2$. Therefore $g(t)$ is strictly increasing function for $t \geq 3/2$ and so $g(t) > g(3/2) = 1/4 > 0$ for all $t \in [3/2, 2]$. Since $18t^2 > 0$ and $t - 2$ is negative for $t < 2$ we conclude that $f(t) - 1/9 < 0$ for all $t \in [3/2, 2)$. $\qquad\square$

Having finished all the necessary preparations we are now in a position to complete the proof of our main result.

Proof of Theorem 1.1. It is easy to see that complete 3-partite graph with parts of size $n/3$ has $(n/3)^3 = n^3/27$ triangles and that every edge of this graph is contained in exactly $n/3$ of them. To make this graph bipartite we need to destroy all these triangles. Since deletion of one edge can destroy at most $n/3$ of them, altogether we need to delete at least $\frac{n^3/27}{n/3} = n^2/9$ edges. To finish the proof it remains to show that deletion of $\leq n^2/9$ edges is sufficient to make every K_4-free graph bipartite.

Let G be a K_4-free graph on n vertices with e edges. Turán's theorem [18] says that $e \leq n^2/3$, with equality only when G is a complete 3-partite graph with parts of size $n/3$. By Lemma 2.4, we need to delete at most $e - b(G) \leq 5e/7 - 8e^2/(7n^2) = \left(\frac{5}{7}(e/n^2) - \frac{8}{7}(e/n^2)^2\right)n^2$ edges to make G bipartite. The function $g(t) = 5t/7 - 8t^2/7$ is increasing in the interval $t \leq 1/4$ and so $g(t) \leq g(1/4) = 3/28$. Therefore if $e \leq n^2/4$ we can delete at most $3n^2/28 < n^2/9$ edges to make G bipartite.

Next, consider the case when $n^2/4 \leq e \leq n^2/3$ and let m be the number of triangles in G. By Lemma 2.3, we can delete at most $e - b(G) \leq e - \left(4e^2/n^2 - 6m/n\right)$ edges to make G bipartite. So we can assume that $e - 4e^2/n^2 + 6m/n \geq n^2/9$ or we are done. Then the number of triangles in G satisfies $m \geq \frac{n}{6}\left(n^2/9 + 4e^2/n^2 - e\right)$ and Lemma 2.2 implies that G contains a triangle $\triangle = \{u, v, w\}$ with

$$d(u,v) + d(u,w) + d(v,w) \geq \frac{9m}{e} \geq 6e/n + n^3/(6e) - 3n/2.$$

Let $V_1 = N(u,v), V_2 = N(u,w), V_3 = N(v,w)$ and let $X = V(G) \setminus (\cup_{i=1}^3 V_i)$. Since G is K_4-free and $(u,v), (u,w), (v,w)$ are edges of G we have that sets $V_i, 1 \leq i \leq 3$ are independent and disjoint. Consider a 4-partite subgraph G' of G with parts V_1, V_2, V_3 and X. This graph has $e(G') = e - e(X)$ edges where $e(X)$ is the number of edges spanned by X. By Turán's theorem $e(X) \leq |X|^2/3$ and we also know that

$$|X| = n - \sum_i |V_i| = n - \left(d(u,v) + d(u,w) + d(v,w)\right) \leq 5n/2 - 6e/n - n^3/(6e).$$

5

Since G' is 4-partite we can now use Lemma 2.1 to deduce that $b(G) \geq b(G') \geq 2e(G')/3 = \frac{2}{3}(e - e(X))$. Therefore the number of edges we need to delete to make G bipartite is bounded by

$$
\begin{aligned}
e - b(G) &\leq e - 2(e - e(X))/3 = e/3 + 2e(X)/3 \leq e/3 + 2|X|^2/9 \\
&\leq e/3 + \frac{2}{9}\left(5n/2 - 6e/n - n^3/(6e)\right)^2 \\
&= \left(\frac{1}{18}(6e/n^2) + \frac{2}{9}\left(5/2 - 6e/n^2 - (6e/n^2)^{-1}\right)^2\right)n^2 \\
&= f(6e/n^2) \cdot n^2,
\end{aligned}
$$

where $f(t) = t/18 + \frac{2}{9}(5/2 - t - 1/t)^2$. As $n^2/4 \leq e \leq n^2/3$ we have that $3/2 \leq t = 6e/n^2 \leq 2$. Then, by Lemma 2.5, $f(6e/n^2) \leq 1/9$ with equality only if $e = n^2/3$. This shows that we can delete at most $n^2/9$ edges to make G bipartite and we need to delete that many edges only when $e(G) = n^2/3$, i.e., G is a complete 3-partite graph with parts of size $n/3$. . $\square$

2.2 Forbidding fixed 4-chromatic subgraph

In this short subsection we show how to use Theorem 1.1 to deduce a similar statement about graphs with any fixed forbidden 4-chromatic subgraph. The proof is a standard application of Szemerédi's Regularity Lemma and we refer the interested reader to the excellent survey of Komlós and Simonovits [14], which discusses various results proved by this powerful tool.

