Probabilistic Techniques

Problem set #1 – The basics

Assignment:	30. 9. 2021
Hints:	9. 10. 2021 (!!)
Deadline:	14. 10. 2021
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By *classical probability space* we denote the probability space $(\Omega, 2^\Omega, \Pr)$ where Ω is a finite set and $\Pr[A] = |A|/|\Omega|$. We define $[n] = \{0, 1, \ldots, n-1\}$.

If you do not understand something (a term, notation, the whole problem, ...), **please** send me an e-mail. Even if you in fact should know it from some previous course. It is not fun to grade homework solving a different problem :).

1. Consider a classical probability space on p elements, where p is a prime number. Let A and B be two events. Show that A and B are independent **if and only if** one of them is $\emptyset$ or Ω. **[1 point]**

2. Compute the probability that in a random permutation of $[n]$, the elements 0 and 1 are in one cycle. **[3 points]**

3. Prove that there exists an absolute constant $c > 0$ such that for every n and every $n \times n$ matrix A with pairwise distinct entries, there is a permutation of columns of A such that no row contains an increasing subsequence of length greater than $c\sqrt{n}$. **[4 points]**

4. Consider the classical probability space on an underlying set with 8 elements. Find an example of four events A, B, C, D such that:
   - all triples of them are independent,
   - the four events are not independent. **[2 points]**

5. Find an example of events A, B, C in a classical probability space such that they are not independent, but it holds that
$$\Pr[A \cap B \cap C] = \Pr[A] \Pr[B] \Pr[C].$$
[1 point]

6. Recall that $G(n,p)$ is a random graph of n vertices such that every pair of vertices forms an edge with probability p independently of every other pair. Show that
$$\lim_{n \to \infty} \Pr[G(n, 1/2) \text{ is connected}] = 1.$$
[4 points]

7. Prove that there exist constants $c_1, c_2 > 0$ such that for every pair of integers n and m:
   (a) If $m \geq c_1 n^2$ then a random mapping from $[n]$ to $[m]$ is injective[1] (1-to-1) with probability at least 0.99. **[1 point]**
   (b) If $m \leq c_2 n^2$ then a random mapping from $[n]$ to $[m]$ is injective with probability at most 0.01. **[2 points]**

https://kam.mff.cuni.cz/~matej/teaching/2122/pt

[1] in Czech "prosté"