Probabilistic techniques
Problem set #5 - Markov chains

Deadline: 6 January 2022
Please send your solutions to dbulavka@kam.mff.cuni.cz

1. By homogeneous discrete time Markov chain we mean a sequence of random variables $X_0, X_1, \dots$ such that for every n we have $\Pr[X_{n+1} = i_{n+1} | X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0] = \Pr[X_{n+1} = i_{n+1} | X_n = i_n]$.
   - Let $Y_0, Y_1, \dots$ be integers chosen independently uniformly from $\{-1, 0, 1\}$ and let $X_n = \max\{Y_0, \dots, Y_n\}$. Decide whether $X_0, X_1, \dots$ is a homogeneous discrete time Markov chain. **[1 point]**
   - Let $Y_0, Y_1, \dots$ be integers chosen independently uniformly from $\{-1, 0, 1\}$ and let $X_n = \max\{Y_n, Y_{n-1}\}$. Decide whether $X_0, X_1, \dots$ is a homogeneous discrete time Markov chain. **[1 point]**
2. Construct satisfiable 2-SAT formulae with n variables (for arbitrary large n) such that the randomized 2-SAT algorithm from the lecture takes $\Omega(n^2)$ steps (in expectation) to find a satisfying assignment. (Recall that in each step the algorithm picks an arbitrary unsatisfied clause and flips one of its variables) **[2 points]**
3. We say that a graph G is triangle 2-colorable if there is a 2-coloring of the vertices such that no triangle in G is monochromatic. (For example, a 3-colorable graph is triangle 2-colorable.) Let G be a 3-colorable graph. Start with $c\colon V(G) \to \{0, 1\}$ being constant 0, and then do the following: while there is a monochromatic triangle in G (with respect to c), pick one of its vertices at random and flip its color. Show that the expected number of steps for this process to find a suitable triangle 2-coloring is polynomial in $|V(G)|$. **[3 points]**
4. Show that in one communicating class of a Markov chain, there cannot be a state which is recurrent and another state which is transient. **[2 points]**
5. In a finite Markov chain show that the following holds:
   - There is at least one recurrent state. **[2 points]**
   - all recurrent state are positive recurrent (i. e., the expected time until the process returns to this state is less than infinity). **[2 points]**
6. Let X be a Poisson random variable with mean $\mu \in \mathbb{Z}$. Prove that $\Pr[X \geq \mu] \geq 1/2$. (Hint: show that $\Pr[X = \mu + h] \geq \Pr[X = \mu - h - 1]$ for every $0 \leq h \leq \mu - 1$.) **[2 points]**
7. Throw m balls uniformly independently into n bins, where the bins are numbered from 0 to $n - 1$. We say that there is a k-gap starting at j if bins $j, j + 1, \dots, j + k - 1$ are all empty.
   - Compute the expected number of k-gaps. **[1 points]**
   - Prove a Chernoff-like bound for the number of k-gaps X, that is, the inequality

     $$\Pr[X \geq (1 + \delta)\mathbb{E}[X]] \leq e^{-c\mathbb{E}[X]\delta^2}$$

     where $0 < \delta < 1$ and c is a positive constant. **[4 points]**