

Combinatorics

Exercise 7 – Hamiltonian graphs

Homework

Deadline: **7. 12. 2022**

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1. Let Q_n be the n -dimensional hypercube graph, i.e. the vertices of Q_n are all $\{0, 1\}$ -strings of length n and two strings are connected by an edge if and only if they differ on exactly one position. Prove that if $n \geq 2$, Q_n has a Hamiltonian cycle
2. Prove that in a bipartite graph whose all vertices of one class of bipartition have odd degrees, the total number of Hamiltonian cycles is even.
3. Let G be a graph on n vertices and let \bar{G} be its complement (i.e. uv is an edge of \bar{G} if and only if it is not an edge of G). Prove that $ch(G) + ch(\bar{G}) \leq n + 1$, where ch denotes the choosability.

Problems

1. Prove that in a bipartite graph whose all vertices of one class of bipartition have odd degrees, every edge belongs to an even number of Hamiltonian cycles.
2. Let G be a cubic graph with a Hamiltonian cycle. Prove that G has at least 3 Hamiltonian cycles.
3. Prove that a planar triangulation with more than 3 vertices cannot contain exactly one Hamiltonian cycle:
 - (a) Let $G = (V, E)$ be such a graph and let $H = x_1, x_2, \dots, x_n$ be a Hamiltonian cycle in G . For every edge e of H , let $M(e)$ be the two triangles incident with e . Prove that M has a system of distinct representatives s .
 - (b) Let F be the domain of s (i.e. the assigned faces) and consider a bipartite graph G' with one partition V , the other partition F and vertex $v \in V$ connected to a face $f \in F$ if and only if $v \in f$. Observe that each Hamiltonian cycle in G' gives a Hamiltonian cycle in G .
 - (c) Prove that G' has at least two Hamiltonian cycles. If at least one of them does not correspond to H , we are done, otherwise do some extra work to prove the main statement anyway.