Probabilistic Techniques

Problem set #1 – The basics

Assignment:	29. 9. 2020
Hints:	6. 10. 2020
Deadline:	13. 10. 2020
Send to:	Matěj (*matej@kam.mff.cuni.cz*)

By *classical probability space* we denote the probability space $(\Omega, 2^\Omega, \Pr)$ where Ω is a finite set and $\Pr[A] = |A|/|\Omega|$. We define $[n] = \{0, 1, \ldots, n-1\}$.

If you do not understand something (a term, notation, the whole problem, ...), **please** send me an e-mail. Even if you in fact should know it from some previous course. It is not fun to grade homework solving a different problem :).

1. You flip a coin 6 times. Compute the probability of the event "There is an even number of heads or there are exactly 3 heads and 3 tails". **[1 point]**

2. Compute the probability that in a random permutation of $[n]$, the elements 0 and 1 are in one cycle. **[3 points]**

3. Prove that there exists an absolute constant $c > 0$ such that for every n and every $n \times n$ matrix A with pairwise distinct entries, there is a permutation of columns of A such that no row contains an increasing subsequence of length greater than $c\sqrt{n}$. **[4 points]**

4. Consider the classical probability space on an underlying set with 8 elements. Find an example of four events A, B, C, D such that:

   - all triples of them are independent,
   - the four events are not independent.

   [2 points]

5. Find an example of events A, B, C in a classical probability space such that they are not independent, but it holds that

$$\Pr[A \cap B \cap C] = \Pr[A] \Pr[B] \Pr[C].$$

   [1 point]

6. Recall that $G(n, p)$ is a random graph of n vertices such that every pair of vertices forms an edge with probability p independently of every other pair. Show that

$$\lim_{n\to\infty} \Pr[G(n, 1/2) \text{ is connected}] = 1.$$

   [4 points]

7. Chad's favourite number is k. He recently bought a coin with probability $0 \le p \le 1$ for heads and decided to toss it n times. Before doing that, he did some calculations and realised that the events "*a head is obtained on the first toss*" and "*exactly k heads are obtained*" are independent. Determine all possibilities for k (depending on p and n). **[2 points]**