Probabilistic techniques
Problem set #4 - Lovász and Chernoff

Hints: 09 December 2021
Deadline: 16 December 2021
Please send your solutions to dbulavka@kam.mff.cuni.cz

1. Prove that for every $0 < p < 1$ it holds that

$$\lim_{n\to\infty} \Pr[G(n,p) \text{ is } \frac{np^2}{2}\text{-vertex connected}] = 1.$$

[2 points]

2. The van der Waerden number $W(r,k)$ is the smallest number n such that for every coloring of integers $\{1,\ldots,n\}$ using r colors there exists an arithmetic progression of length k that is monochromatic. Prove that $W(2,k) \in \Omega(2^k/k)$. **[2 points]**

3. Let $G = (V,E)$ be an undirected graph where each vertex $v \in V$ is assigned a list $L(v)$ of allowed colors. A proper list coloring of G assigns to each vertex $v \in V$ a color from $L(v)$ such that adjacent vertices have different colors. Let $k \geq 1$ be an integer and suppose the following conditions hold,

   - For each $v \in V$ we have that $|L(v)| \geq 8k$ and
   - For each $v \in V$ and $c \in L(V)$, there are at most k neighbors u of v such that $L(u)$ contains c.

   Show that G has a proper list coloring. **[4 points]**

4. The company Škoda has n^2 employees and it is time to select a new executive board composed of no more than n employees. The election goes as follows, each employee chooses an integer between 1 and n, without knowing the other employees choice. Then, the employees with the least frequent number are elected - if there are more least frequent numbers, we pick one of them randomly.

   Unfortunately, among the n^2 employees there are $n^2/10$ bad employees who want to ruin the company, the rest are good employees. The good employees choose the number during the election uniformly at random, while the bad employees can agree on a strategy and cooperate. Show that it is unlikely that more than one fifth of the board is composed of bad employees, i.e the probability of this happening goes to zero as n goes to infinity. **[4 points]**

5. Let σ be a uniformly random permutation of $[n] = \{1,\ldots,n\}$. Denote $X = |\{i \in [n] : (\forall j < i)\sigma(j) < \sigma(i)\}|$. Prove that for every $\epsilon > 0$ it holds that

$$\lim_{n\to\infty} \Pr[(1-\epsilon)H_n < X < (1+\epsilon)H_n] = 1,$$

   where $H_n = \sum_{i=1}^{n} \frac{1}{i}$. **[2 points]**

6. Consider $G(n,p)$ and let T_v be the number of triangles containing the vertex v. Prove that for every $0 < \epsilon < 1$ and for every vertex v it holds that

$$\lim_{n\to\infty} \Pr[(1-\epsilon)n^2p^3/2 \leq T_v \leq (1+\epsilon)n^2p^3/2] = 1$$

   . **[4 points]**