**MASTER THESIS (Draft)**

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# Semigroup-valued metric spaces

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# 1. Introduction

In 2007 Nešetřil [Neš07] proved that the class of all linearly ordered finite metric spaces is Ramsey (see Section 2.2.1). For graphs (and isometric embeddings) a similar result was obtained by Dellamonica and Rödl [DR12] in 2012, it also follows from a more general result of Hubička and Nešetřil [HN16]. Mašulović [Maš16] gave a simpler proof by a reduction to partially ordered sets which are known to be Ramsey [NR84, PTW85]. Solecki [Sol05] and Vershik [Ver08] independently proved that the class of all finite metric spaces has EPPA (see Definition 2.24).

Sauer [Sau13b] in 2013 classified the sets $S \subseteq \mathbb{R}^{\geq 0}$ for which there is a universal homogeneous metric space with distances from $S$. Ramsey expansions of Sauer's $S$-metric spaces were then fully determined by Hubička and Nešetřil [HN16] (extending partial results by Nguyen Van Thé [NVT10]).

Conant [Con15] studied EPPA in the context of generalised metric spaces where the distances come from a linearly ordered monoid and Hubička, Nešetřil and the author [HKN17] later found Ramsey expansions for all such spaces.

Braunfeld [Bra17], motivated by his classification of generalised permutation structures [Bra18a] found Ramsey expansions of $\Lambda$-ultrametric spaces which are "metric spaces" where the distances come from a finite distributive lattice.

Aranda, Bradley-Williams, Hubička, Karamanlis, Kompatscher, Pawliuk and the author [ABWH$^+$17c] studied Ramsey expansions of metric spaces from Cherlin's list of metrically homogeneous graphs [Che16]. These are metric spaces with distances from $\{0, 1, \ldots, \delta\}$ with some other families of triangles also forbidden.

The Ramsey property of all of these (and also other) classes follows, sometimes directly, sometimes less so, from the fact that they admit some form of the *shortest path completion* (see Chapter 3). In this thesis we explore the boundaries of this method and introduce the concept of semigroup-valued metric spaces, a unifying framework for the aforementioned results. We prove that under certain assumptions, such classes have the strong amalgamation property, the Ramsey property and the extension property for partial automorphisms (EPPA).

This thesis is based on joint work with Hubička and Nešetřil.

## 1.1 Our results

A *commutative semigroup* is a tuple $(M, \oplus)$, where $\oplus\colon M^2 \to M$ is a commutative and associative operation.

**Definition 1.1** (Partially ordered commutative semigroup)**.** We say that a tuple $\mathfrak{M} = (M, \oplus, \preceq)$ is a *partially ordered commutative semigroup* if

1. $(M, \oplus)$ is a commutative semigroup;
2. $(M, \preceq)$ is a partial order which is reflexive ($a \preceq a$ for every $a \in M$);
3. For every $a, b \in M$ it holds that $a \preceq a \oplus b$;
4. For every $a, b, c \in M$ it holds that if $b \preceq c$ then $a \oplus b \preceq a \oplus c$ ($\oplus$ is monotone with respect to $\preceq$).

Note that point 4 implies that if $a \preceq b$ and $c \preceq d$, then $a \oplus c \preceq b \oplus d$.

The following definition is motivated by [Con15, Bra17, KPR18]. It is one of the key definitions of this thesis.

**Definition 1.2.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup. Given a set $A$ and a function $d\colon \binom{A}{2} \to M$, we call $(A, d)$ an *$\mathfrak{M}$-valued metric space* (or just *$\mathfrak{M}$-metric space*) if for every $a, b, c \in A$ it holds that $d(\{a,c\}) \preceq d(\{a,b\}) \oplus d(\{b,c\})$.

Given a partially ordered commutative semigroup $\mathfrak{M}$, we let $\mathcal{M}_\mathfrak{M}$ denote the class of finite $\mathfrak{M}$-metric spaces.

The function $d$ being defined on the set of all pairs instead of on $A^2$ is simply due to the fact that there is no identity in $\mathfrak{M}$ to set $d(a,a)$ to. In the rest of the paper we will treat $d$ as a two-parameter symmetric function not defined on any pair $a, a$.

**Example 1.3.** The following are some examples of partially ordered commutative semigroups:

1. Let $S$ be a topologically closed set of positive reals and for $a, b \in S$ define $a \oplus_S b = \sup\{c \in S; c \leq a + b\}$. Sauer [Sau13b] classified the sets $S$ for which $\oplus_S$ is an associative operation and hence $(S, \oplus, \leq)$ is a partially ordered commutative semigroup.
2. Consider the set of nonnegative real numbers extended by infinitesimal elements, i.e. $R^* = \left\{a + b \cdot \mathrm{dx} \mid a, b \in \mathbb{R}_0^+\right\}$ with piece-wise addition $+$ and order $\preceq$ given by the standard order of reals and $\mathrm{dx} \prec a$ for every positive real number $a$. Then $(R^*, +, \preceq)$ is also a partially ordered commutative semigroup.
3. The ultrametric $(\{1, \ldots, n\}, \max, \leq)$, where $\leq$ is the linear order of integers is a partially ordered commutative semigroup.
4. A distributive lattice $\Lambda = (L, \wedge, \vee, 0)$ with minimum 0 can be viewed as a partially ordered commutative semigroup, where the operation is $\vee$ and the order is the standard partial order of $\Lambda$. The $\Lambda$-valued metric spaces are essentially Braunfeld's $\Lambda$-ultrametric spaces [Bra17] (see Remark 1.4).
5. The multiplicative monoid $(\mathbb{Z}^{\geq 1}, \cdot, |)$, where the order is given by the "is a divisor of" relation is also a partially ordered commutative semigroup. The multiplicative metric spaces behave differently from the standard real-valued ones and one of the original contributions of this thesis is finding their Ramsey expansions.

*Remark* 1.4. For Definition 1.2 it would be more convenient to work with monoids instead of semigroups (that is, semigroups which contain a neutral element 0) and demand that if $d(x,y) = 0$, then $x = y$. It would also simplify Sections 4.2 and 5.1. On the other hand, it would make some other parts of this thesis more notationally complicated (if 0 represents the identity, then it necessarily needs different treatment than all the other distances).

In the context of this thesis, the advantages of not having any special distances outweighed the disadvantages. However, all the mentioned results work with monoids (and, unlike this thesis, have the neutral element 0 represent the identity — in this sense our metric spaces are sometimes only pseudo-metric spaces). In particular, Braunfeld's definition of $\Lambda$-ultrametric spaces is stronger than Definition 1.2 for $\mathfrak{M}$ being a distributive lattice, because it allows for the neutral element (representing the identity in the $\Lambda$-ultrametric spaces) to be *meet-reducible* (that is, to be the meet of two non-neutral elements, see Definition 4.9). Such classes do not have the *strong amalgamation property* (see Definition 2.2) and are out of this thesis' scope.

Now we state the main theorem of this thesis. There are several undefined notions, they will be defined in the subsequent chapters.

**Theorem 1.5.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathcal{F}$ be a family of $\mathfrak{M}$-edge-labelled cycles (Definition 3.1). Suppose that the following conditions hold:*

1. *$\mathcal{F}$ is $\mathfrak{M}$-omissible (Definition 3.4);*
2. *$\mathcal{F}$ contains all $\mathfrak{M}$-disobedient cycles (Definition 3.3);*
3. *$\mathcal{F}$ synchronizes meets (Definition 4.18); and*
4. *$\mathcal{F}$ is confined (Definition 4.21).*

*Then the class $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ of all finite $\mathfrak{M}$-valued metric spaces $\mathbf{A}$ such that there is no $\mathbf{F} \in \mathcal{F}$ with a homomorphism $\mathbf{F} \to \mathbf{A}$ has the strong amalgamation property and the class $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ of all convexly ordered (Definition 5.1) finite $\mathfrak{M}$-valued metric spaces $\mathbf{A}$ such that there is no $\mathbf{F} \in \mathcal{F}$ with a homomorphism $\mathbf{F} \to \mathbf{A}$ has the Ramsey property.*

This theorem looks quite technical, but often some of the conditions are trivial to satisfy. In particular when the order is linear, any family $\mathcal{F}$ including the empty one contains all disobedient cycles and synchronizes meets and hence we obtain a strengthening of the results of [HKN17] on Conant's generalised metric spaces and the results of [HN16] on Sauer's $S$-metric spaces. On the other hand, when $\mathfrak{M}$ is a distributive lattice (i.e. $\vee$ is the operation) and $\mathcal{F}$ is empty, one obtains the results of [Bra17] on $\Lambda$-ultrametric spaces (only ones with the strong amalgamation property, see Remark 1.4). In Section 6.1 we shall see that for a suitable partially ordered commutative semigroup $\mathfrak{M}$ (the *ordered magic semigroup*) and a suitable family $\mathcal{F}$, Theorem 1.5 also contains the results of [ABWH+17c] on Cherlin's primitive 3-constrained metrically homogeneous graphs.

*Remark.* In our proofs we make use of all the conditions of $\mathcal{F}$: $\mathfrak{M}$-omissibility ensures that the family $\mathcal{F}$ behaves consistently with the shortest path completion. $\mathcal{F}$ containing all $\mathfrak{M}$-disobedient cycles is a weaker version of requiring $\preceq$ to be a lattice and $\oplus$ to distribute with the infima. We need this weaker variant in order to be able to represent the metrically homogeneous graphs. Because we want to prove local finiteness (see Definition 2.21) of structures omitting $\mathcal{F}$, we have to put some conditions on $\mathcal{F}$, namely confinedness, to ensure that it does not break the local finiteness.

Meet synchronization is slightly more problematic. It ensures that the shortest path completion behaves consistently with the definable equivalences on the $\mathfrak{M}$-metric spaces. It seemed more convenient to allow the semigroup not to have this property by itself but to let $\mathcal{F}$ ensure it. However, we are not aware of any situations when $\mathcal{F}$ containing all disobedient cycles and being confined would not also ensure that it synchronizes meets:

**Question 1.** In the setting of Theorem 1.5, is the meet synchronization requirement necessary? In other words, if $\mathfrak{M}$ is a partially ordered commutative semigroup and $\mathcal{F}$ is a confined $\mathfrak{M}$-omissible family of $\mathfrak{M}$-edge-labelled cycles which contains all disobedient ones, does $\mathcal{F}$ necessary synchronize meets?

To illustrate the flexibility of our techniques we also prove EPPA.

**Theorem 1.6.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathcal{F}$ be a family of $\mathfrak{M}$-edge-labelled cycles (Definition 3.1). Suppose that the following conditions hold:*

1. *$\mathcal{F}$ is $\mathfrak{M}$-omissible (Definition 3.4);*
2. *$\mathcal{F}$ contains all $\mathfrak{M}$-disobedient cycles (Definition 3.3);*
3. *$\mathcal{F}$ synchronizes meets (Definition 4.18); and*
4. *$\mathcal{F}$ is confined (Definition 4.21).*

*Then the class $\mathcal{M}_{\mathfrak{M}} \cap \operatorname{Forb}(\mathcal{F})$ has EPPA (Definition 2.24).*

Further applications are discussed in Chapter 6.

## 1.2 Organization of the thesis

In Chapter 2 we review the history of the Ramsey theory and the study of homogeneous structures with emphasis on different metric-space-like structures. We also present all the necessary notions and definitions including the results of [HN16] on multiamalgamation classes, which are going to be a key ingredient for proving the Ramsey property.

Chapter 3 is dedicated to introducing the setting which we are going to work in, stating all the necessary definitions and proving some basic theorems. In Chapter 4 we study *blocks* (maximal archimedean subsemigroups) and the corresponding definable equivalences in $\mathfrak{M}$-valued metric spaces. We define an expansion which explicitly represents these equivalences and prove that these expanded classes have the strong amalgamation property while being locally finite. These results suffice to prove EPPA (Section 4.5).

Because for the Ramsey property one needs order, we add such an order in Chapter 5 (in a way very similar to Braunfeld [Bra17]) and again prove the strong amalgamation property and local finiteness. This enables us to prove Theorem 1.5 and also the expansion property. In Chapter 6 we sketch how Cherlin's metrically homogeneous graphs relate to $\mathfrak{M}$-valued metric spaces and discuss two other extensions or applications of our results.

Throughout the thesis we introduce many new notions and several different classes derived from the $\mathfrak{M}$-valued metric spaces. Appendix A contains the important notions together with short (often informal) definitions and links to the proper definitions, Appendix B contains a list of the classes which we introduced and work with. Again, their descriptions are informal; these appendices should be understood as cheatsheets while reading the thesis, not as replacements for reading and understanding the definitions.

## 1.3 Outline of the methods

The ideas of this thesis are quite intuitive. However, to carry them out correctly takes some effort and perhaps obscures the key insights a bit. In this section we try to give a very intuitive outline of what is going to happen on the upcoming pages. We omit many details and technical complications and for simplicity only talk about edge-labelled graphs (that is, binary symmetric structures).

Recently (cf. Theorems 2.22 and 2.25), the following meta-question became of interest in the combinatorial model theory: "Given a finite graph $\mathbf{G}$ with edges labelled by symbols from some set $L$, is it possible to add the remaining edges and their symbols so that the resulting complete edge-labelled graph is from a given class $\mathcal{C}$?" The answer is often found constructively, that is, by finding an explicit *completion procedure* which, given a graph $\mathbf{G}$, produces a complete edge-labelled graph $\mathbf{G}'$ such that $\mathbf{G}'$ is from $\mathcal{C}$ unless $\mathbf{G}$ has no *completion* in $\mathcal{C}$.

A prime example is the *shortest path completion* which works for $\mathcal{C}$ being a class of, say, *integer-valued metric spaces*, that is, complete graphs with edges labelled by positive integer which contain no *non-metric triangle* (triple of vertices such that the label of one of the edges is larger than the sum of the labels of the other two edges). The shortest path completion sets the *distance* (label of the edge) of each pair of vertices to be the *length* (sum of the labels) of the shortest path connecting them.

The shortest path completion procedure has been adapted for many different classes of edge-labelled complete graphs. In this thesis, we study the boundaries of this method. Instead of integers, we allow for the labels to form an arbitrary commutative semigroup $\mathfrak{M}$ with operation $\oplus$. And instead of the linear order of the integers, we work with a partial order $\preceq$ of $\mathfrak{M}$. Our classes thus consist of finite $\mathfrak{M}$-edge-labelled complete graphs such that in every triangle every edge is smaller (in $\preceq$) than the $\oplus$-sum of the other two; we call such graphs *$\mathfrak{M}$-valued metric spaces*. In the shortest path completion, instead of setting the distance of two vertices to the length of the shortest path connecting them, we set it to the infimum of the semigroup-lengths of all the paths connecting the two vertices.

The last sentence was deliberately slightly imprecise. The partial order on the semigroup surely cannot be arbitrary. One should at least require $\oplus$ to be monotone. And in order for the previous paragraph to describe a well-defined procedure, the partial order would actually have to be a semi-lattice. This, however, turns out to be too strong a condition (cf. Section 6.1). Instead, we refine the relevant classes and allow to further forbid (homomorphic images of) a family of $\mathfrak{M}$-edge-labelled cycles $\mathcal{F}$. In other words the classes which we are interested in are the intersections of all finite $\mathfrak{M}$-valued metric spaces and $\mathrm{Forb}(\mathcal{F})$ (the class of all finite $\mathfrak{M}$-edge-labelled graphs containing no homomorphic image of a member of $\mathcal{F}$) for some partially ordered commutative semigroup $\mathfrak{M}$ and a family $\mathcal{F}$.

Such family $\mathcal{F}$ then ensures that all infima of paths which the shortest path completion procedure encounters are well-defined. Of course, not every family of cycles can be forbidden, the least we have to demand is that it is *compatible* with the shortest path completion, that is, if the input graphs $\mathbf{G}$ is from $\mathrm{Forb}(\mathcal{F})$, then its shortest path completion $\mathbf{G}'$ must also be from $\mathrm{Forb}(\mathcal{F})$.

In Chapter 3 we describe precisely the conditions on $\mathcal{F}$ and study the basic properties of the shortest path completion. In particular, we observe that a graph from $\mathrm{Forb}(\mathcal{F})$ has a completion to a $\mathfrak{M}$-metric space if and only if it contains no non-$\mathfrak{M}$-metric cycle (a cycle which has no completion).

While the completion procedure is an important ingredient, for some applications, for example for finding Ramsey expansions, it is not enough. In order to use the Hubička–Nešetřil theorem (Theorem 2.22), one needs for every finite set of distances $S \subseteq \mathfrak{M}$ to find a finite family of *obstacles* $\mathcal{O}_S$ such that an $S$-edge-labelled graph has a completion in the given class if and only if it is from $\mathrm{Forb}(\mathcal{O}_S)$.

Such families, however, do not exist for every partially ordered commutative semigroup $\mathfrak{M}$ and family $\mathcal{F}$. For example, for $\mathfrak{M} = (\{1, 2\}, \max, \leq)$ and $\mathcal{F} = \emptyset$, every cycle with one edge labelled 2 and all the other edges labelled 1 has no completion. The reason for this is that "being in distance 1" is an equivalence relation. In Chapter 4 we deal with this phenomenon in general and study the *block structure* of partially ordered commutative semigroups, or in other words, we study the equivalence relations given by "being in a distance from a given subset of $\mathfrak{M}$".

