Probabilistic Techniques

Problem set #1 – The basics

Assignment:	3.10.2018
Hints:	10.10.2018
Deadline:	17.10.2018

By *classical probabilistic space* we denote the probabilistic space $(\Omega, 2^{\Omega}, \Pr)$ where Ω is a finite set and $\Pr[A] = |A|/|\Omega|$.

1. Consider a classical probability space on p elements, where p is a prime number. Let A and B be two events. Show that A and B are independent if and only if one of them is $\emptyset$ or Ω. **[1 point]**

2. Compute the probability that in a random permutation of $1, 2, \ldots, n$, the elements 1 and n are in one cycle. **[3 points]**

3. Prove that there exists an absolute constant $c > 0$ such that for every n and every $n \times n$ matrix A with pairwise distinct entries, there is a permutation of columns of A such that no row contains an increasing subsequence of length greater than $c\sqrt{n}$. **[4 points]**

4. Consider the classical probability space on an underlying set with 8 elements. Find an example of four events A, B, C, D such that:
   - all triples of them are independent,
   - the four events are not independent. **[2 points]**

5. Find an example of events A, B, C in a classical probability space such that they are not independent, but it holds that
$$\Pr[A \cap B \cap C] = \Pr[A]\Pr[B]\Pr[C].$$
[1 point]

6. Recall that $G(n, p)$ is a random graph of n vertices such that every pair of vertices forms an edge with probability p independently of every other pair. Show that
$$\lim_{n\to\infty} \Pr[G(n, 1/2) \text{ is connected}] = 1.$$
[4 points]

7. Chad's favourite number is k. He recently bought a coin with probability $0 \le p \le 1$ for heads and decided to toss it n times. Before doing that, he did some calculations and realised that the events "*a head is obtained on the first toss*" and "*exactly k heads are obtained*" are independent. Determine all possibilities for k (depending on p and n). **[2 points]**