Probabilistic Techniques

Problem set #4 — Lovász and Chernoff

Assignment: 28. 11. 2018
Hints: 12. 12. 2018
Deadline: 19. 12. 2018

Theorem (Chernoff inequality)**.** *Suppose $X_1, \dots, X_n$ are independent random variables with values in $\{0,1\}$, $X = \sum_i X_i$ and $\mu = \mathbb{E}[X]$. Then*

$$\Pr[X \le (1-\delta)\mu] \le e^{-\frac{\delta^2\mu}{2}}, \qquad 0 \le \delta \le 1,$$

$$\Pr[X \ge (1+\delta)\mu] \le e^{-\frac{\delta^2\mu}{2+\delta}}, \qquad 0 \le \delta.$$

Note that $2+\delta \le 3$ for $\delta \le 1$ and $2+\delta \le 3\delta$ for $\delta \ge 1$, this can further simplify the second bound.

1. Chad bought $n \ge 1$ new crayons and coloured (not necessarily properly) the vertices of his favourite C_{11n} (cycle on $11n$ vertices) such that each colour is used on precisely 11 vertices. Prove that no matter how he coloured them, he can choose one point of each colour such that no two are adjacent. **[2 points]**
   Bonus: Explain Chad's motivation for doing it. **[0 points]**

2. Prove that for every $0 < p < 1$ it holds that **[2 points]**

$$\lim_{n\to\infty} \Pr[G(n,p) \text{ is } \frac{np^2}{2}\text{-vertex connected }] = 1.$$

3. The van der Waerden theorem says that there are arbitrarily long finite arithmetic sequences in every finite colouring of $\mathbb{N}$. The van der Waerden number $W(k,n)$ is the smallest number N such that in every k-colouring of $\{1,\dots,N\}$ there is a monochromatic arithmetic sequence of length n. Prove that $W(2,n) \in \Omega(2^n/n)$. **[3 points]**

4. Let π be a (uniformly) random permutation of $\{1,2,\dots,n\}$. Denote $X = |\{i \in [n] : (\forall j < i)(\pi_j < \pi_i)\}|$. Prove that for every $\varepsilon > 0$ it holds that ($H_n = \sum_{i=1}^n \frac{1}{i}$) **[2 points]**

$$\lim_{n\to\infty} \Pr[(1-\varepsilon)H_n < X < (1+\varepsilon)H_n] = 1.$$

5. Let $G = (V,E)$ be a graph and consider a function L which assigns each vertex a list $L(v)$ of colours. Let $k \ge 1$ be an integer and suppose that the following hold for every $v \in V$:

   (a) $|L(v)| = 8k$, and
   (b) for every colour $c \in L(v)$ it holds that $|\{u \in V : uv \in E \text{ and } c \in L(u)\}| \le k$.

   Prove that under these assumptions there is a proper colouring of G assigning each vertex v a colour from $L(v)$. **[4 points]**

6. Consider $G(n,p)$ and let T_v be the number of triangles containing vertex v. Prove that for every $0 < \varepsilon < 1$ and for every vertex v it holds that **[4 points]**

$$\lim_{n\to\infty} \Pr[(1-\varepsilon)n^2p^3/2 \le T_v \le (1+\varepsilon)n^2p^3/2] = 1.$$

7. By mistake, you signed up for the Probabilistic techniques III class, however you are totally lost. And the exam is comming closer and closer! Fortunately, the students are allowed to take the exam in pairs. Unfortunately, Chad is the only other person taking the class. The test will consist of n yes/no answers. And by a complicated neurosurgical study you found out that for this type of questions, Chad is slightly better than a random generator: He gives the correct answer with probability $1/2 + 1/p(n)$, where $p(n)$ is some given polynomial in n which is positive for $n \ge 1$. (Note that, quite remarkably, Chad's answers are completely independent, even if you ask him the same question multiple times.)

   Realize (and prove) that you can use Chad to get at least $(1-2^{-n})n$ answers correctly (in expectation). Note that due to time constraints, you will only be able to ask him polynomially many (in n) questions. **[2 points]**