We prove (in Section 4.3) that for a fixed finite $S \subseteq \mathfrak{M}$, one can select a bounded number of *important* edges from each non-$\mathfrak{M}$-metric cycle such that the other edges effectively only say that some vertices are block-equivalent for some block. The next thing to do then is to remove the unimportant edges and replace them by something which can concisely represent the "these vertices are block-equivalent" relation.

If the reader is familiar with model theory, they know that we want to find an expansion with *elimination of imaginaries*. Specifically, we will add new vertices (*ball vertices*) to represent the equivalence classes of all the block equivalences and link the *original vertices* to them by unary functions. Two vertices are equivalent if and only if they "point" (by the unary functions) to the same vertex corresponding to the given equivalence. This means that one can indeed forget about the unimportant edges and only keep the ball vertices certifying that some original vertices are equivalent, and hence making the family $\mathcal{O}_S$ of obstacles finite.

The previous paragraphs summarised the ideas from Chapter 4, but to carry it out correctly takes some effort, we ignored several important details here.

While being interesting and useful by itself, Chapter 4 is not enough to get Ramsey expansions of the $\mathfrak{M}$-valued metric spaces, because a Ramsey class has to have an order, which is the topic of Chapter 5. In particular, one also has to order the ball vertices. And as is standard in the structural Ramsey theory, the order of the ball vertices has to depend on the order of the original vertices. Braunfeld [Bra17] has found such an ordering for his $\Lambda$-ultrametric spaces and we use the same ideas.

# 2. Preliminaries and background

We first review some standard model-theoretic notions of structures with relations and functions (see e.g. [Hod93]) with a small variation that our functions will be partial.

A *language* $L$ is a collection $L = L_{\mathcal{R}} \cup L_{\mathcal{F}}$ of relational symbols $R \in L_{\mathcal{R}}$ and function symbols $F \in L_{\mathcal{F}}$ each having associated arities. For relations the arity is denoted by $a(R) > 0$ for relations, for functions $a(F)$ is the arity of the domain, in this thesis range will always have arity one.

An $L$-*structure* $\mathbf{A}$ is then a tuple $(A, \{R_{\mathbf{A}}\}_{R\in L_{\mathcal{R}}}, \{F_{\mathbf{A}}\}_{F\in L_{\mathcal{F}}})$, where $A$ is the *vertex set*, $R_{\mathbf{A}} \subseteq A^{a(R)}$ is an interpretation of $R$ for each $R \in L_{\mathcal{R}}$ and $F_{\mathbf{A}} \colon A^{a(F)} \to A$ is a partial function for each $F \in L_{\mathcal{F}}$. We denote by $\mathrm{Dom}(F_{\mathbf{A}})$ the domain of $F$ (i.e. the set of tuples of vertices of $\mathbf{A}$ for which $F$ is defined). An $L$-structure is *finite* (or has *finite support*) if its vertex set is finite.

An $L$-structure $\mathbf{A}$ is *connected* if for every two vertices $u, v \in A$ there exists a sequence of vertices $u = v_0, v_1, \ldots, v_k = v$ such that for every $i$ there is a tuple $\bar{x} \in A^k$ and a relation $R \in L$ or a function $F \in L$ such that $\bar{x} \in R$ or $\bar{x} \in F$ respectively (where we understand a function as a relation with some outdegree condition).

Notationally, we shall distinguish structures from their underlying sets by typesetting structures in bold font. When the language $L$ is clear from the context, we will use it implicitly.

Let $\mathbf{A}$ and $\mathbf{B}$ be $L$-structures. A *homomorphism* $f\colon \mathbf{A} \to \mathbf{B}$ is a mapping $f\colon A \to B$ satisfying for every $R \in L_{\mathcal{R}}$ and for every $F \in L_{\mathcal{F}}$ the following three statements:

(a) $(x_1, \ldots, x_{a(R)}) \in R_{\mathbf{A}} \Rightarrow (f(x_1), \ldots, f(x_{a(R)})) \in R_{\mathbf{B}}$;

(b) $f(\mathrm{Dom}(F_{\mathbf{A}})) \subseteq \mathrm{Dom}(F_{\mathbf{B}})$; and

(c) $f(F_{\mathbf{A}}(x_1, \ldots, x_{a(F)})) = F_{\mathbf{B}}(f(x_1), \ldots, f(x_{a(F)}))$ for every $(x_1, \ldots, x_{a(F)}) \in \mathrm{Dom}(F_{\mathbf{A}})$.

For a subset $A' \subseteq A$ we denote by $f(A')$ the set $\{f(x); x \in A'\}$ and by $f(\mathbf{A})$ the homomorphic image of a structure.

If $f$ is and injective homomorphism, it is a *monomorphism*. A monomorphism is an *embedding* if for every $R \in L_{\mathcal{R}}$ and $F \in L_{\mathcal{F}}$ the following holds:

(a) $(x_1, \ldots, x_{a(R)}) \in R_{\mathbf{A}} \iff (f(x_1), \ldots, f(x_{a(R)})) \in R_{\mathbf{B}}$, and,

(b) $(x_1, \ldots, x_{a(F)}) \in \mathrm{Dom}(F_{\mathbf{A}}) \iff (f(x_1), \ldots, f(x_{a(F)})) \in \mathrm{Dom}(F_{\mathbf{B}})$.

If $f$ is a bijective embedding, then it is an *isomorphism* and we say that $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic*. An isomorphism $\mathbf{A} \to \mathbf{A}$ is called *automorphism*. If the inclusion $\mathbf{A} \subseteq \mathbf{B}$ is an embedding then $\mathbf{A}$ is a *substructure* of $\mathbf{B}$. For $\mathbf{A}$ and $\mathbf{B}$ structures, we denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all embeddings of $\mathbf{A}$ to $\mathbf{B}$. Note that while for purely relational languages every set $A \subseteq B$ gives a substructure of $\mathbf{B}$, it does not hold in general for languages with functions (we need $A$ to be *closed* on functions).

For a family $\mathcal{F}$ of $L$-structures, we denote by $\mathrm{Forb}(\mathcal{F})$ the class of all finite $L$-structures (or $L^+$-structures with $L^+ \supseteq L$ if this is clear from the context) $\mathbf{A}$

such that there is no $\mathbf{F} \in \mathcal{F}$ with a homomorphism $\mathbf{F} \to \mathbf{A}$. If $f\colon X \to Z$ is a function and $Y \subseteq X$, we denote by $f|_Y$ the restriction of $f$ on $Y$.

Let $\mathbf{A}, \mathbf{B}$ be $L$-structures and $f\colon A \to B$ an arbitrary injective map. We say that $f$ is *automorphism-preserving* if for every automorphism $\alpha\colon \mathbf{A} \to \mathbf{A}$ there is an automorphism $\beta\colon \mathbf{B} \to \mathbf{B}$ such that $f \circ \alpha = \beta \circ f$.

## 2.1 Homogeneous structures

The main reference for this section is the survey on homogeneous structures by Macpherson [Mac11].

We say that an $L$-structure $\mathbf{M}$ is *homogeneous* (sometimes called *ultrahomogeneous*) if for every finite $\mathbf{A}, \mathbf{B} \subseteq \mathbf{M}$ and every isomorphism $g\colon \mathbf{A} \to \mathbf{B}$ there is an automorphism $f$ of $\mathbf{M}$ with $f|_A = g$.

Let $\mathcal{C}$ be a class of (not necessarily all) $L$-structures and let $\mathbf{M}$ be an $L$-structure. We say that $\mathbf{M}$ is *universal* for $\mathcal{C}$ if for every $\mathbf{C} \in \mathcal{C}$ there exists an embedding $\mathbf{C} \to \mathbf{M}$.

**Example 2.1.** The structure $(\mathbb{Q}, <)$, where $<$ is the standard linear order of the rationals, is homogeneous and universal for countable linear orders.

*Proof.* First we observe that $(\mathbb{Q}, <)$ satisfies the *extension property*, which is a very direct generalisation of density and means that for every finite $S \subset \mathbb{Q}$ and $B \subset \mathbb{Q}$ such that for every $s \in S$ and $b \in B$ it holds that $s < b$ there is $x \in \mathbb{Q}$ such that for every $s \in S$ it holds that $s < x$ and for every $b \in B$ it holds that $x < b$. For both $S$ and $B$ nonempty this follows by using density of $(\mathbb{Q}, <)$ for $\max S$ and $\min B$, for $S$ or $B$ empty is follows by $(\mathbb{Q}, <)$ having no endpoints.

Let $(X, \prec)$ be a countable (i.e. finite or countably infinite) linear order. We will construct an embedding $f\colon (X, \prec) \to (\mathbb{Q}, <)$. Enumerate vertices of $X$ arbitrarily as $x_1, x_2, \ldots$. Define $f(x_1)$ to be an arbitrary vertex of $\mathbb{Q}$. Now assume that $f$ is already defined on $x_1, \ldots, x_{i-1}$. By the extension property there is a vertex $y \in \mathbb{Q}$ different from all already defined $f(x_j)$ such that for all $j < i$ it holds that $y < f(x_j)$ if and only if $x_i < x_j$. Thus we can set $f(x_i) = y$. Formally we are creating an increasing sequence of partial embeddings and the embedding $(X, \prec) \to (\mathbb{Q}, <)$ will then be their union.

To prove that $(\mathbb{Q}, <)$ is homogeneous, we will again construct an embedding of $(\mathbb{Q}, <)$ into $(\mathbb{Q}, <)$ step-by-step, but now we have to make sure that it is a surjection and that it extends the given partial automorphism. In order to do this, we use the so-called *back-and-forth* argument.

Let $g$ be an isomorphism of finite substructures of $(\mathbb{Q}, <)$. We will define a nested sequence $g = f_0 \subset f_1 \subset \cdots$ of partial automorphisms of $(\mathbb{Q}, <)$ as follows: Enumerate vertices of $\mathbb{Q}$ arbitrarily as $q_1, q_2, \ldots$. Assume that $g = f_0, \ldots, f_{i-1}$ are already defined and we want to extend $f_{i-1}$ by one point to construct $f_i$.

If $i$ is odd, let $j$ be the smallest integer such that $q_j \notin \mathrm{Dom}(f_{i-1})$ where $\mathrm{Dom}(f_{i-1})$ is the domain of $f_{i-1}$. By the extension property there is $y \in \mathbb{Q} \setminus \mathrm{Range}(f_{i-1})$ such that for all $x \in \mathrm{Dom}(f_{i-1})$ it holds that $q_j < x$ if and only if $y < f_{i-1}(x)$. Then we can set $f_i(q_j) = y$.

If $i$ is even, let $j$ be the smallest integer such that $q_j \notin \mathrm{Range}(f_{i-1})$. Again, by the extension property there is $x \in \mathbb{Q} \setminus \mathrm{Dom}(f_{i-1})$ such that for all $x' \in \mathrm{Dom}(f_{i-1})$ it holds that $x < x'$ if and only if $q_j < f_{i-1}(x')$. Then we can set $f_i(x) = q_j$.

From the construction it follows that $f = \bigcup_{i=0}^{\infty} f_i$ is an automorphism of $(\mathbb{Q}, <)$ such that $g \subset f$. $\square$

In the early 1950s Fraïssé ([Fra53, Fra86]) noticed that while $(\mathbb{Q}, <)$ is homogeneous and universal for countable linear orders, $(\mathbb{N}, <)$ is neither of those (though it is universal for finite linear orders) and as a (very successful) attempt to extract the necessary properties and generalise this phenomenon he proved Theorem 2.4. The extracted properties are summarised in the following definition.

**Definition 2.2** (Amalgamation property [Fra53])**.** Fix a language $L$ and let $\mathcal{C}$ be a non-empty class of finite $L$-structures. We say that $\mathcal{C}$ is an *amalgamation class* if it has the following properties:

1. $\mathcal{C}$ is closed under isomorphisms and substructures;
2. $\mathcal{C}$ has the *joint embedding property (JEP)*: For all $\mathbf{B}_1, \mathbf{B}_2 \in \mathcal{C}$ there is $\mathbf{C} \in \mathcal{C}$ and embeddings $\beta_1 \colon \mathbf{B}_1 \to \mathbf{C}$ and $\beta_2 \colon \mathbf{B}_2 \to \mathbf{C}$; and
3. $\mathcal{C}$ has the *amalgamation property (AP)*: For all $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{C}$ and embeddings $\alpha_1 \colon \mathbf{A} \to \mathbf{B}_1$ and $\alpha_2 \colon \mathbf{A} \to \mathbf{B}_2$, there is $\mathbf{C} \in \mathcal{C}$ and embeddings $\beta_1 \colon \mathbf{B}_1 \to \mathbf{C}$ and $\beta_2 \colon \mathbf{B}_2 \to \mathbf{C}$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$. We call such $\mathbf{C}$ the *amalgamation* (or *amalgam*) of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ (with respect to $\alpha_1$ and $\alpha_2$, but these embeddings are often assumed implicitly).

It is common in the area to identify members of $\mathcal{C}$ with their isomorphism types.

Let $\mathbf{C}$ be an amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$. We say that $\mathbf{C}$ is a *strong amalgamation* (or *strong amalgam*) of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ if $\beta_1(x_1) = \beta_2(x_2)$ if and only if $x_1 \in \alpha_1(\mathbf{A})$ and $x_2 \in \alpha_2(\mathbf{A})$, which means that in the amalgam one only glues over the necessary substructure $\mathbf{A}$ and nothing more.

A strong amalgamation is a *free amalgamation* (*free amalgam*) if $C = \beta_1(x_1) \cup \beta_2(x_2)$ and furthermore for every relation $R$ and every tuple $\bar{x} = (x_1, \ldots, x_{a(R)}) \in R_{\mathbf{C}}$ it holds that $\bar{x} \in \beta_i(B_i)$ for some $i \in \{1, 2\}$ and similarly for every function $F$, every tuple $\bar{x} = (x_1, \ldots, x_{a(F)}) \in \mathrm{Dom}(F_{\mathbf{C}})$ and $y \in C$ such that $F_{\mathbf{C}}(\bar{x}) = y$ it holds that $\bar{x}, y \in \beta_i(\mathbf{B}_i)$ for the same $i \in \{1, 2\}$.

The joint embedding property says that for every two structures in $\mathcal{C}$ there is a structure in $\mathcal{C}$ containing both of them. If present in $\mathcal{C}$, their disjoint union could play such a role. But at the other extreme, if $\mathbf{B}_1 = \mathbf{B}_2$ then one can also put $\mathbf{C} = \mathbf{B}_1$.

The amalgamation property says that if one glues two structures $\mathbf{B}_1$ and $\mathbf{B}_2$ over a common substructure $\mathbf{A}$, there is a structure $\mathbf{C}$ in $\mathcal{C}$ which contains this patchwork, see Figure 2.1. By definition it is possible that the embeddings of $\mathbf{B}_1$ and $\mathbf{B}_2$ in $\mathbf{C}$ overlap by more vertices than just the vertices of $\mathbf{A}$ and also that $\mathbf{C}$ has more vertices than just $\beta_1(B_1) \cup \beta_2(B_2)$.

If $\mathcal{C}$ contains the empty structure, then AP for $\mathbf{A}$ being the empty structure is precisely JEP.

**Example 2.3.** Let $\mathcal{C}$ be the class of all finite metric spaces (with distances from, say, $\mathbb{R}^{\geq 0}$) and suppose that $\mathbf{A} = (A, d_{\mathbf{A}})$, $\mathbf{B}_1 = (B_1, d_{\mathbf{B}_1})$ and $\mathbf{B}_2 = (B_2, d_{\mathbf{B}_2})$ are structures from $\mathcal{C}$ such that $\mathbf{A} \subseteq \mathbf{B}_1$ and $\mathbf{A} \subseteq \mathbf{B}_2$. Define the metric space $\mathbf{C}$

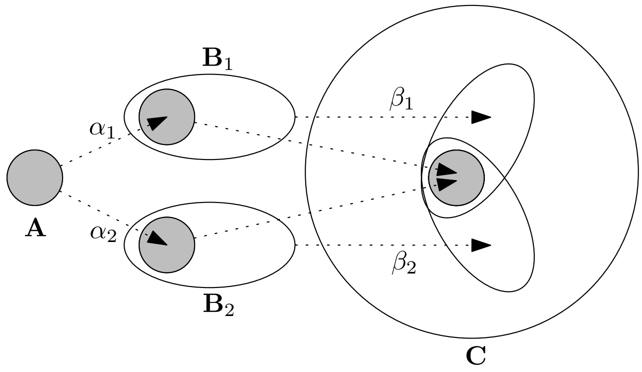



Figure 2.1: An amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$.

such that its vertex set is $C = B_1 \cup B_2$ and the metric is defined as

$$d_{\mathbf{C}}(u,v) = \begin{cases} d_{\mathbf{B}_1}(u,v) & \text{if } u,v \in B_1 \\ d_{\mathbf{B}_2}(u,v) & \text{if } u,v \in B_2 \\ \min_{w\in A} d_{\mathbf{B}_1}(u,w) + d_{\mathbf{B}_2}(v,w) & \text{if } u \in B_1, v \in B_2 \\ \min_{w\in A} d_{\mathbf{B}_2}(u,w) + d_{\mathbf{B}_1}(v,w) & \text{if } u \in B_2, v \in B_1. \end{cases}$$

Then $\mathbf{C}$ is the strong amalgam of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ where all the embeddings are inclusions. Furthermore, out of all strong amalgams of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$, each distance in $\mathbf{C}$ is as large as possible.

