# Combinatorics

Exercise 1 – Hall's theorem & basics

## Homework

Deadline: **19. 10. 2022**
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1. Prove or find a counterexample to the following statement: Let $I$ and $X$ be **infinite** sets and let $\mathcal{M} = \{M_i\}_{i \in I}$ be a set system such that $\bigcup \mathcal{M} = X$. If $\mathcal{M}$ satisfies the Hall condition ($\forall J \subseteq I$ : $|\bigcup_{j \in J} M_j| \geq |J|$), then $\mathcal{M}$ has SDR.
2. Let $G$ be a bipartite graph with 42 vertices such that whenever you pick 31 vertices, they will contain at least one edge. Show that $G$ has a matching with at least 12 edges.
3. Prove that a tree has a perfect matching if and only if deleting any vertex creates exactly one component with an odd number of vertices.

## Problems

1. What if I asked about more iterations?
   (a) Does the system of all 3-elements subsets of $\{1, 2, 3, 4\}$ SDR?
   (b) Does the system of all 3-elements subsets of $\{1, 2, 3, 4, 5\}$ SDR?
2. Dilworth's theorem says that if a finite poset $(P, \prec)$ has the largest antichain of size $r$, then $P$ can be decomposed into $r$ chains. Prove that Dilworth's theorem implies the harder implication of Hall's theorem.
3. For every graph $G = (V, E)$ it holds that
$$\sum_{v \in V} \deg_G(v) = 2|E|.$$
4. If $G = (V, E)$ is a connected graph with $|V| \geq 2$ then there are $u \neq v \in V$ such that both $G - u$ and $G - v$ are connected.
5. Is there a graph with at least two vertices such that all of its vertices have pairwise different degrees?
6. Construct infinitely many graphs isomorphic to their complements.
7. Let $\sigma$ be a permutation of $\{1, \ldots, 2n\}$. We say that $\sigma$ is *splendid* if there is $i$ such that $|\sigma(i) - \sigma(i+1)| = n$, otherwise $\sigma$ is *plain*. Prove that there are more splendid permutations than plain ones (without explicitly counting them if possible).
8. There are $n$ points labelled $1, 2, \ldots, n$. Between every pair of them there is an arrow going from the smaller to the larger one. A red-blue colouring of the arrows is *delicious* if there is no pair of points $A, B$ with both blue and red oriented paths from $A$ to $B$. Prove that there are $n!$ delicious colourings (without using paper and pen if possible).
9. At the Rector's Sports Day there is a prestigious football tennis match between the students and the professors. The rules are as follows: Each time, one of the teams serves the ball and then one of the teams wins a point. The first team to reach $n$ points wins. The professors are the first to serve and then there are two service schemes: Either the teams alternate, or the team to win the last point serves the next ball. Assuming that the probability of winning a point only depends on which team is serving, prove that the probability of winning the match is independent from the choice of the service scheme.