Probabilistic Techniques

Problem set #2 – Expectation and the method of alteration

Assignment:	14. 10. 2021
Hints:	28. 10. 2021
Deadline:	4. 11. 2021

The method of alteration, which might be needed for some (at most two) of the problems, should be presented at the lecture on 19. 10. I will briefly introduce it on some examples at the tutorial session

1. Chad owns n safes and for each of them he has exactly one key. Very late last night he thought that it would be a fun idea to randomly lock one key into each safe.[1] (Randomly means that out of all the possibilities, he picked one uniformly at random.) Now he regrets his decision and wants his keys back. Compute the expected number of safes he will have to break into in order to retrieve all the keys. **[3 points]**

2. Let M be an $n \times n$ matrix with entries uniformly independently chosen from $\{-1, 1\}$. Determine
   (a) $\mathbb{E}[\det(M)]$, **[1 point]**
   (b) $\mathbb{E}[\det(M^2)]$. **[2 points]**

3. Let $G = (V, E)$ be a bipartite graph with n vertices such that each vertex $v \in V$ is assigned a list $L(v)$ of $\lceil \log_2(n) \rceil + 1$ colours. Prove that there is a proper colouring of G such that each vertex v gets a colour from $L(v)$. **[2 points]**

4. Let G be a graph with n vertices, let d be its average degree, Δ its maximum degree and $\alpha(G)$ the size of its largest independent set (that is, induced subgraph with no edges). Show that:
   (a) $\alpha(G) \geq \frac{n}{\Delta+1}$, **[1 point]**
   (b) $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v)+1}$, **[3 points]**
   (c) $\alpha(G) \geq \frac{n}{d+1}$. **[1 point]**

   You can get the point for (c) even if you only prove the implication (b)⇒(c) without proving (b).

5. Chad is taking this year's incarnation of *Probabilistic Techniques I* and is quite hopeless with the homework. He asked for further hints and Matěj, being his magnanimous self, gave Chad the following offer:

   > *"Yes, I will give you further hints, but in order to get one hint, you must do 10 push-ups. But there is a catch: Each time you do 10 push-ups, I shall only give you a hint to a problem which is chosen uniformly independently randomly from all n of them."*

   Chad is now wondering what the expected number of push-ups he needs to do in order to get at least one hint for every problem is, and because he really sucks with probability, he asked you for help.

   But because cooperation is forbidden for this course, you have to send your solutions to Matěj instead! **[2 points]**

6. Prove that for every n there is a bipartite graph with both parts of size n, at least $\Omega\left(n^{4/3}\right)$ edges, but with no $K_{2,2}$ as a subgraph. **[2 points]**

7. Prove that $R(4, t) \in \Omega\left((t/\log t)^2\right)$, where $R(k, \ell)$ is the smallest number of vertices of a complete graph such that in every 2-colouring of its edges one can find either a K_k in the first colour or a K_ℓ in the second colour. **[2 points]**

   Hint: You might want to first prove (and you will get partial points for it) that for every $n, k, \ell \in \mathbb{N}$ and every $p \in [0, 1]$ it holds that

   $$R(k, l) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell} (1-p)^{\binom{\ell}{2}}.$$

https://kam.mff.cuni.cz/~matej/teaching/2122/pt

[1] The safes can be locked without the keys. Unfortunately, this is not true for unlocking them.