Let $\mathbf{M}$ be an $L$-structure. Then the *age of* $\mathbf{M}$ is defined as

$$\mathrm{Age}(\mathbf{M}) = \{\mathbf{A} \mid \mathbf{A} \text{ is a } \textbf{finite } L\text{-structure with an embedding } \alpha\colon \mathbf{A} \to \mathbf{M}\}.$$

Again, it is common to identify the age with the class of all isomorphism types of structures in the age.

If $\mathbf{M}$ is a countable homogeneous $L$-structure such that for every finite $X \subseteq M$ there is a finite substructure $\mathbf{Y} \subseteq \mathbf{M}$ with $X \subseteq Y$ then we call $\mathbf{M}$ a *Fraïssé structure*. A *Fraïssé class* is an amalgamation class with only countably many members up to isomorphism. The main condition on a homogeneous structure to be Fraïssé says that finite sets of vertices generate only finite substructures. This is sometimes called local finiteness (for example in group theory), but unfortunately in the structural Ramsey theory this name is reserved for something else (see Definition 2.21).

**Theorem 2.4** (Fraïssé [Fra53])**.**

1. *Let* $\mathbf{M}$ *be a Fraïssé structure. Then* $\mathrm{Age}(\mathbf{M})$ *is a Fraïssé class.*
2. *For every Fraïssé class* $\mathcal{C}$ *there is a Fraïssé structure* $\mathbf{M}$ *such that* $\mathrm{Age}(\mathbf{M}) = \mathcal{C}$*. Furthermore, if* $\mathbf{N}$ *is a countable homogeneous* $L$*-structure such that* $\mathrm{Age}(\mathbf{N}) = \mathcal{C}$*, then* $\mathbf{M}$ *and* $\mathbf{N}$ *are isomorphic.*

We call the structure $\mathbf{M}$ from the second point the *Fraïssé limit of* $\mathcal{C}$.

Fraïssé's theorem gives a correspondence between amalgamation classes and homogeneous structures. Besides the age of $(\mathbb{Q}, <)$, which is the class of all finite linear orders, there are many more known homogeneous structures and amalgamation classes. A prominent example is the random graph (often called the Rado graph), which we understand as a structure with one binary relation $E$ whose age is the class of all finite graphs and which is actually universal for all countable graphs. Or the generic triangle-free graph, which is the Fraïssé limit of all finite triangle-free graphs and again is universal for all countable triangle-free graphs. An example from a very different area is Hall's universal group [Hal59] (a nice exposition is in [Sin17]). Hall's universal group is universal for all countable locally finite groups and it is a countable homogeneous group, which means that every isomorphism between finite subgroups can be extended to an automorphism of the whole group.

Take the random graph $\mathbf{R}$[1]. From homogeneity and universality it follows that it satisfies the *extension property* which says that for every two disjoint finite sets of vertices $U, V \subset \mathbf{R}$ there is a vertex $x \in \mathbf{R}$ such that $x$ is connected by an edge to all vertices in $U$ and no vertex in $V$: There clearly exists a finite graph $\mathbf{G}$ whose vertex set can be partitioned into disjoint sets $U' \cup V' \cup \{x'\}$ such that $\mathbf{G}[U'] \cong \mathbf{R}[U]$ ($\mathbf{G}[U']$ means the subgraph of $\mathbf{G}$ induced on $U'$), $\mathbf{G}[V'] \cong \mathbf{R}[V]$ and $x'$ is connected to all members of $U'$ and no member of $V'$. Hence, by universality, we can assume that $\mathbf{G} \subseteq \mathbf{R}$. But by homogeneity one can extend the natural partial isomorphism sending $U \cup V$ to $U' \cup V'$ to an automorphism $g$ of $\mathbf{R}$ and then just let $x = g(x')$.

This extension property for $\mathbf{R}$ is an analogue of the extension property for $(\mathbb{Q}, <)$. And it turns out that it defines $\mathbf{R}$ — in model theory $\mathbf{R}$ is usually axiomatized by having the extension property. By a back-and-forth argument one can show that every two countable graphs having this property are isomorphic. The extension property thus implies homogeneity and universality in the same way as it does for $(\mathbb{Q}, <)$.

A suitable variant of the extension property can be defined for every homogeneous countable $L$-structure $\mathbf{M}$. And when $L$ is for example a finite relational language, a back-and-forth argument can be utilized to prove that every countable structure in the same language with the extension property is isomorphic to $\mathbf{M}$.[2]

Another example of a homogeneous structure is the Urysohn space, which is a homogeneous separable metric space universal for all countable separable metric spaces. It was constructed by Urysohn in 1924 [Ury27]. For a historical and metric-space-theoretical context, see [Hus08] where the author compares Urysohn's, Hausdorff's and later Katětov's approaches to the problem.

The Urysohn space $\mathbb{U}$ is constructed as the completion (in the metric space sense) of the rational Urysohn space $\mathbb{U}_{\mathbb{Q}}$, which is a homogeneous countable metric space with rational distances which is universal for all finite metric spaces with rational distances. $\mathbb{U}_{\mathbb{Q}}$ is constructed by a procedure in principle not very different from what Fraïssé used to prove Theorem 2.4. Hence Urysohn was ahead of Fraïssé by roughly 30 years and the whole theory should perhaps be called Urysohn–Fraïssé theory instead of just Fraïssé theory.

[1] For a proper review of the history of the random graph see Peter Cameron's blog post History of the Random Graph [Cam15].

[2] Theories with the property that they have only one countable model up to isomorphism are called $\omega$-categorical and are very important and interesting in the model theory context. This observation means that the theory of every homogeneous structure in a finite relational language is $\omega$-categorical.

So far we have seen a couple of examples of homogeneous structures, which are homogeneous for "good reasons". On the other hand, $(\mathbb{N}, <, (U_i)_{i\in\mathbb{N}})$, the structure consisting of natural numbers with the standard order plus infinitely many unary relations such that $U_i^{\mathbb{N}} = \{i\}$ (each vertex gets its own unary relation), is also homogeneous, but for "stupid reasons", namely the complete lack of isomorphisms between substructures.

### 2.1.1 Classification results

Homogeneous structures are being studied from several different perspectives. In this thesis we promote the combinatorial one. Other possible perspective is one of group theory: The automorphism groups of homogeneous structures are very rich (unless, of course, the structure is for example $(\mathbb{N}, <, (U_i)_{i\in\mathbb{N}})$). There are many results and notions connected to automorphism groups of homogeneous structures and a survey by Cameron [Cam99] serves as a very good starting reference. In model theory, homogeneous structures are studied for example from the stability point of view, see [Mac11] for details. And last but not least, the automorphism group can be equipped with a natural topology — the *pointwise-convergence topology* — and then studied from the point of view of topological dynamics. This will be touched upon a little more in Section 2.3.

But the initial direction after the Fraïssé theorem was on classification. In this section, we briefly and partially overview some classification results. We start with a theorem of Lachlan and Woodrow on the classification of all countably infinite homogeneous (undirected) graphs.

**Theorem 2.5** (Classification of countably infinite homogeneous graphs [LW80])**.** *Let* $\mathbf{G}$ *be a countably infinite homogeneous undirected graph. Then* $\mathbf{G}$ *or* $\overline{\mathbf{G}}$ *(the complement of* $\mathbf{G}$*) is one of the following:*

1. *The random graph* $\mathbf{R}$ *(i.e. the Fraïssé limit of the class of all finite graphs);*
2. *the* generic *(that is, universal and homogeneous)* $K_n$*-free graph for some finite clique* $K_n$*, which is the Fraïssé limit of the class of all finite* $K_n$*-free graphs; or*
3. *the disjoint union of complete graphs of the same size (either an infinite union of* $K_n$*'s for some* $n < \infty$*, or a finite or infinite union of* $K_\omega$*'s).*

This theorem implies that there are only countably many countable homogeneous graphs, which is in contrast with an earlier result of Henson [Hen72] who found $2^{\aleph_0}$ non-isomorphic countable homogeneous directed graphs. They are analogues of the $K_n$-free graphs, but while forbidding $K_n$ and $K_m$ is the same as forbidding $K_{\min(m,n)}$, Henson is forbidding tournaments and one can construct infinite sets of pairwise "incomparable" tournaments. Cherlin [Che98], more than 20 years later, gave a full classification of countably infinite homogeneous directed graphs. The proof takes more than 170 pages and even the list itself is too long and complicated for our small historical overview; it contains for example the much older classification of homogeneous partial orders [Sch79]. And recently, Cherlin [Che11, Che16] offered a list of metrically homogeneous graphs, that is, countable graphs which become homogeneous metric spaces if one considers the path-distance metric. They were the topic of the author's Bachelor thesis [Kon18] and we will briefly touch upon them in Section 6.1.

## 2.2 Ramsey theory

Surveying the rich history of the Ramsey theory could easily be a topic for more than one thesis, but not a very good topic as there already are several good references. For this chapter's brief sketch of some of the most important results of Ramsey theory, Nešetřil's chapter in [GGL95, Ch. 25] and Prömel's book [Prö13] were the main references. Some of the more recent developments in the structural Ramsey theory were surveyed by, for example, Bodirsky [Bod15], Nguyen Van Thé [NVT15] and Solecki [Sol13].

In 1930, F. P. Ramsey published a paper where he proves the folowing theorem (which we state in today's language, by $[n]$ we mean the set $\{0, 1, \dots, n-1\}$, and, for a set $A$, by $\binom{A}{p}$ we mean the set of all $p$-elements subsets of $A$):

**Theorem 2.6** (Ramsey's theorem [Ram30])**.** *For every triple of natural numbers $n, p, k$ with $n \geq 0$, $p > 0$ and $k \geq 1$ there is $N$ such that the following holds:*

*For every colouring $c\colon \binom{[N]}{p} \to [k]$ there is an $n$-element subset $H \in \binom{[N]}{n}$ such that $c|_{\binom{H}{p}}$ is constant.*

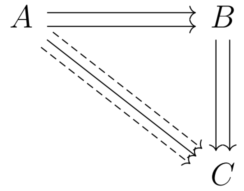
### 2.2.1 Category theory

Historically, the structural Ramsey theory was based on category theory [Lee73]. Although it is no longer the most common way, it will later be useful to define the Ramsey property for categories. In order to do it, recall that a *category* $\mathfrak{C}$ is a tuple $(\mathrm{ob}(\mathfrak{C}), \mathrm{hom}(\mathfrak{C}))$, where $\mathrm{ob}(\mathfrak{C})$ is a class of *objects* and $\mathrm{hom}(\mathfrak{C})$ is a class of *morphisms* (*arrows*, *maps*) between the objects. If $f$ is a morphism from object $A$ to object $B$ we write $f\colon A \to B$ and we write $\mathrm{hom}(A, B)$ for the class of all morphisms $A \to B$. Finally, for every triple of objects $A, B, C$ there is an associative binary operation $\circ\colon \mathrm{hom}(B, C) \times \mathrm{hom}(A, B) \to \mathrm{hom}(A, C)$ (the *composition*), and we require that for every objects there is an identity morphism (with respect to composition).

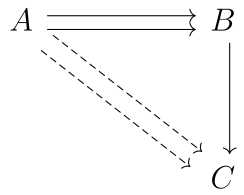
The name hom is rather unfortunate, note that is has nothing to do with homomorphisms as defined at the beginning of this chapter, or more precisely, one can consider categories where $\mathrm{hom}(A, B)$ is the set of all homomorphisms $A \to B$, but it is just a particular example of a category.

Let $\mathfrak{C}$ be a category. We say that an object $C$ is a *Ramsey witness* for objects $A$ and $B$ and $k \in \mathbb{N}$ colours if for every colouring $c\colon \mathrm{hom}(A, C) \to \{0, \dots, k-1\}$ there is a morphism $f\colon B \to C$ and a colour $0 \leq i < k$ such that for every $g \in \mathrm{hom}(A, B)$ it holds that $c(f \circ g) = i$.

Denoted schematically, for every $k$-colouring of $\mathrm{hom}(A, C)$ as in the following diagram (where we for simplicity let $k = 2$ and depict one colour by dashed arrows and the other by full arrows)



there is an arrow $f\colon B \to C$, such that the compositions $f \circ \hom(A, B)$ are monochromatic:



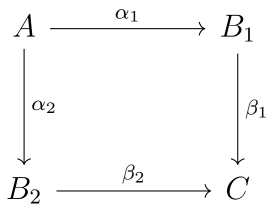

If $C$ is a *Ramsey witness* for $A$, $B$ and $k$, we denote it as

$$C \longrightarrow (B)^A_k,$$

which is called the Erdös–Rado partition arrow. If a category has the property that for every $A$, $B$ and $k$ there is $C$ such that $C \longrightarrow (B)^A_k$, we say that the category has the *Ramsey property.*

Notice now that the Ramsey theorem says that the category $\mathfrak{LO}$ of linear orders where maps are monotone injections has the Ramsey property.

*Remark.* The amalgamation property is also a categorical property: A category has the amalgamation property if for every triple of objects $A$, $B_1$ and $B_2$ with morphisms $\alpha_1\colon A \to B_1$ and $\alpha_2\colon A \to B_2$ there is an object $C$ and morphisms $\beta_1\colon B_1 \to C$ and $\beta_2\colon B_2 \to C$ such that the following diagram commutes:




Notice that it is a weak version of a pushout as $C$ need not have any universal property.

This categorical point of view will be useful later as it is an easy way to transfer the amalgamation and Ramsey properties between isomorphic categories.

### 2.2.2 Some more Ramsey-type results

Ramsey's paper was concerned with a problem of deciding whether satisfaction of certain formulas (the $\exists\forall$-formulas) is decidable. In 1935, Erdös and Szekeres initiated the combinatorial applications of the Ramsey theorem by proving the following result:

**Theorem 2.7** (Erdös–Szekeres [ES35])**.** *For every positive integer $m \geq 3$ there is a positive integer $N$ such that in any set of $N$ points in the Euclidean plane, no three of which are collinear, there are $m$ points which form the vertex set of a convex $m$-gon.*

This theorem can be proved using the Ramsey theorem for quadruples of points coloured based on whether they all lie on the boundary of their convex hull.

Long before Ramsey, Schur, using the following Ramsey-type result about the natural numbers, proved that Fermat's Last Theorem is false modulo large primes:

**Lemma 2.8** (Schur [Sch17]). *For every positive integer $k$ there exists a positive integer $N$ such that for every colouring $c\colon \{1,\dots,2N\} \to [k]$ there are $1 \le x < y \le N$ such that $c(x) = c(y) = c(x+y)$.*

It follows easily from Ramsey's theorem by colouring pairs $x, y$ by $c(|x-y|)$ and looking for a monochromatic triple. Schur also conjectured the following statement which was in 1927 proved by van der Waerden.

**Theorem 2.9** (van der Waerden [vdW27]). *For every pair of positive integers $k$ and $r$ there exists a positive integer $N$ such that for every colouring $c\colon \{1,\dots,N\} \to [k]$ there exist positive integers $a$ and $d$ such that $A = \{a + id \mid 0 \le i < r\} \subseteq \{1,\dots,N\}$ and $c|_A$ is constant.*

The original proof of van der Waerden's theorem goes by double induction and does not give a primitive recursive upper bound. This was an open problem for a long time until Shelah [She88] found such a bound.

There are many more important and interesting results (for example the celebrated Szemerédi theorem [Sze69], the Hales–Jewett [HJ63] or the Graham–Rothschild theorems [GR71]), but we now concentrate on the structural Ramsey theory which this thesis is a contribution to.

### 2.2.3 Structural Ramsey theory

Having defined what the Ramsey property for a category is, a natural question for a combinatorist would be: "What about graphs?"

The answer is that it is an ambiguous question. What are the morphisms? If one looks at graphs with the non-induced-subgraph morphisms (that is, equivalence classes of monomorphisms modulo permutation of vertices), then this category has the Ramsey property directly by the Ramsey theorem, as it is enough to be able to find arbitrarily large monochromatic complete graphs. If one takes the category Gra of finite graphs with embeddings, then the answer is negative.

**Proposition 2.10.** *The category* Gra *of finite graphs with embeddings does not have the Ramsey property.*

*Proof.* It is enough to take $\mathbf{A} = \mathbf{B}$ the graph consisting of two vertices connected by an edge. It has a non-trivial automorphism (one that switches the vertices), hence there are two embeddings of $\mathbf{A}$ into $\mathbf{B}$. Suppose that there is $\mathbf{C}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ and let $c\colon \binom{\mathbf{C}}{\mathbf{A}} \to [2]$ be such that for every edge of $\mathbf{C}$ it will colour one of the embeddings by colour 0 and the other by colour 1. Then, clearly, there is no embedding of $\mathbf{B}$ into $\mathbf{C}$ which is $c$-monochromatic. □

Note that one can repeat the same argument for any category which contains an object with a non-trivial automorphism. Thus a category needs to be *rigid* (no non-trivial automorphisms of any object) in order to have the Ramsey property.

