Combinatorics

Exercise 7 – Hamiltonian graphs

Homework

Deadline: **15. 12. 2021 9:00 (!)**
Send to: matej@kam.mff.cuni.cz (in PDF, preferably LaTeX, of course)

1. Let Q_n be the n-dimensional hypercube graph, i.e. the vertices of Q_n are all $\{0,1\}$-strings of length n and two strings are connected by an edge if and only if they differ on exactly one position. Prove that if $n \geq 2$, Q_n has a Hamiltonian cycle
2. Prove that in a bipartite graph whose all vertices of one class of bipartition have odd degrees, the total number of Hamiltonian cycles is even.
3. Let G be a graph on n vertices and let $\bar{G}$ be its complement (i.e. uv is an edge of $\bar{G}$ if and only if it is not an edge of G). Prove that $ch(G) + ch(\bar{G}) \leq n + 1$, where ch denotes the choosability.

Problems

1. Prove that in a bipartite graph whose all vertices of one class of bipartition have odd degrees, every edge belongs to an even number of Hamiltonian cycles.
2. Let G be a cubic graph with a Hamiltonian cycle. Prove that G has at least 3 Hamiltonian cycles.
3. Prove that a planar triangulation with more than 3 vertices cannot contain exactly one Hamiltonian cycle:

   (a) Let $G = (V, E)$ be such a graph and let $H = x_1, x_2, \ldots, x_n$ be a Hamiltonian cycle in G. For every edge e of H, let $M(e)$ be the two triangles incident with e. Prove that M has a system of distinct representatives s.

   (b) Let F be the domain of s (i.e. the assigned faces) and consider a bipartite graph G' with one partition V, the other partition F and vertex $v \in V$ connected to a face $f \in F$ if and only if $v \in f$. Observe that each Hamiltonian cycle in G' gives a Hamiltonian cycle in G.

   (c) Prove that G' has at least two Hamiltonian cycles. If at least one of them does not correspond to H, we are done, otherwise do some extra work to prove the main statement anyway.