Probabilistic techniques
Problem set #3 - Markov and Chebyshev inequalities

Hints: 17 November 2020
Deadline: 24 November 2020
Please send your solutions to dbulavka@kam.mff.cuni.cz

1. Let $n \geq 2$ be a positive integer and $v_1 = (x_1, y_1), \dots, v_n = (x_n, y_n)$ vectors with x_i and y_i integers such that $x_i^2, y_i^2 \leq \frac{1}{10000}\frac{2^n}{n}$. Prove that there exist two non-empty disjoint subsets $I, J \subset [n]$ such that
$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$
[4 points]

2. Let X denote the number of isolated vertices in $G(n, p(n))$ with $p(n) = c\frac{\ln(n)}{n}$. Show that
   (a) $\lim_{n\to\infty} \Pr[X = 0] = 1$ for $c > 1$. **[2 points]**
   (b) $\lim_{n\to\infty} \Pr[X \geq 1] = 1$ for $0 \leq c < 1$. **[3 points]**

3. Let G be a graph with n vertices, b and m be positive integers. If G has at least $(b+m)n/2$ edges prove that there exists a subset $B \subset V(G)$ of size at least b such that each vertex of B has at least m neighbors. **[3 points]**

4. Let X be a non-negative integer random variable such that $\mathbb{E}[X^2]$ is finite and nonzero. Prove that
$$\Pr[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]}.$$
[2 points]

5. Prove that
$$\lim_{n\to\infty} \Pr[G(n, 1/2) \text{ has an induced cycle of lenght } > 3\log_2 n] = 0.$$
Remember that $(v_1, \dots, v_k)$ is an induced cycle of length k if it satisfies that v_iv_j is an edge if and only if $j = i + 1 \mod (k)$. **[3 points]**

6. Let X be a real random variable with $\mathrm{Var}[X] = \sigma^2$ and $\mathbb{E}[X] = 0$. For every real number $\lambda > 0$ prove the inequality
$$\Pr[X \geq \lambda] \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$
[2 points]