Combinatorics

Exercise 0 – The basics & non-counting

Problems

1. For every graph $G = (V, E)$ it holds that
$$\sum_{v \in V} \deg_G(v) = 2|E|.$$
2. If $G = (V, E)$ is a connected graph with $|V| \geq 2$ then there are $u \neq v \in V$ such that both $G - u$ and $G - v$ are connected.
3. Is there a graph with at least two vertices such that all of its vertices have pairwise different degrees?
4. Construct infinitely many graphs isomorphic to their complements.
5. Let σ be a permutation of $\{1, \ldots, 2n\}$. We say that σ is *splendid* if there is i such that $|\sigma(i) - \sigma(i+1)| = n$, otherwise σ is *plain*. Prove that there are more splendid permutations than plain ones (without explicitly counting them if possible).
6. There are n points labelled $1, 2, \ldots, n$. Between every pair of them there is an arrow going from the smaller to the larger one. A red-blue colouring of the arrows is *delicious* if there is no pair of points A, B with both blue and red oriented paths from A to B. Prove that there are $n!$ delicious colourings (without using paper and pen if possible).
7. At the Rector's Sports Day there is a prestigious football tennis match between the students and the professors. The rules are as follows: Each time, one of the teams serves the ball and then one of the teams wins a point. The first team to reach n points wins. The professors are the first to serve and then there are two service schemes: Either the teams alternate, or the team to win the last point serves the next ball. Assuming that the probability of winning a point only depends on which team is serving, prove that the probability of winning the match is independent from the choice of the service scheme.