*Remark.* If instead of embeddings one considers the category of graphs with the "induced subgraph" morphisms (i.e. equivalence classes of embeddings modulo automorphisms), then the category still doesn't have the Ramsey property. It has the $\mathbf{A}$-Ramsey property (for every $k$ and $\mathbf{B}$ there is a $\mathbf{C}$ with $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_k$) precisely for complete graphs and independent sets, see [GGL95, Ch. 25, Sec. 5].

In the structural Ramsey theory, one is always working with categories of (finite) structures equipped with embeddings. Hence it is appropriate to say that a class $\mathcal{C}$ of finite $L$-structures, where $L$ is a language, has the Ramsey property, or *is Ramsey*, if the category with objects from $\mathcal{C}$ and embeddings as morphisms has the Ramsey property.

In 1977 Nešetřil and Rödl and independently in 1978 Abramson and Harrington proved the following:

**Theorem 2.11** (Nešetřil–Rödl [NR77a, NR77b], Abramson–Harrington [AH78])**.** *The class of all linearly ordered finite graphs is Ramsey.*

Note that the language is $L = \{E, \leq\}$ and the embeddings also have to preserve the order.

The techniques of Nešetřil and Rödl actually prove much more. Let $\mathcal{C}$ be an amalgamation class of $L$-structures. A structure $\mathbf{C} \in \mathcal{C}$ is *reducible* if there are structures $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{C}$ such that $\mathbf{C}$ is the free amalgam of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$. Otherwise $\mathbf{C}$ is *irreducible*.

In the class of graphs the irreducible structures are precisely cliques. If $L$ is a relational language and $\mathcal{C}$ is the class of all finite $L$-structure then irreducibility means that for every pair of vertices there is a relation and a tuple in that relation containing both vertices.

Now we state what is known as the Nešetřil–Rödl theorem:

**Theorem 2.12** (Nešetřil–Rödl [NR77a, NR77b])**.** *Let $L$ be a relational language and let $\mathcal{F}$ be a collection of irreducible finite $L$-structures. Define $\mathcal{F}^{\leq}$ to be the collection of all linear orderings of structures from $\mathcal{F}$. Then the class of all linearly ordered finite $L$-structures $\mathbf{M}$ such that there is no $\mathbf{F} \in \mathcal{F}^{\leq}$ with an embedding $\mathbf{F} \to \mathbf{M}$ is Ramsey.*

Theorem 2.11 is a direct consequence of Theorem 2.12. Theorem 2.12 also implies that the class of ordered $K_n$-free graphs is a Ramsey class or in general that every relational free amalgamation class is Ramsey when linearly ordered. It is known (cf. Proposition 2.19) that every Ramsey class *fixes* an order. Thus Theorem 2.12 is tight in the sense that adding the order is necessary.

The following observation of Nešetřil from 1989 gives (under a mild assumption) a strong necessary condition for Ramsey classes and also connects the Ramsey theory and the theory of homogeneous structures:

**Theorem 2.13** (Nešetřil [Neš89, Neš05])**.** *Let $\mathcal{C}$ be a Ramsey class of finite structures with the joint embedding property. Then $\mathcal{C}$ has the amalgamation property.*

*Proof.* We need to show that for every $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{C}$ and embeddings $\alpha_1\colon \mathbf{A} \to \mathbf{B}_1$ and $\alpha_2\colon \mathbf{A} \to \mathbf{B}_2$ there is $\mathbf{C} \in \mathcal{C}$ and embeddings $\beta_1\colon \mathbf{B}_1 \to \mathbf{C}$ and $\beta_2\colon \mathbf{B}_2 \to \mathbf{C}$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$.

Let $\mathbf{B}$ be a joint embedding of $\mathbf{B}_1$ and $\mathbf{B}_2$ and take such $\mathbf{C} \in \mathcal{C}$ that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. We will prove that $\mathbf{C}$ is the amalgam we are looking for.

Assume the contrary which means that there is no embedding $\alpha\colon \mathbf{A} \to \mathbf{C}$ with the property that there are embeddings $\beta_1\colon \mathbf{B}_1 \to \mathbf{C}$ and $\beta_2\colon \mathbf{B}_2 \to \mathbf{C}$ such that $\beta_i \circ \alpha_i = \alpha$ for $i \in \{1, 2\}$. Hence, for every $\alpha\colon \mathbf{A} \to \mathbf{C}$ there is at most one

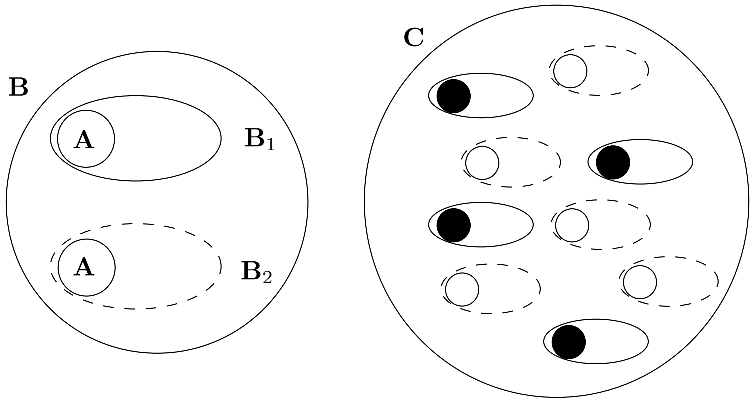



Figure 2.2: Illustration of the proof of Theorem 2.13. Copies of $\mathbf{A}$ from $\mathbf{B}_1$ are coloured black, copies of $\mathbf{A}$ from $\mathbf{B}_2$ are coloured white.

such embedding $\beta_i\colon \mathbf{B}_i \to \mathbf{C}$. Define the colouring $c\colon \binom{\mathbf{C}}{\mathbf{A}} \to \{0,1\}$ by letting

$$c(\alpha) = \begin{cases} 0 & \text{if there is } \beta_1\colon \mathbf{B}_1 \to \mathbf{C} \text{ such that } \alpha = \beta_1 \circ \alpha_1 \\ 1 & \text{otherwise .} \end{cases}$$

For an illustration, see Figure 2.2.

But then, as $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$, there is an embedding $\beta\colon \mathbf{B} \to \mathbf{C}$ such that $c|_{\beta(\mathbf{B})}$ is constant. But there are at least two embeddings of $\mathbf{A}$ into $\beta$ — one is given by $\alpha_1$ and the other is given by $\alpha_2$. And $\alpha_1$ can be extended to an embedding of $\mathbf{B}_1$, while $\alpha_2$ can be extended to an embedding of $\mathbf{B}_2$, hence they got different colours, which is a contradiction. □

Theorem 2.13 gives rise to the question whether every amalgamation class is a Ramsey class, to which the answer is negative, as for example the (unordered) graphs form an amalgamation class which is not Ramsey. A follow-up question might be whether every amalgamation class, when enriched by all possible linear orders (for each structure $\mathbf{M}$ in the original class there will be $n!$ structures in the new, ordered class), is Ramsey. And the answer is, again, no:

**Proposition 2.14.** *Let* $\mathbf{M}$ *be the disjoint union of two infinite cliques* $K_\omega$*.* $\mathbf{M}$ *is on the Lachlan–Woodrow list (Theorem 2.5) and is homogeneous. Let* $\mathcal{C} = \mathrm{Age}(\mathbf{M})$ *be the class of all finite graphs such that they are either a clique or the disjoint union of two cliques. And let* $\mathcal{C}^{\leq}$ *be the class of all possible orderings of members of* $\mathcal{C}$*. Then* $\mathcal{C}^{\leq}$ *has the amalgamation property, but does not have the Ramsey property.*

*Proof.* The amalgamation property is easy: since the order and the graph structure are independent, we can use the amalgamation procedure for the order and the graph structure independently using the fact that both structures have the strong amalgamation property.

Now let $\mathbf{A}$ be a vertex and $\mathbf{B}$ be a pair of vertices not connected by an edge. Any $\mathbf{C} \in \mathcal{C}^{\leq}$ which contains a copy of $\mathbf{B}$ must consist of two cliques. Then one can colour the vertices of one of the cliques red and the vertices of the other clique blue and there will be no monochromatic non-edge in this colouring. □

Such a situation happens often and the following section introduces a way to deal with it.

### 2.2.4 Expansions

Notice that our colouring was based on the fact that the edge relation in the structure $\mathbf{M}$ is actually an equivalence relation with two equivalence classes. If one could, for example, add to the language a unary relation and distinguish the equivalence classes by putting the unary relation on all vertices from one of them, then such a class equipped with an order would be Ramsey (the edge relation would then be redundant and could be formally removed and the Ramseyness of such a class would follow from the Nešetřil–Rödl theorem).

In order to formalize this observation and state another variant of the question whether all classes are Ramsey, we need to give a model-theoretic definition.

**Definition 2.15** (Expansion and reduct)**.** Let $L$ be a language, let $L^+$ be another language such that $L \subseteq L^+$ (i.e. $L^+$ contains all symbols that $L$ contains and they have the same arities). Then we call $L^+$ an *expansion* of $L$ and we call $L$ a *reduct* of $L^+$.

Let $\mathbf{M}$ be an $L$-structure and let $\mathbf{M}^+$ be an $L^+$-structure such that $\mathbf{M}^+|_L = \mathbf{M}$ (by this we mean that $\mathbf{M}$ and $\mathbf{M}^+$ have the same sets of vertices and the interpretations of symbols from $L$ are exactly the same in both structures). Then we call $\mathbf{M}^+$ an *expansion* of $\mathbf{M}$ and we call $\mathbf{M}$ a *reduct* of $\mathbf{M}^+$.

If $\mathcal{C}$ is a class of finite $L$-structures, we say that $\mathcal{C}^+$, a class of finite $L^+$-structures, is its *expansion* if for every $\mathbf{A} \in \mathcal{C}$ there is $\mathbf{A}^+ \in \mathcal{C}^+$ which is its expansion and for every $\mathbf{A}^+ \in \mathcal{C}^+$ there is $\mathbf{A} \in \mathcal{C}$ which is its reduct.

*Remark.* In model theory reduct and expansion often mean something more general, but for our purposes this definition is sufficient.

Historically, expansions are often called *lifts* in the Ramsey-theoretic context ([KN07, HN09]) and reducts are called *shadows*. We say that a class has a *Ramsey expansion* if it has an expansion which is Ramsey.

So far we have only been adding all linear orders, which is clearly a special expansion (and corresponds to adding to the Fraïssé limit the dense linear order with no endpoints which is independent from the rest of the relations). But we also have seen that sometimes it isn't enough.

To sum up, we know that every Ramsey class (with the joint embedding property) has the amalgamation property. Amalgamation classes (of finite structures in a countable language) correspond to homogeneous structures, their Fraïssé limits. And as we have seen, by adding some more structure on top of a homogeneous structure and looking at the age (or in general expanding the class), one can get a Ramsey class.

In 2005, Nešetřil ([Neš05]) started the *classification programme of Ramsey classes* — the counterpart of the Lachlan–Cherlin classification programme of homogeneous structures. Its goal is to classify all possible Ramsey classes, a goal quite ambitious, but in some cases achievable; the classification programme of homogeneous structures offers lists of possible Ramsey classes, or rather base classes for expansions.

In this thesis we extract the similarities of several known Ramsey classes and introduce a more abstract version which is their common generalisation.

Having read this far into the historical introduction, the reader has probably already asked themselves: "Does every amalgamation class have a Ramsey expansion?"

The answer to this question is positive, but by cheating: One can add infinitely many unary predicates and let each vertex have its own predicate. Then every structure has at most one embedding to any other and the Ramsey question becomes trivial. Two different ways of avoiding this cheat have been offered:

**Question 2** (Bodirsky–Pinsker–Tsankov [BPT11])**.** Does every amalgamation class in a finite language have a Ramsey expansion in a finite language?

This question still remains open. The other possible fix is motivated by topological dynamics (which we touch very briefly in Section 2.3). An amalgamation class $\mathcal{C}$ of finite $L$-structures is said to be *$\omega$-categorical* if for every $n$ there are only finitely many non-isomorphic structures in $\mathcal{C}$ on $n$ vertices.[3]

Let $\mathcal{C}$ be a class of $L$-structures and let $\mathcal{C}^+$ be its expansion. We say that $\mathcal{C}^+$ is a *precompact* expansion of $\mathcal{C}$ if for every $\mathbf{A} \in \mathcal{C}$ there are only finitely many non-isomorphic $\mathbf{A}^+ \in \mathcal{C}^+$ which are expansions of $\mathbf{A}$. Thus, precompactness is a relative version of $\omega$-categoricity.

**Question 3** (Melleray–Nguyen Van Thé–Tsankov [MNVTT15])**.** Does every $\omega$-categorical amalgamation class have a *precompact* Ramsey expansion?

This question has recently been answered negatively by Evans, Hubička and Nešetřil [EHN17a].

## 2.3 The KPT correspondence

In 2005, Kechris, Pestov and Todorčević published their famous paper which connected the field of structural Ramsey theory with the seemingly unrelated field of topological dynamics.

*Remark.* While topological dynamics and Ramsey theory are now interconnected, the necessary backgrounds needed to understand the fields are still very different. And I am by no means an expert in topological dynamics.

However, the review would not be complete without mentioning the KPT correspondence. Therefore, this whole section is very sketchy, with emphasis on the intuition and on the consequences for the combinatorics of Ramsey theory. I also dare to expect the reader to know all the used group-theoretical and topological notions. Should this not be the case, the reader can still get the gist of what is happening and understand the historical reasons for the recent growth of the general interest in the Ramsey theory. For a solid overview of the KPT correspondence, the survey by Nguyen Van Thé [NVT15] is a good reference.

Let $\mathbf{M}$ be a structure. By $\mathrm{Aut}(\mathbf{M})$ we denote the automorphism group of $\mathbf{M}$. This group can be viewed as a *topological group* when endowed with the *pointwise convergence topology* (the natural choice in this setting), where by a group being topological we mean that both the operation and the inverse are continuous with respect to the given topology.

Recall that we say that $\mathbf{M}$ is *rigid* if $\mathrm{Aut}(\mathbf{M})$ is trivial, i.e. consist only of the identity.

[3]We have already mentioned what it means to be $\omega$-categorical for countable structures. One can prove that a countable sturcture is $\omega$-categorical if and only if its age is.

For $G$ a topological group a $G$*-flow* is a continuous action of $G$ on a topological space $X$, often denoted as $G \curvearrowright X$. We say that a $G$-flow is *compact* if the space $X$ is compact.

**Definition 2.16** (Extremely amenable group)**.** Left $G$ be a topological group. We say that $G$ is *extremely amenable* if every compact $G$-flow has a fixed point (i.e. $x \in X$ such that $g \cdot x = x$ for every $g \in G$).

*Remark.* One might ask what an amenable group is. Roughly, a (locally compact) topological group is amenable if it admits a finitely additive left-invariant probability measure on its subsets. For extremely amenable groups this measure can be Dirac.[4]

Yet another equivalent definition of extreme amenability is that the *universal minimal flow* (which we did not define here) is a singleton. Computing universal minimal flows is of interest to people in topological dynamics and it turns out that the results of the structural Ramsey theory are an essential ingredient.

Now we can state the theorem of Kechris, Pestov and Todorčević (preprint was published in 2003):

**Theorem 2.17** (KPT correspondence [KPT05])**.** *Let* $\mathbf{M}$ *be a countable homogeneous structure. Then the following are equivalent:*

1. $\mathrm{Aut}(\mathbf{M})$ *is extremely amenable; and*
2. $\mathrm{Age}(\mathbf{M})$ *has the Ramsey property.*

We have defined Ramseyness as a property of an amalgamation class, but thanks to the KPT correspondence the Ramsey property is now witnessed directly by its Fraïssé limit (if it exists). This theorem of Kechris, Pestov and Todorčević ignited a new wave of interest in the structural Ramsey theory.

Nguyen Van Thé later ([NVT13]) built on the ideas used in the proof of the KPT correspondence and introduced a way of computing the universal minimal flows using Ramsey expansions which are in certain sense *minimal.*

**Definition 2.18** (Expansion property [NVT13])**.** Let $\mathcal{C}$ be a class of finite structures and let $\mathcal{C}^+$ be its expansion. We say that $\mathcal{C}^+$ has the *expansion property* (with respect to $\mathcal{C}$) if for every $\mathbf{B} \in \mathcal{C}$ there is $\mathbf{C} \in \mathcal{C}$ such that for every $\mathbf{B}^+ \in \mathcal{C}^+$ and for every $\mathbf{C}^+ \in \mathcal{C}^+$ such that $\mathbf{B}^+$ is an expansion of $\mathbf{B}$ and $\mathbf{C}^+$ is an expansion of $\mathbf{C}$ it holds that there is an embedding $\mathbf{B}^+ \to \mathbf{C}^+$.

An expansion has the expansion property if for every small structure $\mathbf{B}$ in the non-expanded class there is a large structure $\mathbf{C}$ in the non-expanded class such that every expansion of $\mathbf{C}$ contains every expansion of $\mathbf{B}$. The expansion property is a generalization of the ordering property studied by Nešetřil and Rödl in the

[4]Several paragraphs ago we said that Evans, Hubička and Nešetřil [EHN17a] found an $\omega$-categorical class with no precompact Ramsey expansion, one of the so-called *Hrushovski constructions*. It turns out that the difficulty for this class is already in finding an amenable expansion (an expansion whose Fraïssé limit's automorphism group is amenable). They suggested yet another question, namely whether every $\omega$-categorical structure with an amenable automorphism group has a precompact Ramsey expansion. This question has also been asked by Ivanov [Iva15].

