Combinatorics

Exercise 2 – Hall's theorem

Homework

Deadline: **20. 10. 2021**
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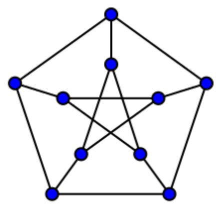
1. Prove or find a counterexample to the following statement: Let I and X be **infinite** sets and let $\mathcal{M} = \{M_i\}_{i \in I}$ be a set system such that $\bigcup \mathcal{M} = X$. If $\mathcal{M}$ satisfies the Hall condition ($\forall J \subseteq I$: $|\bigcup_{j \in J} M_j| \geq |J|$), then $\mathcal{M}$ has SDR.
2. Let G be a bipartite graph with 42 vertices such that whenever you pick 31 vertices, they will contain at least one edge. Show that G has a matching with at least 12 edges.
3. Prove that a tree has a perfect matching if and only if deleting any vertex creates exactly one component with an odd number of vertices.

Problems

1. What if I asked about more iterations?
   (a) Does the system of all 3-elements subsets of $\{1, 2, 3, 4\}$ SDR?
   (b) Does the system of all 3-elements subsets of $\{1, 2, 3, 4, 5\}$ SDR?
2. Find 6 different perfect matchings in the Petersen graph and prove that there are no more of them.



3. Dilworth's theorem says that if a finite poset $(P, \prec)$ has the largest antichain of size r, then P can be decomposed into R chains. Prove that Dilworth's theorem implies the harder implication of Hall's theorem.