We start with a few definitions, most of which follow [14]. Let $G = (V, E)$ be a graph, and let A and B be two disjoint subsets of $V(G)$. If A and B are non-empty, define the *density of edges* between A and B by $d(A, B) = \frac{e(A,B)}{|A||B|}$. For $\epsilon > 0$ the pair (A, B) is called ϵ-*regular* if for every $X \subset A$ and $Y \subset B$ satisfying $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have $|d(X, Y) - d(A, B)| < \epsilon$. An *equitable partition* of a set V is a partition of V into pairwise disjoint classes $V_1, \cdots, V_k$ of almost equal size, i.e., $\big||V_i| - |V_j|\big| \leq 1$ for all i, j. An equitable partition of the set of vertices V of G into the classes $V_1, \cdots, V_k$ is called ϵ-*regular* if $|V_i| \leq \epsilon|V|$ for every i and all but at most ϵk^2 of the pairs (V_i, V_j) are ϵ-regular. The above partition is called *totally* ϵ-*regular* if all the pairs (V_i, V_j) are ϵ-regular. The following celebrated lemma was proved by Szemerédi in [17].

Lemma 2.6 *For every $\epsilon > 0$ there is an integer $M(\epsilon)$ such that every graph of order $n > M(\epsilon)$ has an ϵ-regular partition into k classes, where $k \leq M(\epsilon)$.*

In order to apply the Regularity Lemma we need to show the existence of a complete multipartite subgraph in graphs with a totally ϵ-regular partition. This is established in the following lemma which is a special case of a well-known result, see, e.g., [14].

Lemma 2.7 *For every $\delta > 0$ and integer t there exist an $0 < \epsilon = \epsilon(\delta, t)$ and $n_0 = n_0(\delta, t)$ with the following property. If G is a graph of order $n > n_0$ and $(V_1, \cdots, V_4)$ is a totally ϵ-regular partition of vertices of G such that $d(V_i, V_j) \geq \delta$ for all $i < j$, then G contains a complete 4-partite subgraph $K_4(t)$ with parts of size t.*

Proof of Corollary 1.2. Let H be a fixed 4-chromatic graph of order t and let G be a graph on n vertices not containing H as a subgraph. Suppose $\delta > 0$ and let $\epsilon = \min\big(\delta, \epsilon(\delta, t)\big)$, where $\epsilon(\delta, t)$ is defined in the previous statement. Then, by Lemma 2.6, for sufficiently large n there exists an ϵ-regular partition $(V_1, \cdots, V_k)$ of vertices of G.

Consider a new graph G' on the vertices $\{1, \ldots, k\}$ in which (i, j) is an edge iff (V_i, V_j) is an ϵ-regular pair with density at least δ. We claim that G' contains no K_4. Indeed, any such clique in G' corresponds to 4 parts in the partition of G such that any pair of them is ϵ-regular and has density at least δ. This contradicts our assumption on G, since by Lemma 2.7, the union of these parts will contain a copy of complete 4-partite graph $K_4(t)$ which clearly contains H.

By applying Theorem 1.1 to graph G', we deduce that there is a set D of at most $k^2/9$ edges of G' whose deletion makes it bipartite. Now delete all the edges of G between the pairs (V_i, V_j) with $(i, j) \in D$. Delete also the edges of G that lie within classes of the partition, or that belong to a non-regular pair, or that join a pair of classes of density less than δ. It is easy to see that the remaining graph is bipartite and the number of edges we deleted is at most

$$(k^2/9)(n/k)^2 + \epsilon n^2 + \delta n^2 \leq \big(1/9 + 2\delta\big)n^2 = (1 + o(1))n^2/9. \qquad \square$$

3 Concluding remarks

How many edges do we need to delete to make a K_r-free graph G of order n bipartite? For $r = 3, 4$ this was asked long time ago by P. Erdős. For triangle-free graphs he conjectured that deletion of $n^2/25$ edges is always enough and that extremal example is a blow-up of a 5-cycle. In this paper we answered the question for $r = 4$ and proved that the unique extremal construction in this case is a complete 3-partite graph with equal parts. Our result suggests that a complete $(r - 1)$-partite graph of order n with equal parts is worst example also for all remaining values of r. Therefore we believe that it is enough to delete at most $\frac{(r-2)^2}{4(r-1)^2}n^2$ edges for even $r \geq 5$ and at most $\frac{r-3}{4(r-1)}n^2$ edges for odd $r \geq 5$ to make bipartite any K_r-free graph G of order n. It seems that some of the ideas presented here can be useful to make a progress on this problem for even r.

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