Friday, March 13, 2020 8:25 AM

1) $a_0 = \sqrt{2}$, $a_{n+1} = \sqrt{2 - a_n}$ $\qquad \underbrace{a_n > 0}$

Pokud existuje vlastní limita $L = \lim a_n$

tak splňuje $L = \lim a_n = \lim a_{n+1} = \lim \sqrt{2 - a_n} = \sqrt{2-L}$

$L = \sqrt{2-L}$ ... $L^2 + L - 2 = 0$

$(L+2)(L-1) = 0$ (Viettovy vzorce) $(x-A)(x-B) = 0$

$L = -2 \quad L = 1$ $\qquad x^2 - \underbrace{(A+B)}x + \underbrace{AB} = 0$

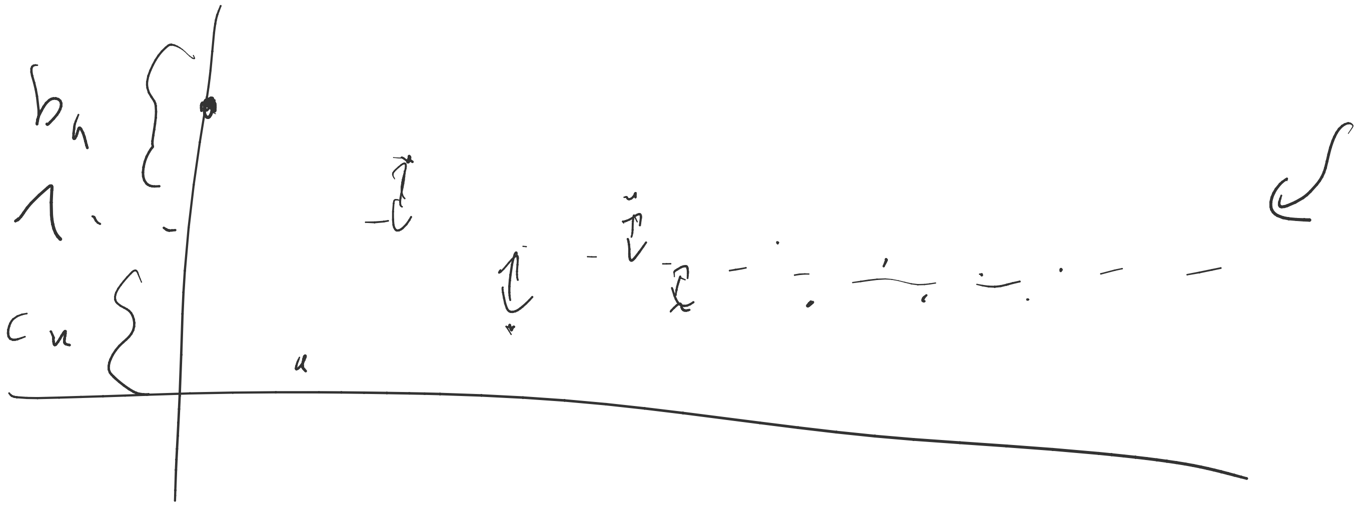
< 0

$a_n \leq \sqrt{2}$. Pro a_0 to platí, jinak $a_{n+1} = \sqrt{\underbrace{2 - a_n}_{>0}} \leq \sqrt{2}$

$\left[\begin{array}{l} a_n > 1 \text{ pro } n \text{ sudé} \\ a_n < 1 \text{ pro } n \text{ liché} \end{array}\right.$ $\qquad \underbrace{a_0 \text{ platí}}$ $\qquad$ Indukcí

$\underbrace{a_n > 1}$, $a_{n+1} = \sqrt{\underbrace{2 - a_n}_{<1}} < 1$ $\qquad n \to n+1$