70's and 80's [NR75] and it turns out that it expresses well what an intuitively "good" expansion is.

Nguyen Van Thé also proves that under certain assumptions (for example finite relational languages satisfy those) Ramsey expansions with the expansion property correspond to the universal minimal flow. This means that, up to bi-definability, there is only one Ramsey expansion with the expansion property. And this is in some sense the best one. It is worth noting that Kechris, Pestov and Todorčević [KPT05] proved this for the special case when the expansion is all linear orders (i.e. the expansion property is the ordering property).

We conclude this section and the whole chapter with a sketch of an application of the KPT correspondence. For finite relational languages this proposition can be proved combinatorially (see [Bod15, Proposition 2.22]) and in a stronger setting where the order will be definable:

**Proposition 2.19** (Kechris–Pestov–Todorčević [KPT05])**.** *Let* $\mathbf{M}$ *be a Ramsey structure (that is, its age has the Ramsey property). Then* $\mathrm{Aut}(\mathbf{M})$ *fixes a linear order, which means that there exists a linear order* $\preceq$ *on the vertices of* $\mathbf{M}$ *such that for every* $g \in \mathrm{Aut}(\mathbf{M})$ *and every* $x, y \in M$ *it holds that* $x \preceq y$ *if and only if* $g(x) \preceq g(y)$*.*

*Sketch of proof.* Let $LO(M)$ be the space of all linear orders on $M$ viewed as a subspace of $\{0,1\}^{M\times M}$. Clearly $LO(M)$ is compact and $\mathrm{Aut}(\mathbf{M})$ acts continuously on it by its standard action: for $L \in LO(M)$ and $g \in \mathrm{Aut}(\mathbf{M})$ we define $g \cdot L$ by $(x, y) \in g \cdot L$ if and only if $(g^{-1}(x), g^{-1}(y)) \in L$. Therefore, as $\mathbf{M}$ is Ramsey, $\mathrm{Aut}(\mathbf{M})$ is extremely amenable and thus this action has a fixed point, which is an order $L$ such that $g \cdot L = L$ for every automorphism $g$. □

## 2.4 The Hubička–Nešetřil theorem

In this section we present the results of [HN16]. We use a slightly different terminology, namely instead of closure descriptions we work explicitly with functions (like [HN17]).

A homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ is a *homomorphism-embedding* if for every irreducible $\mathbf{C} \subseteq \mathbf{A}$ it holds that $f|_C$ is an embedding $\mathbf{C} \to \mathbf{B}$.

**Definition 2.20** (Completion)**.** Let $L$ be a language, $\mathbf{C}$ be a $L$-structure and $\mathbf{C}'$ an irreducible structure. We say that $\mathbf{C}'$ is a *(strong) completion* of $\mathbf{C}$ if there is an injective homomorphism-embedding $f\colon \mathbf{C} \to \mathbf{C}'$. Let further $\mathbf{B}$ be an irreducible $L$-structure. Then $\mathbf{C}'$ is a *completion with respect to copies of* $\mathbf{B}$ if there is a function $f\colon C' \to C$ such that for every $\beta \in \binom{\mathbf{C}}{\mathbf{B}}$ it holds that $f \circ \beta$ is an embedding of $\mathbf{B}$ to $\mathbf{C}'$. A completion is *automorphism-preserving* if the respective map is automorphism-preserving.

All completions in this thesis will be strong, therefore we will sometimes omit the adjective.

**Definition 2.21** (Locally finite subclass [HN16])**.** Let $L$ be a language, $\mathcal{R}$ be a class of finite irreducible structures and $\mathcal{K} \subseteq \mathcal{R}$ a subclass of $\mathcal{R}$. We say that $\mathcal{K}$ is a *locally finite subclass of* $\mathcal{R}$ if for every $\mathbf{C}_0 \in \mathcal{R}$ and $\mathbf{B} \in \mathcal{K}$ there exists an integer $n = n(\mathbf{B}, \mathbf{C}_0)$ such that for every $L$-structure $\mathbf{C}$ there exists $\mathbf{C}' \in \mathcal{K}$ which is a completion of $\mathbf{C}$ with respect to copies of $\mathbf{B}$ provided that:

1. there exists a homomorphism-embedding from $\mathbf{C}$ to $\mathbf{C}_0$; and
2. for every substructure $\mathbf{S} \subseteq \mathbf{C}$ such that $\mathbf{S}$ has at most $n$ vertices there exists $\mathbf{S}' \in \mathcal{K}$ which is a completion of $\mathbf{S}$.

Now we can state the main result of [HN16].

**Theorem 2.22** (Hubička–Nešetřil [HN16], Theorem 2.2)**.** *Let $L$ be a language and $\mathcal{R}$ be a class of finite irreducible $L$-structures which has the Ramsey property. Let $\mathcal{K} \subseteq \mathcal{R}$ be a locally finite subclass of $\mathcal{R}$ which has the strong amalgamation property and is hereditary (if $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{A} \in \mathcal{K}$). Then $\mathcal{K}$ is Ramsey.*

When one works with a relational language $L$, the class $\mathcal{R}$ is usually the class of all ordered finite $L$-structures, which is Ramsey by the Nešetřil–Rödl theorem. For languages involving both relations and functions, we need an analogue of the Nešetřil–Rödl theorem proved recently by Evans, Hubička and Nešetřil.

**Theorem 2.23** (Evans–Hubička–Nešetřil [EHN17b])**.** *Let $L$ be a language and let $\mathcal{K}$ be a free amalgamation class of $L$-structures. Then $\overrightarrow{\mathcal{K}}$, the class of all linearly ordered structures from $\mathcal{K}$, is a Ramsey class.*

## 2.5 EPPA

There is another combinatorial property with a distant analogue of Theorem 2.22 called *extension property for partial automorphisms* (or *EPPA*). We only give the minimum necessary background, for a broader overview of EPPA consult the PhD thesis of Daoud Siniora [Sin17].

**Definition 2.24** (EPPA)**.** Let $L$ be a language and $\mathcal{C}$ be a class of finite $L$-structures. We say that $\mathcal{C}$ has the *extension property for partial automorphisms* (*EPPA*) if for every $\mathbf{A} \in \mathcal{C}$ there exists $\mathbf{B} \in \mathcal{C}$ and an embedding $\alpha\colon \mathbf{A} \to \mathbf{B}$ such that for every *partial automorphism* $f\colon \mathbf{A} \to \mathbf{A}$ (that is, an isomorphism of substructures of $\mathbf{A}$) there exists an automorphism $g$ of $\mathbf{B}$ such that $g$ extends $f$, or formally $\alpha \circ f \subseteq g \circ \alpha$. We call such a $\mathbf{B}$ an *EPPA-witness for* $\mathbf{A}$.

EPPA is sometimes also called the *Hrushovski property*, because Hrushovski was the first to prove that the class of all finite graphs has EPPA [Hru92]. The difficult thing on EPPA is finding a **finite** EPPA-witness $\mathbf{B}$, because, if $\mathcal{C}$ is a Fraïssé class, its Fraïssé limit is clearly the EPPA-witness for every structure in the class.

A distant analogue of the Hubička–Nešetřil theorem for EPPA is the following result by Herwig and Lascar.

**Theorem 2.25** (Herwig–Lascar [HL00])**.** *Let $L$ be a finite relational language, $\mathcal{O}$ be a finite family of finite $L$-structures and $\mathbf{A} \in \mathrm{Forb}(\mathcal{O})$ a finite $L$-structure. There exists a finite structure $\mathbf{B} \in \mathrm{Forb}(\mathcal{O})$ containing $\mathbf{A}$ such that every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.*

It is possible to strengthen the definition of EPPA and also the Herwig–Lascar theorem, see [SS17, HO03, Ott14].

# 3. The shortest path completion

In this section we study the basic properties of the shortest path completion. They are then used implicitly in the rest of the thesis. The ideas used in this chapter are often generalisations of [HKN17]; [HN16], [Neš07], [Sol05] or [Ver08] also proceed similarly.

**Definition 3.1.** Fix a partially ordered commutative semigroup $\mathfrak{M} = (M, \oplus, \preceq)$. An $\mathfrak{M}$-*edge-labelled graph* $\mathbf{G}$ is a pair $(G, d)$ where $G$ is the *vertex set* and $d$ is a partial function from $\binom{G}{2}$ to $M$. For simplicity, we shall write $d(u, v)$ instead of $d(\{u, v\})$ and keep in mind that the function $d$ is symmetric and undefined for $u = v$. A pair of vertices $u, v$ on which $d(u, v)$ is defined is called an *edge* of $\mathbf{G}$. We also call $d(u, v)$ the *length of the edge* $u, v$.

If $\mathfrak{M}$ is clear from context, we may write simply *edge-labelled graph.* We denote by $\mathcal{G}^{\mathfrak{M}}$ the class of all finite $\mathfrak{M}$-edge-labelled graphs.

An edge-labelled graph can also be seen as a relational structure. Hence the standard notions of homomorphism, embedding, and isomorphism extend naturally to edge-labelled graphs. (This is very important for this thesis!) We find it convenient to use notation that resembles the standard notation of metric spaces.

An ($\mathfrak{M}$-edge-labelled) graph $\mathbf{G}$ is *complete* if every pair of vertices forms an edge; a complete $\mathfrak{M}$-edge-labelled graph $\mathbf{G}$ is called an $\mathfrak{M}$-*metric space* if it is an $\mathfrak{M}$-metric space as defined in Definition 1.2. An $\mathfrak{M}$-edge-labelled graph $\mathbf{G} = (G, d)$ is $\mathfrak{M}$-*metric* (or *metric* if $\mathfrak{M}$ is clear from the context) if there exists an $\mathfrak{M}$-metric space $\mathbf{M} = (G, d')$ such that $d(u, v) = d'(u, v)$ for every edge $u, v$ of $\mathbf{G}$. Such a metric space $\mathbf{M}$ is also called a *(strong)* $\mathfrak{M}$-*metric completion* of $\mathbf{G}$. An $\mathfrak{M}$-edge-labelled graph which is not $\mathfrak{M}$-metric is called *non-*$\mathfrak{M}$*-metric.*

We shall adopt the standard graph-theoretic notions, such as the notion of a cycle; an $\mathfrak{M}$-edge-labelled graph $\mathbf{G}$ is a ($\mathfrak{M}$-edge-labelled) cycle if its edges form a cycle.

Given an $\mathfrak{M}$-edge-labelled graph $\mathbf{G}$ the *walk* from $u$ to $v$ is any sequence of vertices $u = v_1, v_2, \dots, v_n = v \in \mathbf{G}$ such that $v_i, v_{i+1}$ form an edge for every $1 \leq i < n$. If the sequence contains no repeated vertices, it is a *path.* The $\mathfrak{M}$-length (or simply a length) of a walk $\mathbf{W}$ is defined and denoted as

$$\|\mathbf{W}\| = d(v_1, v_2) \oplus d(v_2, v_3) \oplus \cdots \oplus d(v_{n-1}, v_n).$$

We say that $\mathbf{G}$ is *connected* if there exists a path from $u$ to $v$ for every choice of $u \neq v \in G$.

To avoid unnecessary notational complications, we shall sometimes treat a walk or a cycle as a (cyclic) sequence of elements of $\mathfrak{M}$ which represent the lengths of the edges of the path/cycle $\mathbf{K}$. In this case, we will say that $\mathbf{K} = (c_1, c_2, \dots, c_k)$, where $c_1, \dots, c_k \in \mathfrak{M}$ and the edges with lengths $c_i, c_{i+1}$ are adjacent (if $\mathbf{K}$ is a cycle, we furter want $c_1, c_k$ adjacent).

In 2007 Nešetřil proved that the class of all linearly ordered metric spaces has the Ramsey property [Neš07], while Solecki [Sol05] and Vershik [Ver08] independently proved that the class of all metric spaces has EPPA. All these results were based on the fact that a finite $\mathbb{R}^{>0}$-edge-labelled graph is metric if and only if it contains no non-metric cycle as a (non-induced) subgraph and that one can complete a metric edge-labelled graph to a metric space by letting the distance betwean each two vertices be the length of the shortest path connecting them.

As mentioned in Example 1.3, Sauer [Sau13b] classified the sets $S$ for which $\oplus_S$ is an associative operation and hence $(S, \oplus, \leq)$ is a partially ordered commutative semigroup. This happens if and only if the class of all finite metric spaces with distances from $S$ is an amalgamation class. Hubička and Nešetřil [HN16] gave Ramsey expansions of all Sauer's $S$-metric spaces (some of their results had been obtained before by Nguyen Van Thé [NVT10]), again using the shortest path completion (where now the lengths of paths are measured using the $\oplus_S$ operation). Their techniques also directly imply the same results for classes of $S$-metric spaces where one further forbids "short odd cycles", that is, triangles with distances $a, b, c$ such that $a + b + c$ is odd and "small enough".

Conant [Con15] studied EPPA in the context of generalised metric spaces which are $\mathfrak{M}$-metric spaces where $\mathfrak{M}$ is a linearly ordered commutative monoid. Hubička, Nešetřil and the author [HKN17] later found Ramsey expansions for all such spaces.

Braunfeld [Bra17], motivated by his classification of generalised permutation structures [Bra18a], found Ramsey expansions of $\Lambda$-ultrametric spaces for a distributive lattice $\Lambda = (\Lambda, \vee, \wedge, 0)$, which are essentially (see Remark 1.4) $\mathfrak{M}$-valued metric spaces, where $\mathfrak{M} = (\Lambda, \vee, \preceq)$ for $\preceq$ the standard lattice order.

Finally, Aranda, Bradley-Williams, Hubička, Karamanlis, Kompatscher, Pawliuk and the author [ABWH+17c, ABWH+17a, ABWH+17b] studied Ramsey expansion of metric spaces from Cherlin's catalogue of metrically homogeneous graphs [Che11, Che16]. These are metric spaces with distances from $\{0, 1, \ldots, \delta\}$ with $\delta \geq 3$, such that one can further forbid several families of cycles (such as cycles of short odd perimeter or cycles of long perimeter). A key ingredient was finding an explicit procedure to complete $\delta$-edge-labelled graphs. This was done by introducing a binary commutative operation $\oplus$ on $\{1, \ldots, \delta\}$ and an order of $\{1, \ldots, \delta\}$ and then in stages according to the order complete *forks* (that is, triples of vertices $u, v, w$ such that $uv$ and $vw$ are edges and $uw$ is not an edge) using the $\oplus$ operation. An exposition can be found in the bachelor thesis of the author [Kon18].

All the mentioned results, some rather directly, some not at all directly, satisfy some form of triangle inequality and admit some form of the shortest path completion.

Let $\mathbf{G}$ be an $\mathfrak{M}$-edge-labelled graph, $u, v \in G$ vertices of $\mathbf{G}$ and $\mathcal{W}$ be a finite family of walks in $\mathbf{G}$ from $u$ to $v$. Then we define $\inf(\mathcal{W}) = \inf_{\mathbf{W} \in \mathcal{W}} \|\mathbf{W}\|$.

**Definition 3.2** (Shortest path completion)**.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathbf{G} = (G, d)$ be a finite connected $\mathfrak{M}$-edge-labelled graph. Denote by $\mathcal{P}(u, v)$ the (finite) family of all paths in $\mathbf{G}$ from $u$ to $v$. Assume that $\inf(\mathcal{P}(u, v))$ is defined for every $u \neq v$.

Define $d'(u, v) = \inf(\mathcal{P}(u, v))$ for every $u \neq v$. Then we call the complete $\mathfrak{M}$-metric graph $\mathbf{A} = (G, d')$ the *shortest path completion* of $\mathbf{G}$.

Note that the shortest path completion assumes the existence of all infima in encounters. The motivation for the following definition is to be able to ensure that this assumption is satisfied. We also take care of the distributivity of inf and $\oplus$ which will be needed later when proving the strong amalgamation property.

For sets of distances $A, B$, denote by $A \oplus B$ the set $\{a \oplus b : a \in A, b \in B\}$. If $\mathbf{A}$ is an $\mathfrak{M}$-edge-labelled graph, $u, v, w$ are its distinct vertices, $\mathcal{W}$ is a family of walks from $u$ to $v$ and $\mathcal{W}'$ is a family of walks from $v$ to $w$, then by $\mathcal{W} \oplus \mathcal{W}'$ we mean the family of all walks from $u$ to $w$ which one can get by concatenating a walk from $\mathcal{W}$ with a walk from $\mathcal{W}'$ by the vertex $v$.

**Definition 3.3** (Disobedient cycles)**.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathcal{F}$ be a family of $\mathfrak{M}$-edge-labelled cycles. We say that $\mathcal{F}$ *contains all disobedient cycles* if the following holds for every $\mathfrak{M}$-metric space $\mathbf{A} \in \mathcal{M}_{\mathfrak{M}} \cap \operatorname{Forb} \mathcal{F}$:

Let $u, v, w \in A$ be a triple of vertices of $\mathbf{A}$, $\mathcal{W}$ be a finite family of walks from $u$ to $v$ in $\mathbf{A}$ and $\mathcal{W}'$ be a finite family of walks from $v$ to $w$ such that every walk in $\mathcal{W}$ and $\mathcal{W}'$ has at least two edges. Then it holds that

1. $\inf(\mathcal{W})$ and $\inf(\mathcal{W}')$ are defined; and
2. $\inf(\mathcal{W} \oplus \mathcal{W}') = \inf(\mathcal{W}) \oplus \inf(\mathcal{W}')$.

