Probabilistic Techniques

Problem set #3 — Неравенства Маркова и Чебышёва

Assignment:	6. 11. 2019
Hints:	20. 11. 2019
Deadline:	27. 11. 2019

1. During the first tutorial session, we proved that each graph G of minimum degree δ has a dominating set of size at most $n\frac{1+\ln(\delta+1)}{\delta+1}$. But who would be satisfied with just a dominating set! Prove that there exists a constant $c > 0$, such that in every graph G on n vertices with minimum degree $\delta > 1$, there exists a subset $A \subseteq V(G)$ of size $|A| \leq cn\frac{\log(\delta)}{\delta}$, such that for each vertex $u \in V(G) \setminus A$, there are vertices $v \in A$ and $w \in V(G) \setminus A$, such that u is connected by an edge to both v and w. **[2 points]**

2. Let x be a point chosen uniformly randomly from the unit square $[0,1]^2$, that is, each of its coordinates is a random number chosen uniformly independently from the unit interval $[0,1]$. Let $X = \|x\|_2^2$ be the random variable computing the square of the Euclidean norm of x.
   (a) Compute $\Pr[X > 1]$. **[1 point]**
   (b) Compute $\mathbb{E}[X]$. What does Markov's inequality say about $\Pr[X > 1]$? **[1 point]**
   (c) Compute $\mathrm{Var}[X]$. **[1 point]**

3. Let $n \geq 2$ be a positive integer, and let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ be two-dimensional vectors, such that all x_i's and y_i's are integers, and furthermore $x_i^2, y_i^2 \leq \frac{1}{10000}\frac{2^n}{n}$. Prove that there exist two **non-empty disjoint** subsets $I, J \subseteq [n]$, such that
$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$
[4 points]

4. Let X denote the number of isolated vertices in $G(n, p(n))$ where $p(n) = c\frac{\ln(n)}{n}$. Show that:
   (a) $\lim_{n\to\infty} \Pr[X = 0] = 1$ for $c > 1$. **[2 points]**
   (b) $\lim_{n\to\infty} \Pr[X \geq 1] = 1$ for $0 \leq c < 1$. **[3 points]**

5. Show that for every $\varepsilon > 0$ it holds that
$$\lim_{n\to\infty} \Pr[\omega(G(n, 1/2)) \geq (2+\varepsilon)\log_2(n)] = 0,$$
where $\omega(G)$ is the size of the largest clique in graph G. **[3 points]**
Note: In fact, this holds even with $\varepsilon = 0$.

6. Let X be a non-negative integer random variable such that $\mathbb{E}[X^2]$ is finite and nonzero (and thus $\mathbb{E}[X]$ is finite, too). Prove that
$$\Pr[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]}.$$
[2 points]