Combinatorics

Exercise 1 – Stalling for time (there still hasn't been a lecture - or has it?)

Problems

1. Prove that the vertices of every k-uniform hypergraph with less than 2^k hyperedges ($k \geq 2$) can be 2-coloured so that no hyperedge is monochromatic.
2. Prove that the vertices of every k-uniform hypergraph with less than $\frac{4^{k-1}}{3^k}$ hyperedges ($k \geq 2$) can be 4-coloured so that every hyperedge contains all colours.
3. Let G be a graph with m edges. Prove that G contains a bipartite subgraph with at least $\frac{m}{2}$ edges.
4. Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be unit vectors. Prove that there are $c_1, \ldots, c_n \in \{-1, 1\}$ such that
$$\|c_1 v_1 + \cdots + c_n v_n\| \geq \sqrt{n}.$$
5. Let G be a bipartite graph with n vertices and suppose that each vertex is given a personalised list of more that $\log_2(n)$ possible colours. Prove that it is possible to pick a colour for each vertex from its list so that no two adjacent vertices share the same colour.
6. Prove that for every $n \geq 2$ there exists a graph on $2^{\frac{n}{2}}$ vertices containing no cliques and independent sets on n vertices.
7. (Kraft inequality) Let $\mathcal{F}$ be a finite subset of $\{0, 1\}^{<\omega}$ such that no member of $\mathcal{F}$ is a prefix of another one. Let N_i denote the number of strings of length i in $\mathcal{F}$. Prove that $\sum_i \frac{N_i}{2^i} \leq 1$.
8. (Erdös–Ko–Rado theorem) Let $n \geq 2k$ be positive integers and let $\mathcal{C}$ be a collection of pairwise intersecting k-element subsets of $\{0, \ldots, n-1\}$. Prove that $|\mathcal{C}| \leq \binom{n-1}{k-1}$ and construct such a collection attaining this bound.
9. (IMO 2011, Windmill problem – not a probabilistic proof) Let $\mathcal{P}$ be a finite set of points in the plane in general position. Given $p \in \mathcal{P}$ and a line ℓ passing through p, we start rotating the line clockwise centered in p. Whenever the line hits another point $q \in \mathcal{P}$, we continue rotating it around q (then it hits yet another point, so that points becomes the new *pivot* etc). Prove that it is possible to select a starting $p \in \mathcal{P}$ and a line ℓ such that this process hits every member of $\mathcal{P}$ infinitely many times.