Combinatorics

Exercise 5 – List colouring

Problems

1. For every k, find a bipartite graph G_k with $ch(G_k) \geq k$. Conclude that one cannot bound $ch(G)$ from above by any function of $\chi(G)$.
2. Let G be a d-degenerate graph (i.e. for every $G' \subseteq G$ we have $\delta(G') \leq d$). Prove that $ch(G) \leq d + 1$.
3. Prove that planar triangle-free graphs have choosability at most 4.
4. Prove that planar bipartite graphs have choosability at most 3. The following might be useful intermediate steps:

   (a) Let G be a graph such that every subgraph $H \subseteq G$ satisfies $|E(H)| \leq k|V(H)|$. Prove that G has an orientation with maximum outdegree $\leq k$

   (b) Prove that every orientation of a bipartite graph has a kernel.

5. Let G be a bipartite graph with n vertices. Prove that $ch(G) \leq \lceil \log_2(n) \rceil + 1$.
6. Let G_n be the star with $n + 1$ vertices (i.e. the graph $K_{1,n}$).

   (a) Observe that $ch(G_n) = 2$.

   (b) Let G be a graph, let k be an integer and consider the following "game" played on G by two players, *Painter* and *Paint shop clerk*. There are k rounds. In each of the k rounds, Painter points at a vertex v of her choice and Paint shop clerk offers her a new colour which has not yet been offered for v. That is, after k rounds, Painter has a list of some colours for every vertex and the sum of the sizes of these lists is k. After this, Painter wins if G has a list colouring using these lists. If k is the smallest number for which Painter has a winning strategy on G in k rounds, we write $ich(G) = \frac{k}{|V(G)|}$.

      i. Observe that $ich(G) \leq ch(G)$.

      ii. Prove that $ich(G_n) \leq 1 + \frac{2\sqrt{n}}{n+1}$ (in other words, one needs just a little bit more than 1 colour per vertex on average).