We want to show that the class of all finite $\mathfrak{M}$-metric spaces which omit homomorphic images of members of $\mathcal{F}$ has the strong amalgamation property. For it we need to ensure that the amalgamation process and the shortest path completion do not introduce any cycles from $\mathcal{F}$. We ensure it by putting some conditions on $\mathcal{F}$.

**Definition 3.4** (Omissible family)**.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathcal{F}$ be a family of $\mathfrak{M}$-labelled cycles. We say that $\mathcal{F}$ is *omissible* if for every cycle $(a_1, a_2, \ldots, a_k) \in \mathcal{F}$ the following conditions hold:

1. For every $1 \leq i \leq k$ it holds that $a_i \prec \bigoplus_{j \neq i} a_j$ (that is, $\mathcal{F}$ contains no *geodesic* or non-$\mathfrak{M}$-metric cycles).
2. For every $1 \leq i \leq k$ and every family of paths $\mathcal{P}$ such that $\inf(\mathcal{P}) = a_i$ and no two members of $\mathcal{P}$ together form a non-metric cycle or a cycle from $\mathcal{F}$ there is $\mathbf{P} \in \mathcal{P}$ such that the cycle $\mathbf{C}'$ is in $\mathcal{F}$, where $\mathbf{C}'$ is obtained from $\mathbf{C}$ by replacing $a_i$ by the path $\mathbf{P}$. We say that $\mathcal{F}$ is *closed under inverse steps of the shortest path completion* or *upwards closed.*
3. If the following two conditions are satisfied, we say that $\mathcal{F}$ is *downwards closed*:

   (a) For every $1 \leq i < j \leq k$ such that $j - i \geq 2$ and $(i, j) \neq (1, k)$ it holds that one of the cycles $(a_i, \ldots, a_j)$ and $(a_1, \ldots, a_{i-1}, a_{j+1}, \ldots, a_k)$ is in $\mathcal{F}$ or is non-$\mathfrak{M}$-metric (two edges of different lengths are also a non-$\mathfrak{M}$-metric cycle); and

   (b) For every $i, j$ such that $1 \leq i < j \leq k$, $j - i \geq 2$ and $(i, j) \neq (1, k)$ and an arbitrary distance $c \in \mathfrak{M}$ it holds that one of the cycles $(c, a_i, a_{i+1}, \ldots, a_j)$ and $(a_1, \ldots, a_{i-1}, c, a_{j+1}, \ldots, a_k)$ is in $\mathcal{F}$ or is non-$\mathfrak{M}$-metric (if $c$ is too small or large).

**Example 3.5.** For $\mathfrak{M} = (\mathbb{Z}^{\geq 1}, +, \leq)$ and an arbitrary positive integer $p$, every family $\mathcal{F}_p$ consisting of all metric cycles of odd perimeter smaller than $p$ is omissible. Indeed, the perimeter being odd ensures that $\mathcal{F}_p$ contains no geodesic cycles.

Because $\mathfrak{M}$ is linearly ordered, upwards closedness amounts to checking that if one replaces the edge of length $a$ by a path of length $a$, the cycle has still short odd perimeter and for downwards closedness it is enough to realise that if one cuts an odd cycle in two (by, possibly, an edge of length 0 which is interpreted as gluing two vertices together) one of them will have odd perimeter and either one will be non-metric, or both will have perimeter less than $p$.

The notion of an omissible family of cycles is a generalisation of Example 3.5. Cherlin and Shi [CS96] proved that metric spaces without short odd cycles have the strong amalgamation property, Ramsey property and EPPA follow in the same way as they do when one does not forbid any short odd cycles.

The main result of this section is the following theorem, but the auxilliary lemmas will be used later as well.

**Theorem 3.6.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and let $\mathcal{F}$ be an omissible family of $\mathfrak{M}$-edge-labelled cycles containing all disobedient ones. Let $\mathbf{G}$ be a finite* ***connected*** *$\mathfrak{M}$-edge-labelled graph such that it contains no homomorphic image of any member of $\mathcal{F}$.*

1. *Suppose that $\mathbf{G}$ is metric and let $\mathbf{G}' = (G, d')$ be its shortest path completion. Then $\mathbf{G}'$ is well defined and it is a completion of $\mathbf{G}$ in $\mathcal{M}_{\mathfrak{M}} \cap \operatorname{Forb} \mathcal{F}$ (in the sense of Definition 2.20).*
2. *If $\mathbf{G}$ is metric then every automorphism of $\mathbf{G}$ is also an automorphism of $\mathbf{G}' = (G, d')$, the shortest path completion of $\mathbf{G}$, if it is defined.*
3. *$\mathcal{M}_{\mathfrak{M}} \cap \operatorname{Forb} \mathcal{F}$ is a strong amalgamation class.*

The first part says that the shortest path completion is "complete" (can complete everything which can be completed at all), the second part says that it is "canonical" (this will be useful in Section 4.5) and the third part suggests that we are interested in "reasonable" classes.

In order to prove Theorem 3.6, we first prove several auxilliary results. All the proofs are straightforward.

**Lemma 3.7.** *Let $\mathfrak{M}$, $\mathcal{F}$ and $\mathbf{G}$ be as in Theorem 3.6.*

1. *Let $X \subseteq \mathfrak{M}$ be a set of distances and $a \in \mathfrak{M}$ be a distance such that $\inf X$ is defined and $\inf X \not\succeq a$. Then there is $x \in X$ with $x \not\succeq a$.*
2. *If $\mathbf{G}$ is $\mathfrak{M}$-metric, then the shortest path completion is defined for $\mathbf{G}$ (that is, for every pair of vertices $u, v \in G$ the infimum of the $\mathfrak{M}$-lengths of any family of paths from $u$ to $v$ is defined).*

*Proof.*

**1.** For a contradiction suppose that this is not true. Then this means that $x \succeq a$ for every $x \in X$. But then $a \preceq \inf X$, a contradiction.

**2.** Suppose that the shortest path completion of $\mathbf{G}$ is not defined. That means that there is a pair of vertices $u, v \in G$ and a collection of paths $\mathcal{P}$ from $u$ to $v$ such that $\inf_{P \in \mathcal{P}} \|P\|$ is not defined. Take a minimal such $\mathcal{P}$. As $\mathbf{G}$ omits all homomorphic images from $\mathcal{F}$ which contains all disobedient cycles, it means that one of the paths has length one, hence it is the edge $uv$. From minimality of $\mathcal{P}$ it follows that $i = \inf_{\mathbf{P} \in \mathcal{P} \setminus \{uv\}} \|\mathbf{P}\|$ is defined, but this in particular means that $i$ and $d(u, v)$ are incomparable. And from part 1 we get a path $\mathbf{P} \in \mathcal{P}$ such that $\|\mathbf{P}\| \not\succeq d(u, v)$, a contradiction with $\mathbf{G}$ being $\mathfrak{M}$-metric. $\square$

All the parts of the following lemma will be used not only in the proof of Theorem 3.6, but also implicitly in the rest of the thesis.

**Lemma 3.8.** *Let $\mathfrak{M}$, $\mathcal{F}$ and $\mathbf{G}$ be as in Theorem 3.6.*

1. *If defined, the shortest path completion $\mathbf{G}' = (G, d')$ of $\mathbf{G}$ is an $\mathfrak{M}$-metric space.*
2. *A cycle $\mathbf{C} = (a_1, \ldots, a_k)$ such that $\mathbf{C} \notin \mathcal{F}$ is non-$\mathfrak{M}$-metric if and only if for some $i$ it holds that $a_i \not\preceq \bigoplus_{j \neq i} a_j$.*
3. *$\mathbf{G}$ contains a homomorphic image of a non-$\mathfrak{M}$-metric cycle if and only if it contains a non-$\mathfrak{M}$-metric cycle as a (non-induced) subgraph.*
4. *If $\mathbf{G}$ is $\mathfrak{M}$-metric and $\mathbf{G}' = (G, d')$ is its shortest path completion, then $d'(u, v) = d(u, v)$ whenever $d(u, v)$ is defined (i.e. $\mathbf{G}'$ is a completion of $\mathbf{G}$ in the sense of Definition 2.20).*
5. *$\mathbf{G}$ is $\mathfrak{M}$-metric if and only if it contains no homomorphic image of a non-$\mathfrak{M}$-metric cycle $\mathbf{C}$.*

*Proof.*

**1.** We need to verify that $d'$ satisfies the triangle inequality. Take any three vertices $u, v, w \in \mathbf{G}$. Combine the paths from $\mathcal{P}(u, w)$ and $\mathcal{P}(w, v)$ in every possible way to get a family of walks $\mathcal{W}$ from $u$ to $v$ in $\mathbf{G}$. But then $\inf(\mathcal{W}) = \inf(\mathcal{P}(u, w) \oplus \mathcal{P}(w, v)) = d'(u, w) \oplus d'(w, v)$, because $\mathcal{F}$ contains all disobedient cycles. Clearly $\inf(\mathcal{P}(u, v)) \preceq \inf(\mathcal{W})$ as one can get a path from a walk by removing the "loops", thus we get $d'(u, v) = \inf(\mathcal{P}(u, v)) \preceq \inf(\mathcal{W})$.

**2.** Clearly such a cycle is non-$\mathfrak{M}$-metric. On the other hand, if for every $i$ it holds that $a_i \preceq \bigoplus_{j \neq i} a_j$, then the shortest path completion (which is defined as $\mathbf{C} \notin \mathcal{F}$) is a completion of $\mathbf{C}$ in the sense of Definition 2.20; it is an $\mathfrak{M}$-metric space by part 1 and it preserves the edges of $\mathbf{C}$ by the assumption on $\mathbf{C}$.

**3.** One implication is trivial. The other follows from $\oplus$ being monotone with respect to $\preceq$: Denote the cycle $\mathbf{C} = (a_1, \ldots, a_n)$ with $a_1 \not\preceq \bigoplus_{i=2}^{n} a_i$. Then it is enough to take the minimal subcycle of the homomorphic image of $\mathbf{C}$ containing the edge $a_1$.

**4.** Clearly $d'(u,v) \preceq d(u,v)$, as the edge $u,v$ is a path between $u$ and $v$. So it suffices to show $d'(u,v) \succeq d(u,v)$. We show a stronger claim: if $\mathbf{G}'' = (G, d'')$ is a completion of $\mathbf{G}$ then $d''(u,v) \preceq d'(u,v)$ for every $u \neq v \in G$.

Suppose, for a contradiction, that there are vertices $u \neq v \in G$ such that for some completion $\mathbf{G}''$ of $\mathbf{G}$ it holds that $d''(u,v) \not\preceq d'(u,v)$. By definition of $d'$, there is a family of paths $\mathcal{P}(u,v)$ in $\mathbf{G}$ with $d'(u,v) = \inf(\mathcal{P}(u,v))$. But these paths are also present in $\mathbf{G}''$ and hence $\mathbf{G}''$ contains a non-$\mathfrak{M}$-metric cycle by part 1 of Lemma 3.7, which is a contradiction.

**5.** If $\mathbf{G}$ contains a homomorphic image of a non-$\mathfrak{M}$-metric cycle then by part 3 it contains a non-$\mathfrak{M}$-metric cycle as a non-induced substructure. But a completion of $\mathbf{G}$ is in particular a completion of that cycle, which contradicts it being non-$\mathfrak{M}$-metric.

We claim that if $\mathbf{G}$ is non-$\mathfrak{M}$-metric, then there are vertices $u, v \in G$ such that $d(u,v) \not\preceq \inf(\mathcal{P}(u,v))$ where $\mathcal{P}(u,v)$ is the family of all paths from $u$ to $v$. By part 1 of Lemma 3.7 the statement then follows.

If $\mathbf{G}$ is non-$\mathfrak{M}$-metric and $\mathbf{G}' = (G, d')$, its shortest path completion, is defined, from non-metricity we get vertices $u \neq v \in G$ with $d'(u,v) \prec d(u,v)$. But that means that the infimum of the $\mathfrak{M}$-lengths of $\mathcal{P}(u,v)$ is not greater than or equal to $d(u,v)$. Otherwise $\mathbf{G}'$ is undefined and we proceed as in the proof of part 2 of Lemma 3.7. □

Now we are ready to prove Theorem 3.6.

*Proof of Theorem 3.6.*

**1.** The fact that $\mathbf{G}'$ is well-defined follows from Lemma 3.7; it is a metric completion of $\mathbf{G}$ by Lemma 3.8. It remains to check that $\mathbf{G}' \in \mathrm{Forb}(\mathcal{F})$.

Suppose for a contradiction that there is $\mathbf{C} \in \mathcal{F}$ and a homomorphism $f\colon \mathbf{C} \to \mathbf{G}'$. For every edge $uv \in \mathbf{C}$ there is a family of paths $\mathcal{P}_{uv}$ in $\mathbf{G}$ such that $\inf(\mathcal{P}_{uv}) = d'(f(u), f(v)) = d_{\mathbf{C}}(u,v)$. Because $\mathcal{F}$ is closed under inverse steps of the shortest path completion, there is $\mathbf{P}$ in $\mathcal{P}_{uv}$ such that the cycle which one gets from $\mathbf{C}$ by exchanging the edge $uv$ by the path $\mathbf{P}$ is in $\mathcal{F}$. Doing this for every edge $uv \in \mathbf{C}$, we get $\mathbf{C}' \in \mathcal{F}$ with a homomorphism to $\mathbf{G}$, which is a contradiction.

**2.** Let $\alpha$ be an automorphism of $\mathbf{G}$. We need to prove that for every $u, v \in G$ it holds that $d'(u,v) = d'(\alpha(u), \alpha(v))$. By definition there is a family $\mathcal{P}(u,v)$ of paths from $u$ to $v$ such that $\inf(\mathcal{P}(u,v)) = d'(u,v)$ for every pair of vertices $u, v$. Because $\alpha$ is automorphism, there also has to be a family of paths from $\alpha(u)$ to $\alpha(v)$ with the same infimum and vice versa, hence indeed $d'(u,v) = d'(\alpha(u), \alpha(v))$.

**3.** Now it is easy to see that part 5 of Lemma 3.8 also holds for $\mathbf{G}$ which is not connected, because it can be turned into a connected one by adding new edges connecting individual components without introducing any new cycles.

We will observe that for every $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ the free amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ contains no embedding of any non-$\mathfrak{M}$-metric cycle and no embedding of any cycle from $\mathcal{F}$. Then we can take the amalgam to be the shortest path completion of the free amalgam (possibly after connecting the components if $\mathbf{A} = \emptyset$).

First note that if we let $\mathcal{F}'$ be the union of $\mathcal{F}$ and the family of all non-$\mathfrak{M}$-metric cycles, then $\mathcal{F}'$ is still downwards closed (cf. Definition 3.4). Suppose for a contradiction that there is a cycle $\mathbf{F} \in \mathcal{F}'$ with a homomorphism $f$ to the free amalgam of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ and among all such situations take one where $\mathbf{F}$ has the smallest number of vertices.

In the range of $f$ there must be vertices from both $B_1 \setminus A$ and $B_2 \setminus A$. And because cycles are 2-connected, there are at least two vertices $u \neq v \in F$ with $f(u), f(v) \in A$. Either $f(u) = f(v)$, but then from downwards closedness of $\mathcal{F}'$ we get a contradiction with minimality of $|F|$. Otherwise $f(u) \neq f(v)$ and because $\mathbf{A}$ is complete, the distance $d_{\mathbf{A}}(f(u), f(v))$ is defined. Denote by $\mathbf{F}'$ and $\mathbf{F}''$ the two cycles one gets from $\mathbf{F}$ by adding the edge $uv$ of length $d_{\mathbf{A}}(f(u), f(v))$. From downwards closedness of $\mathcal{F}'$ it follows that one of them is in $\mathcal{F}'$, so suppose without loss of generality that $\mathbf{F}' \in \mathcal{F}'$. Clearly $f|_{F'}$ is a homomorphism from $\mathbf{F}'$ to the amalgam, which is a contradiction with minimality of $|F|$. □

# 4. Locally finite description of the unordered classes

For Theorem 2.22, one needs to be able to describe structures with a completion to the given class by a bounded number of forbidden substructures. However, as we have observed in Section 1.3, this is not always true for $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$.

Now we make a small detour into a little more advanced model theory topic, the elimination of imaginaries. It very well describes the fact that some semigroups have arbitrarily large non-metric cycles and suggests a way of dealing with it (which we indeed take in Section 4.2). However, the notions introduced here will not be used in the proofs, only in their motivation. We only give the minimum necessary definitions, for more see [Hod93].

**Definition 4.1** (Definable equivalences)**.** Let $\mathbf{A}$ be an $L$-structure and $R \subseteq A^n$ be a relation. We say that $R$ is a *definable* if there is a first-order formula $\varphi(\bar{x})$ (in language $L$) such that for every $\bar{x} \in A^n$ it holds that $\bar{x} \in R$ if and only if $\varphi(\bar{x})$ holds. A *definable equivalence* is a definable relation which is an equivalence. A *definable function* is a definable relation $f \subseteq A^{k+\ell}$ such that for every $\bar{x} \in A^k$ there is exactly one $\bar{y} \in A^\ell$ such that $\bar{x}^\frown\bar{y} \in f$ (where $\bar{x}^\frown\bar{y}$ is the concatenation of $\bar{x}$ and $\bar{y}$).