$a_n < 1$, $a_{n+1} = \sqrt{\underbrace{2 - a_n}_{>1}} > 1$



$b_n = a_{2n}$

$c_n = a_{2n+1}$

$b_{n+1} = a_{2n+2} = \sqrt{2 - a_{2n+1}} = \underbrace{\sqrt{2-\sqrt{2-b_n}}}_{\parallel \; a_{2n}}$

$c_{n+1} = \sqrt{2-\sqrt{2-c_n}}$

Cíl: $b_n \to 1$. Víme, že $b_n \le \sqrt{2}$

že $b_n > 1$

Chceme b_n monotonní, tj. $b_n > \sqrt{2-\sqrt{2-b_n}}$

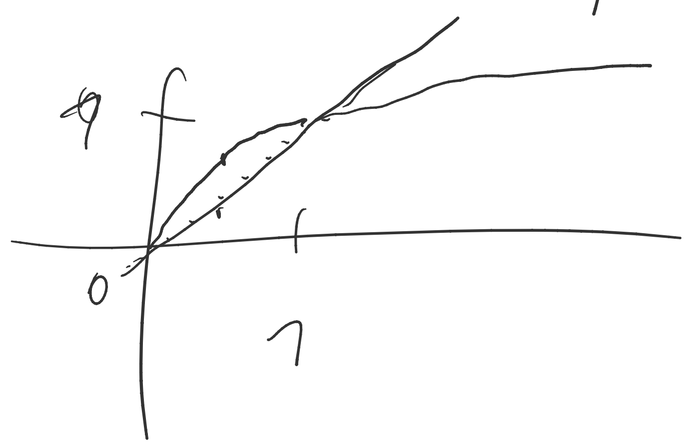
$b_n > b_{n+1}$

$b_n > b_{n+1}$

Dokážeme, že $2 > x > 1 \Rightarrow \boxed{x > \sqrt{2-\sqrt{2-x}}}$

$\Leftrightarrow x^2 > 2 - \sqrt{2-x} \Leftrightarrow \sqrt{2-x} > 2 - x^2$

Všimneme si, že $0 < 2-x < 1 \Rightarrow \boxed{\sqrt{2-x} > 2-x}$

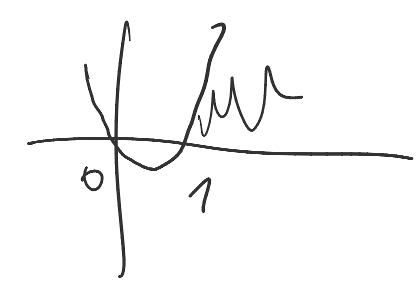


$2-x > 4-4x+x^2 \dashrightarrow$

$x > 1 \Rightarrow x^2 > x \Rightarrow 2-x > 2-x^2$

$x^2 - x > 0$

$x(x-1)$



$\sqrt{2-x} > 2-x > 2-x^2$

b_n monotonní & omezená $\Rightarrow$ má limitu L

$b_n > 1 \Rightarrow L \geq 1$. $b_n \leq \sqrt{2} \Rightarrow L \leq \sqrt{2} < 2$

Víme tedy $L = \sqrt{2-\sqrt{2-L}}$. kdyby $L > 1 \Rightarrow$ spor S

$\Rightarrow L = 1$. Tedy $\underbrace{b_n \to 1}$

Zbývá dokázat, že $\lim c_n = 1$. z toho dostaneme $\lim a_n = 1$

Dokážeme $|1-c_n| < |b_n - 1| \quad \forall n$

$$\Leftrightarrow 1 - c_n < b_n - 1$$

$$\Leftrightarrow 1 - a_{2n+1} < a_{2n} - 1$$

$$a_{2n+1} > 2 - a_{2n}$$

$$\sqrt{2 - a_{2n}}$$

víme $a_{2n} > 1$

$$\Rightarrow 2 - a_{2n} < 1$$

$$\Rightarrow 2 - a_{2n} < \sqrt{2 - a_{2n}}$$

$\square$