

Probabilistic Techniques

Problem set #1 – The basics

Assignment: 3.10.2019
Hints: 9.10.2019
Deadline: 16.10.2019

By *classical probability space* we denote the probability space $(\Omega, 2^\Omega, \Pr)$ where Ω is a finite set and $\Pr[A] = |A|/|\Omega|$. We define $[n] = \{0, 1, \dots, n-1\}$.

If you do not understand something (a term, notation, the whole problem, ...), **please** send me an e-mail. Even if you in fact should know it from some previous course. It is not fun to grade homework solving a different problem :).

1. Consider a classical probability space on p elements, where p is a prime number. Let A and B be two events. Show that A and B are independent **if and only if** one of them is $\emptyset$ or Ω . [1 point]
2. Compute the probability that in a random permutation of $[n]$, the elements 0 and 1 are in one cycle. [3 points]
3. Prove that there exists an absolute constant $c > 0$ such that for every n and every $n \times n$ matrix A with pairwise distinct entries, there is a permutation of columns of A such that no row contains an increasing subsequence of length greater than $c\sqrt{n}$. [4 points]
4. Consider the classical probability space on an underlying set with 8 elements. Find an example of four events A, B, C, D such that:
 - all triples of them are independent,
 - the four events are not independent.[2 points]

5. Find an example of events A, B, C in a classical probability space such that they are not independent, but it holds that

$$\Pr[A \cap B \cap C] = \Pr[A] \Pr[B] \Pr[C].$$

[1 point]

6. Recall that $G(n, p)$ is a random graph of n vertices such that every pair of vertices forms an edge with probability p independently of every other pair. Show that

$$\lim_{n \rightarrow \infty} \Pr[G(n, 1/2) \text{ is connected}] = 1.$$

[4 points]

7. Prove that there exist constants $c_1, c_2 > 0$ such that:

- (a) If $m \geq c_1 n^2$, then a random mapping from $[n]$ to $[m]$ is injective¹ (1-to-1) with probability at least 0.99. [1 point]
- (b) If $m \leq c_2 n^2$, then a random mapping from $[n]$ to $[m]$ is injective with probability at most 0.01. [2 points]