**Definition 4.2** (Elimination of imaginaries)**.** Let $\mathbf{A}$ be an $L$-structure. Every equivalence class of every definable equivalence relation on $A^n$ is an *imaginary element.* We say that $\mathbf{A}$ *eliminates imaginaries* if for every definable equivalence relation $E$ on $A^n$ there is a definable[1] function $f\colon A^n \to A^k$ such that for every $\bar{u}, \bar{v} \in A^n$ it holds that

$$\bar{u}E\bar{v} \Leftrightarrow f(\bar{u}) = f(\bar{v}).$$

**Example 4.3.** Consider an arbitrary expansion of the ordered natural numbers $\mathbb{N}$. It eliminates imaginaries: Indeed, let $E \subseteq \mathbb{N}^{2n}$ be an arbitrary definable equivalence. To each tuple $\bar{x} \in \mathbb{N}^n$ we can definably assign the lexicographically smallest tuple $\bar{y} \in \mathbb{N}^n$ such that $\bar{x}E\bar{y}$.

On the other hand, the $\mathfrak{M}$-valued metric spaces for $\mathfrak{M} = (\{1,2\}, \max, \leq)$, which we have seen as an example of arbitrary large non-metric cycles, do not eliminate imaginaries due to the "being in distance 1" definable equivalence. A possible way to fix this is to add a new vertex for each "ball of diameter 1" and link every "original vertex" to its "ball vertex" by an explicitly added unary function.

The goal of this chapter is to generalise the construction from Example 4.3 and expand the classes $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ to obtain classes (isomorphic with $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ as categories) which have a locally finite description.

In order to do that, we first (in Section 4.1) study the *block structure* of partially ordered semigroups. Then (in Section 4.2) we define the $L^\star_{\mathfrak{M}}$-expansion

[1] Note that, in model theory, these definitions are applied on infinite structures. In order to avoid some pathologies, when talking about formulas and structures in the following definitions, we should actually talk about classes of structures and require the formulas to be fixed for the whole class.

where we add new vertices as explicit representants of each *block equivalence class* and link the original vertices to them by unary functions, thereby eliminating imaginaries. In Section 4.3 we introduce an approximation of maxima of blocks (because true maxima might not exist for infinite blocks) which then allow us to select from each non-$\mathfrak{M}$-metric cycle a bounded number of *important* edges and then in Section 4.4 we prove that indeed the $L^{\star}_{\mathfrak{M}}$-expansions have a locally finite description.

In Chapter 5 we further use the results of this chapter to obtain Ramsey expansions of $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. Besides the Ramsey property, the results of this chapter by themselves suffice to prove EPPA (Section 4.5).

## 4.1 Blocks, block lattice and block equivalences

Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup. For a positive integer $n$ and $a \in \mathfrak{M}$ we denote by $n \times a$ the expression $a \oplus \cdots \oplus a$ with $n$ summands. We say that $\mathfrak{M}$ is *archimedean* if for every $a, b \in \mathfrak{M}$ there is a positive integer $n$ such that $n \times a \succeq b$.

**Example 4.4.** There are many examples of both archimedean and non-archimedean partially ordered commutative semigroups which everyone encounters in their everyday life. All of $(\mathbb{Z}^{>0}, +, \leq)$, $(\mathbb{R}^{>0}, +, \leq)$ or $(\mathbb{R}^{>1}, \cdot, \leq)$ are archimedean. On the other hand the multiplicative monoid $(\mathbb{Z}^{>0}, \cdot, |, 1)$, where by $|$ we mean the "is a divisor of" relation, is non-archimedean because, for example, no power of 2 is divisible by 3.

Note that the monoid $(\mathbb{N}, +, \leq, 0)$, when treated as a partially ordered commutative semigroup, is also non-archimedean, because $n \times 0 = 0$ for every $n$.

As we will show later, the definable equivalences on $\mathfrak{M}$-metric spaces are related to archimedean subsemigroups of $\mathfrak{M}$. The following is a generalisation of a definition by Sauer [Sau13a].

**Definition 4.5.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup. A *block* $\mathcal{B}$ of $\mathfrak{M}$ is either a subset of $\mathfrak{M}$ such that it induces a maximal archimedean subsemigroup of $\mathfrak{M}$ or a special block $\mathbf{0}$ (corresponding to the empty set).

The introduction of the block $\mathbf{0}$ will be useful later. Note that $\mathbf{0}$ is added even if $\mathfrak{M}$ contains a neutral element which one would intuitively expect to represent the identity (it does not). The basic properties of blocks can be summarized as follows.

**Lemma 4.6.** *Given a partially ordered commutative semigroup $\mathfrak{M} = (M, \oplus, \preceq)$ it holds that:*

1. *For every $a \in \mathfrak{M}$ there exists a unique block $\mathcal{B}(a)$ containing $a$.*
2. *Let $a, b \in \mathfrak{M}$. Then $a, b$ are in the same block if and only if there exist $m, n$ such that $m \times a \succeq b$ and $n \times b \succeq a$.*

*Proof.* Let

$$\mathcal{B}(a) = \{b \in \mathfrak{M} \mid (\exists n)(n \times a \succeq b) \wedge (\exists n)(n \times b \succeq a)\}.$$

It is easy to check that $(\mathcal{B}(a), \oplus, \preceq)$ is an archimedean subsemigroup of $\mathfrak{M}$ containing $a$. Maximality and uniqueness follow from the fact that no $b \in \mathfrak{M} \setminus \mathcal{B}(a)$ can be in the same archimedean subsemigroup as $a$. Parts 1 and 2 now follow. $\square$

Given a partially ordered commutative semigroup $\mathfrak{M} = (M, \oplus, \preceq)$ and $a \in \mathfrak{M}$, we will always denote by $\mathcal{B}(a)$ the unique block of $\mathfrak{M}$ containing $a$ given by part 1 of Lemma 4.6.

**Lemma 4.7.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathcal{B}$ be a block of $\mathfrak{M}$. Whenever $a, c \in \mathcal{B}$ and $b \in \mathfrak{M}$ such that $a \prec b \prec c$, then $b \in \mathcal{B}$.*

*Proof.* Take arbitrary $a, c \in \mathcal{B}$ and $b \in \mathfrak{M}$ with $a \prec b \prec c$. As $a, c \in \mathcal{B}$, there is $n$ such that $n \times a \succeq c$. But then also $n \times a \succeq b$ and hence by part 2 of Lemma 4.6 $a$ and $b$ are in the same block. $\square$

If $\preceq$ is a linear order, Lemma 4.7 means that blocks form intervals. And this motivates the following definition:

**Definition 4.8.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup. By the same symbol $\preceq$ we denote the order on blocks of $\mathfrak{M}$ putting $\mathcal{B} \preceq \mathcal{B}'$ if for every $a \in \mathcal{B}$ there is $b \in \mathcal{B}'$ such that $a \preceq b$.

Note that $\mathbf{0} \preceq \mathcal{B}$ for every block $\mathcal{B}$. By Lemma 4.7 $\preceq$ is a partial order of blocks of $\mathfrak{M}$. Note that an equivalent definition could just ask for a pair $a \in \mathcal{B}, b \in \mathcal{B}'$ with $a \preceq b$.

Recall from the abstract algebra course that a structure $(A, \vee, \wedge)$ is a *lattice* if $\vee$ (*join*) and $\wedge$ (*meet*) are binary operations on $A$ satisfying the following equations.

$$
\begin{aligned}
a \wedge a &= a & \qquad a \vee a &= a \\
a \wedge b &= b \wedge a & a \vee b &= b \vee a \\
a \wedge (b \wedge c) &= (a \wedge b) \wedge c & a \vee (b \vee c) &= (a \vee b) \vee c \\
\\
a \wedge (a \vee b) &= a & a \vee (a \wedge b) &= a.
\end{aligned}
$$

Recall also that if $(B, \leq)$ is a partial order, then for $a, b \in B$ we call $c \in B$ their infimum and write $c = \inf(a, b)$ if $c \leq a, b$ and for every $x \in B$ such that $x \leq a, b$ it holds that $x \leq c$ and analogously we call $d \in B$ their supremum ($d = \sup(a, b)$) if $d \geq a, b$ and for every $x \geq a, b$ it holds that $x \geq d$.

It is a well-known fact that lattices and partial orders where all infima and suprema are defined are in 1-to-1 correspondence (in one direction just let $\vee$ be the supremum and $\wedge$ the infimum, in the other direction put $a \leq b$ if $a \wedge b = a$). Also, clearly, from the existence of infima of pairs we get the existence of infima of any finite sets, the same for suprema.

In Definition 4.8 we introduced the order $\preceq$ of blocks, which is inherited from the order $\preceq$ of the elements of $\mathfrak{M}$. It is natural to study the infima and suprema in the block order. For suprema it is quite straightforward. Let $\mathcal{B}_1, \mathcal{B}_2$ be blocks of $\mathfrak{M}$ and take arbitrary $a \in \mathcal{B}_1$ and $b \in \mathcal{B}_2$. Then $\mathcal{B}(a \oplus b)$ is the supremum of

$\mathcal{B}_1, \mathcal{B}_2$. Indeed, clearly $\mathcal{B}(a \oplus b) \succeq \mathcal{B}_1, \mathcal{B}_2$. And if $\mathcal{B}' \succeq \mathcal{B}_1, \mathcal{B}_2$, then in particular there are $x_a, x_b \in \mathcal{B}'$ with $x_a \succeq a$ and $x_b \succeq b$, hence $x_a \oplus x_b \succeq a \oplus b$, thus $\mathcal{B}' \succeq \mathcal{B}(a \oplus b)$ and we are done. We denote the join of blocks by $\vee$.

On the other hand, not all infima of blocks need to be defined. However, in our applications we will need the block order to be a lattice (in fact, a distributive lattice) and thus we choose to denote the block infima by the $\wedge$ symbol nonetheless. From the definition of $\preceq$ it follows that a block $\mathcal{B} \neq \mathbf{0}$ is the meet of blocks $\mathcal{B}_1, \mathcal{B}_2 \neq \mathbf{0}$ if and only if $\mathcal{B} \preceq \mathcal{B}_1, \mathcal{B}_2$ and for all $a \in \mathcal{B}_1, b \in \mathcal{B}_2$ and every $x \in \mathfrak{M}$ such that $x \preceq a, b$ there is $c \in \mathcal{B}$ such that $c \succeq x$.

**Definition 4.9.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup. We say that a block $\mathcal{B}$ is *meet-reducible* if there are blocks $\mathcal{B}_1, \mathcal{B}_2$ such that $\mathcal{B} = \mathcal{B}_1 \wedge \mathcal{B}_2$ and $\mathcal{B} \notin \{\mathcal{B}_1, \mathcal{B}_2\}$. Otherwise $\mathcal{B}$ is *meet-irreducible.*

**Definition 4.10.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup, $\mathbf{A}$ be an $\mathfrak{M}$-metric space and $\mathcal{B}$ be a block of $\mathfrak{M}$.

1. A *block equivalence* $\sim_{\mathcal{B}}$ on vertices of $\mathbf{A}$ is given by $u \sim_{\mathcal{B}} v$ whenever there exists $a \in \mathcal{B}$ such that $d(u, v) \preceq a$.
2. A *ball of diameter* $\mathcal{B}$ in $\mathbf{A}$ is any equivalence class of $\sim_{\mathcal{B}}$ in $\mathbf{A}$.

To verify that for every block $\mathcal{B}$ the relation $\sim_{\mathcal{B}}$ is indeed an equivalence relation it suffices to check transitivity. Given a triangle with distances $a, b, c$, if there exist $a' \in \mathcal{B}$ and $b' \in \mathcal{B}$ such that $a \preceq a'$ and $b \preceq b'$, it also holds that $c \preceq a \oplus b \preceq a' \oplus b' \in \mathcal{B}$. Note that $u \sim_{\mathbf{0}} v$ if and only if $u = v$. It will be convenient later in Chapter 5 to have the identity represented by a block.

For a finite set of blocks $B$ we denote the meet of $B$ as $\bigwedge B = B_1 \wedge \cdots \wedge B_k$ if it exists.

**Observation 4.11.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathbf{A}$ be an $\mathfrak{M}$-metric space. Then for every two blocks $\mathcal{B}_1, \mathcal{B}_2 \neq \mathbf{0}$ such that their meet is defined and different from $\mathbf{0}$, every ball $B_1$ of diameter $\mathcal{B}_1$ and every ball $B_2$ of diameter $\mathcal{B}_2$ it holds that $B_1 \cap B_2$ is either empty or it is a ball of diameter $\mathcal{B}_1 \wedge \mathcal{B}_2$.*

*Proof.* Denote $B = B_1 \cap B_2$ and suppose that $B$ is nonempty. We need to prove that $B$ is a ball of diameter $\mathcal{B}_1 \wedge \mathcal{B}_2$.

Let $u, v \in A$ be vertices of $\mathbf{A}$ such that $u \sim_{\mathcal{B}_1} v$ and $u \sim_{\mathcal{B}_2} v$. Denote $a = d_{\mathbf{A}}(u, v)$. There are $b_1 \in \mathcal{B}_1$ and $b_2 \in \mathcal{B}_2$ such that $a \preceq b_1, b_2$, but then by the definition of meet there is $c \in \mathcal{B}_1 \wedge \mathcal{B}_2$ such that $c \succeq a$ and hence $u \sim_{\mathcal{B}_1 \wedge \mathcal{B}_2} v$. This means that for every $u, v \in B$ it holds that $u \sim_{\mathcal{B}_1 \wedge \mathcal{B}_2} v$.

On the other hand if there are $u \in B$ and $v \in A$ such that $u \sim_{\mathcal{B}_1 \wedge \mathcal{B}_2} v$, then in particular $u \sim_{\mathcal{B}_i} v$ for $i \in \{1, 2\}$, so $v \in B_1 \cap B_2$. □

## 4.2 Explicit representation of balls

The existence of block equivalences is an obstacle for local finiteness because every cycle such that one edge says $u \nsim_{\mathcal{B}} v$, but the rest of the cycle says $u \sim_{\mathcal{B}} v$ is clearly non-$\mathfrak{M}$-metric. In order to deal with this, we explicitly represent the balls of all meet-irreducible diameters by new vertices and link every *original vertex* to its corresponding *ball vertices* by unary functions, thereby eliminating the block imaginaries.

For the rest of the section, fix a partially ordered commutative semigroup $\mathfrak{M} = (M, \oplus, \preceq)$ with **finitely many blocks** where the meet of every pair of blocks is defined and different from $\mathbf{0}$ (unless, of course, one of them is $\mathbf{0}$).

**Example 4.12** (Running example — introduction)**.** In order to obtain a Ramsey expansion, we are going to need to work with several expansions of $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. While all of them are adaptations of rather standard model-theoretical concepts (that is, eliminating imaginaries and convex ordering), we aim for this text to also be accessible to the combinatorial audience. Therefore we develop these notions and classes without refering to the model-theoretical concepts and we also find it helpful to illustrate them on an example.

Let $\mathfrak{D} = (\mathbb{N}^3, +, \leq)$ be the partially ordered commutative semigroup whose vertices are triples of natural numbers (that is, nonnegative integers), the addition is coordinate-wise and $(a, b, c) \leq (u, v, w)$ if and only if $a \leq u$, $b \leq v$ and $c \leq w$. In other words, $\mathfrak{D}$ is the third power of the standard monoid of natural numbers.

$\mathfrak{D}$ has nine blocks: $\mathbf{0}$ and eight "standard" blocks, each corresponding to a subset $I \subseteq \{1, 2, 3\}$ by

$$\mathcal{B}_I = \{(x_1, x_2, x_3) \in \mathbb{N}^3; x_i \neq 0 \Leftrightarrow i \in I\}.$$

The block lattice is distributive and is isomorphic to the subset lattice of a three-element set with a new identity added (to accomodate $\mathbf{0}$).

Note that in the $\mathfrak{D}$-metric spaces there can be distinct vertices in distance $(0, 0, 0)$.

Let $R_{\mathfrak{M}}$ be the set of all non-maximal meet-reducible blocks of $\mathfrak{M}$ and $I_{\mathfrak{M}}$ the set of all non-maximal meet-irreducible blocks of $\mathfrak{M}$. Note that $\mathbf{0} \in I_{\mathfrak{M}}$ (because by our assumption the meet of every pair of non-$\mathbf{0}$ blocks is defined and different from $\mathbf{0}$).

**Example 4.13** (Running example — $I_{\mathfrak{D}}$ and $R_{\mathfrak{D}}$)**.** $I_{\mathfrak{D}} = \{\mathbf{0}, \mathcal{B}_{\{1,2\}}, \mathcal{B}_{\{1,3\}}, \mathcal{B}_{\{2,3\}}\}$, $R_{\mathfrak{D}} = \{\mathcal{B}_{\emptyset}, \mathcal{B}_{\{1\}}, \mathcal{B}_{\{2\}}, \mathcal{B}_{\{3\}}\}$. $\mathcal{B}_{\{1,2,3\}}$ is the maximum block of $\mathfrak{D}$.

