Combinatorics

Exercise 2 – Hall's theorem & matchings

Homework (reminder)

Deadline: **19. 10. 2022**
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1. Prove or find a counterexample to the following statement: Let I and X be **infinite** sets and let $\mathcal{M} = \{M_i\}_{i \in I}$ be a set system such that $\bigcup \mathcal{M} = X$. If $\mathcal{M}$ satisfies the Hall condition ($\forall J \subseteq I : |\bigcup_{j \in J} M_j| \geq |J|$), then $\mathcal{M}$ has SDR.
2. Let G be a bipartite graph with 42 vertices such that whenever you pick 31 vertices, they will contain at least one edge. Show that G has a matching with at least 12 edges.
3. Prove that a tree has a perfect matching if and only if deleting any vertex creates exactly one component with an odd number of vertices.

Problems

1. Dilworth's theorem says that if a finite poset $(P, \prec)$ has the largest antichain of size r, then P can be decomposed into r chains. Prove that Dilworth's theorem implies the harder implication of Hall's theorem.
2. How many minimum vertex covers and how many minimal vertex covers does the star on n vertices have?[1]
3. Let G be a graph and let $\mu(G)$ be the size of its maximum matching. Prove that every maximal matching in a graph has at least $\frac{\mu(G)}{2}$ edges.
4.
   (a) How many perfect matchings does K_n have? How many of them contain a given fixed edge e?
   (b) Let n be even. Prove that every graph on n vertices with more than $\binom{n-1}{2}$ edges has a perfect matching.
5. There are n points labelled $1, 2, \ldots, n$. Between every pair of them there is an arrow going from the smaller to the larger one. A red-blue colouring of the arrows is *delicious* if there is no pair of points A, B with both blue and red oriented paths from A to B. Prove that there are $n!$ delicious colourings (without using paper and pen if possible).
6. At the Rector's Sports Day there is a prestigious football tennis match between the students and the professors. The rules are as follows: Each time, one of the teams serves the ball and then one of the teams wins a point. The first team to reach n points wins. The professors are the first to serve and then there are two service schemes: Either the teams alternate, or the team to win the last point serves the next ball. Assuming that the probability of winning a point only depends on which team is serving, prove that the probability of winning the match is independent from the choice of the service scheme.

[1] Just understand and remember the difference between minimum (nejmenší) and (inclusion) minimal (minimální vzhledem k inkluzi).