Later it will turn out to be useful to only represent irreducible blocks. The following lemma shows that it is in some sense enough.

**Lemma 4.14.** *Let $\mathcal{B}$ be a block of $\mathfrak{M}$. Then there is a set $B \subseteq I_{\mathfrak{M}}$ such that $\mathcal{B}$ is the meet of $B$.*

*Proof.* If $\mathcal{B} \in I_{\mathfrak{M}}$, then $B = \{\mathcal{B}\}$ is a suitable choice, if $\mathcal{B}$ is the maximum block, then take $B = \emptyset$. Enumerate all blocks as $\mathcal{B}_1, \ldots, \mathcal{B}_b$ such that there are no $i < j$ with $\mathcal{B}_i \preceq \mathcal{B}_j$ and let $i$ be the smallest integer such that the statement does not hold for $\mathcal{B}_i$.

It follows that $\mathcal{B}_i \in R_{\mathfrak{M}}$, hence there are two blocks $\mathcal{B}_j, \mathcal{B}_k$ different from $\mathcal{B}_i$ such that $\mathcal{B}_j \wedge \mathcal{B}_k = \mathcal{B}_i$. In particular, $\mathcal{B}_i \preceq \mathcal{B}_j, \mathcal{B}_k$, hence $j, k < i$. Thus, by minimality of $i$, we have $B_j, B_k \subseteq I_{\mathfrak{M}}$ with $\bigwedge B_j = \mathcal{B}_j$ and $\bigwedge B_k = \mathcal{B}_k$. It follows that $\mathcal{B} = \bigwedge B$ for $B = B_j \cup B_k$. $\square$

Recall that we interpret an $\mathfrak{M}$-metric space as a relational structure $\mathbf{A}$ in the language $L_{\mathfrak{M}}$ with (possibly infinitely many) binary relations $R^s$, $s \in \mathfrak{M}$. Now we are going to add explicit representants for balls which will later make it possible to ensure that each non-$\mathfrak{M}$-metric cycle has a "non-$\mathfrak{M}$-metric substructure" of bounded size, which is necessary for Theorem 2.22. This construction is by now standard in the structural Ramsey theory (cf. [HN16] or [Bra17]), however, unlike Braunfeld [Bra17], for our expansion it is enough to use unary functions thanks to Lemma 4.14 and Observation 4.11.

**Definition 4.15.** Denote by $L^\star_{\mathfrak{M}}$ the expansion of $L_{\mathfrak{M}}$ adding

1. unary functions $F^{\mathcal{B}}$ for every $\mathcal{B} \in I_{\mathfrak{M}} \setminus \{\mathbf{0}\}$; and
2. unary functions $F^{\mathcal{B},\mathcal{B}'}$ for every pair of blocks $\mathcal{B}, \mathcal{B}' \in I_{\mathfrak{M}} \setminus \{\mathbf{0}\}$ such that $\mathcal{B} \prec \mathcal{B}'$.

For a given metric space $\mathbf{A} \in \mathcal{M}_{\mathfrak{M}}$, denote by $L^\star(\mathbf{A})$ the $L^\star_{\mathfrak{M}}$-structure $\mathbf{A}^\star$ created by the following procedure:

1. Start with $\mathbf{A}^\star$ being an exact copy of $\mathbf{A}$.
2. For every $\mathcal{B} \in I_{\mathfrak{M}} \setminus \{\mathbf{0}\}$ enumerate balls of diameter $\mathcal{B}$ in $\mathbf{A}$ as $E^1_{\mathcal{B}}, \dots, E^{n_{\mathcal{B}}}_{\mathcal{B}}$.
3. For every $\mathcal{B} \in I_{\mathfrak{M}} \setminus \{\mathbf{0}\}$ and $1 \leq i \leq n_{\mathcal{B}}$ add a new vertex $v^i_{\mathcal{B}}$ to $\mathbf{A}^\star$. We call these vertices *ball vertices* (in contrast to the *original vertices*).
4. For every $\mathcal{B} \in I_{\mathfrak{M}} \setminus \{\mathbf{0}\}$, $1 \leq i \leq n_{\mathcal{B}}$ and $v \in E^i_{\mathcal{B}}$ put $F^{\mathcal{B}}_{\mathbf{A}^\star}(v) = v^i_{\mathcal{B}}$.
5. For every pair of blocks $\mathcal{B}, \mathcal{B}' \in I_{\mathfrak{M}} \setminus \{\mathbf{0}\}$ such that $\mathcal{B} \prec \mathcal{B}'$ and every $1 \leq i \leq n_{\mathcal{B}}$ put $F^{\mathcal{B},\mathcal{B}'}_{\mathbf{A}^\star}(v^i_{\mathcal{B}}) = F^{\mathcal{B}'}_{\mathbf{A}^\star}(v)$ where $v$ is some vertex of $E^i_{\mathcal{B}}$.

Denote by $\mathcal{M}^\star_{\mathfrak{M}}$ the class of all $L^\star(\mathbf{A})$, $\mathbf{A} \in \mathcal{M}_{\mathfrak{M}}$ and by $\mathcal{H}^\star_{\mathfrak{M}}$ the smallest hereditary superclass of $\mathcal{M}^\star_{\mathfrak{M}}$ (see Remark 4.20).

Note that the assumption that $\mathfrak{M}$ has finitely many blocks was necessary to ensure that $L^\star(\mathbf{A})$ has finitely many vertices if $\mathbf{A}$ has. Also note that the original vertices could be understood as ball vertices for $\mathbf{0}$ (and indeed, Braunfeld does exactly this and it makes it possible to deal with lattices where $\mathbf{0}$ is meet-reducible), but in our case they need to carry the distances and thus, for clarity, we treat them completely separately.

**Example 4.16** (Running example — $L^\star_{\mathfrak{D}}$)**.** In Figure 4.1, the $L^\star$ expansion of the triangle with distances $(1, 0, 0)$, $(0, 2, 3)$ and $(1, 2, 3)$ is depicted. The $F^{\mathcal{B}}$ functions are depicted by dashed arrows and their labels are moved to the ball vertices. There are no $F^{\mathcal{B},\mathcal{B}'}$ functions in $L^\star_{\mathfrak{D}}$.

**Proposition 4.17.** *The categories $\mathcal{M}_{\mathfrak{M}}$ and $\mathcal{M}^\star_{\mathfrak{M}}$ are isomorphic. In other words, $L^\star$ is a bijection between the structures from $\mathcal{M}_{\mathfrak{M}}$ and $\mathcal{M}^\star_{\mathfrak{M}}$ which preserves embeddings and their compositions.*

*Proof.* It is straightforward to check that $L^\star$ (with its inverse, which forgets the extra structure and the ball vertices) gives such an isomorphism. □

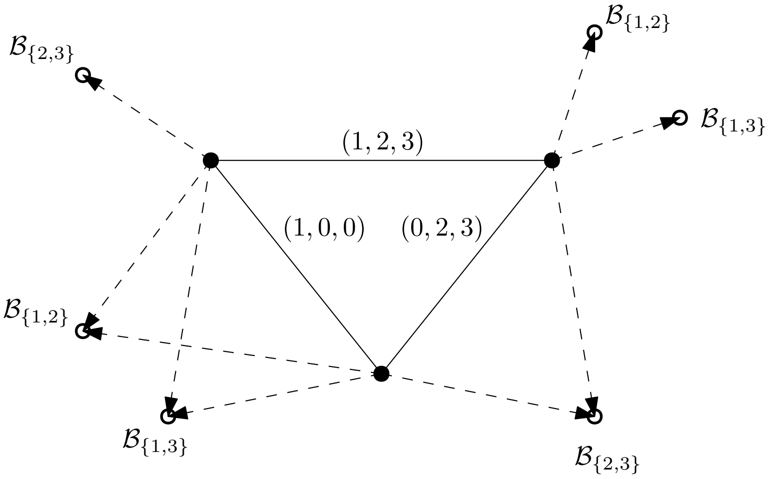



Figure 4.1: The $L^{\star}_{\mathfrak{D}}$-expansion, see Example 4.16

Proposition 4.17 implies that $\mathcal{M}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ has the amalgamation property. However, the strong amalgamation property is not a property expressible only in the categorical language. And, in fact, in order to get the strong amalgamation property for $\mathcal{M}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$, we need to ensure that the shortest path completion is consistent with the block structure (that is, the shortest path completion does not glue any balls which are not explicitly glued in the free amalgam).

**Definition 4.18** (Meet synchronization)**.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and let $\mathcal{F}$ be a family of $\mathfrak{M}$-edge-labelled cycles containing all disobedient ones. We say that $\mathcal{F}$ *synchronizes meets* if for every $\mathbf{A} \in \mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}\,\mathcal{F}$, every $u, v, w, x \in A$, every finite family $\mathcal{P}$ of paths in $\mathbf{A}$ from $u$ to $v$ and every finite family $\mathcal{P}'$ of paths in $\mathbf{A}$ from $w$ to $x$ the following holds:

Suppose that there is a bijection $f\colon \mathcal{P} \to \mathcal{P}'$ such that $\mathcal{B}(\|\mathbf{P}\|) = \mathcal{B}(\|f(\mathbf{P})\|)$ for every $\mathbf{P} \in \mathcal{P}$ and every $\mathbf{P} \in \mathcal{P} \cup \mathcal{P}'$ has at least two edges. Then

$$\mathcal{B}(\inf(\mathcal{P})) = \mathcal{B}(\inf(\mathcal{P}')).$$

If all meets of blocks are defined, meet synchronization means that whenever we encounter a set of paths in $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}\,\mathcal{F}$, then the block where their infimum lies only depends on the blocks where the lengths of the paths lie, that is,

$$\mathcal{B}(\inf(\mathcal{P})) = \bigwedge_{\mathbf{P} \in \mathcal{P}} \mathcal{B}(\|\mathbf{P}\|).$$

*Remark.* It is possible that the definition of meet synchronization is redundant, see Question 1.

**Corollary 4.19.** *If $\mathcal{F}$ is an omissible family of $\mathfrak{M}$-edge-labelled cycles containing all disobedient ones which synchronizes meets, then $\mathcal{M}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$, the subclass of $\mathcal{M}^{\star}_{\mathfrak{M}}$ which omits homomorphic images of members of $\mathcal{F}$, is a strong amalgamation class.*

*Proof.* The isomorphism of categories preserves the distances and it is easy to check that since $\mathcal{F}$ synchronizes meets, the shortest path completion does not introduce any unnecessary $\sim_{\mathcal{B}}$ relations if one chooses a distance from the largest block to connect the connected components of the amalgam (cf. proof of part 3 of Theorem 3.6). $\square$

*Remark.* The functions $F^{\mathcal{B},\mathcal{B}'}$ were added to restrict what the substructures of a structure from $\mathcal{M}^\star_{\mathfrak{M}}$ are. In particular, if a substructure contains a ball vertex for diameter $\mathcal{B}$, it also needs to contain all ball vertices of larger diameters which represent the "superballs". This ensures that amalgamation behaves reasonably, saying that two vertices are in the same ball of diameter $\mathcal{B}$, but in different balls of diameter $\mathcal{B}' \succ \mathcal{B}$ has no reasonable interpretation in the metric space.

*Remark* 4.20. Unfortunately, Theorem 2.22 requires the class $\mathcal{K}$ to be hereditary. $\mathcal{H}^\star_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ differs from $\mathcal{M}^\star_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ by adding the structures which further contain some ball vertices to which no original vertex is linked. This hereditary class also has the strong amalgamation property (for a *confined* family $\mathcal{F}$) which we will prove later (see Corollary 4.32).

### 4.2.1 Semigroups with infinitely many blocks

We have already seen (and will see again) that one sometimes needs to assume that $\mathfrak{M}$ has finitely many blocks. The following Theorem says that one can, to some extent, assume it without loss of generality. In order to state it, we first need to give one more definition which will be useful later.

**Definition 4.21** ($S$-locally finite and confined families)**.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup, let $\mathcal{F}$ be a family of $\mathfrak{M}$-edge-labelled graphs and let $S$ be a finite subset of $\mathfrak{M}$. We say that $\mathcal{F}$ is *$S$-locally finite* if there are only finitely many $S$-edge-labelled graphs in $\mathcal{F}$. If $\mathcal{F}$ is $S$-locally finite for every finite $S \subseteq \mathfrak{M}$, we say that $\mathcal{F}$ is *confined.*

**Theorem 4.22.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathcal{F}$ be a family of $\mathfrak{M}$-edge-labelled cycles. Suppose that the following conditions hold:*

1. *$\mathcal{F}$ is $\mathfrak{M}$-omissible;*
2. *$\mathcal{F}$ contains all $\mathfrak{M}$-disobedient cycles;*
3. *$\mathcal{F}$ is confined;*
4. *all meets of blocks of $\mathfrak{M}$ are defined and $\mathbf{0}$ is meet-irreducible; and*
5. *$\mathcal{F}$ synchronizes meets.*

*Then for every* ***finite*** *$S \subseteq \mathfrak{M}$ there is a countable partially ordered commutative semigroup $\mathfrak{M}' \subseteq \mathfrak{M}$ and a family of $\mathfrak{M}'$-edge-labelled cycles $\mathcal{F}'$ such that $S \subseteq \mathfrak{M}'$ and the following hold:*

1. *$\mathcal{F}'$ is $\mathfrak{M}$-omissible;*
2. *$\mathcal{F}'$ contains all $\mathfrak{M}'$-disobedient cycles;*
3. *$\mathcal{F}'$ is confined;*
4. *all meets of blocks of $\mathfrak{M}'$ are defined and $\mathbf{0}$ is meet-irreducible;*
5. *$\mathcal{F}'$ synchronizes meets; and*
6. *$\mathfrak{M}'$ has finitely many blocks.*

*Furthermore it holds that*

$$\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F}) \cap \mathcal{G}^S \subseteq \mathcal{M}_{\mathfrak{M}'} \cap \mathrm{Forb}(\mathcal{F}') \subseteq \mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F}),$$

*where* $\mathcal{G}^S$ *is the class of all finite* $S$*-edge-labelled graphs.*

Informally, Theorem 4.22 says that "in a problem about finitely many finite structures from $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ one can assume that $\mathfrak{M}$ has finitely many blocks". From $\mathfrak{M}'$ being countable it furthermore follows that $\mathcal{M}_{\mathfrak{M}'} \cap \mathrm{Forb}(\mathcal{F}')$ is a Fraïssé class.

*Proof.* Let $\mathcal{A}$ be the class of all $\mathfrak{M}$-metric $S$-edge-labelled graphs from $\mathrm{Forb}(\mathcal{F})$ and define $\mathcal{A}^+$ to be the class containing the $\mathfrak{M}$-shortest path completions of graphs from $\mathcal{A}$ (they exist by Theorem 3.6). Finally let $S^+$ be the set of all distances appearing in $\mathcal{A}^+$.

Clearly $S^+$ is closed on $\oplus$: If $a, b \in S^+$ then there are $S$-edge-labelled graphs $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and vertices $u, v \in A$, $x, y \in B$ such that $uv$ gets completed to $a$ and $xy$ gets completed to $b$. Then the free amalgamation of $\mathbf{A}$ and $\mathbf{B}$ identifying only $v$ and $x$ is also in $\mathcal{A}$ and in its shortest path completion $uy$ gets completed to $a \oplus b$, hence $a \oplus b \in S^+$.

Denote by $\mathfrak{M}'$ the partially ordered commutative semigroup induced by $\mathfrak{M}$ on $S^+$.

Let $a, b \in \mathfrak{M}'$ and suppose that $\inf(a, b)$ does not exist in $\mathfrak{M}'$ or is different in $\mathfrak{M}'$ than in $\mathfrak{M}$. If it does not exist even in $\mathfrak{M}$, then $\mathcal{F}$ takes care of such a pair. Suppose that it exists in $\mathfrak{M}$. By definition $a$ is in $S^+$ because there is a family $\mathcal{P}_a$ of $S$-edge-labelled paths such that $a = \inf(\mathcal{P}_a)$. By the same argument we get $\mathcal{P}_b$ for $b$. Because $\inf(a, b)$ does not exist in $\mathfrak{M}'$ or is different in $\mathfrak{M}'$ than in $\mathfrak{M}$ we know that for every such family $\mathcal{P}_a$ and every such family $\mathcal{P}_b$ it holds that $\mathcal{P}_a \cup \mathcal{P}_b$ contains either a path of length one (hence a non-$\mathfrak{M}$-metric cycle) or a cycle from $\mathcal{F}$ (where we assume that all the paths in $\mathcal{P}_a \cup \mathcal{P}_b$ share the same two endpoints).

Let $\mathcal{F}'$ be the subset of $\mathcal{F}$ consisting of the $S^+$-edge-labelled cycles from $\mathcal{F}$. Clearly $\mathcal{F}'$ is $\mathfrak{M}'$-omissible and it is confined. It also holds that $\mathcal{F}'$ contains all disobedient cycles: We checked it for undefined infima, for non-distributivity it follows from $\mathcal{F}$ containing all disobedient cycles and the fact that when infimum in $\mathfrak{M}'$ is different than in $\mathfrak{M}$, it is also $\mathcal{F}'$-forbidden. The same argument can also be done for meet synchronization.

This implies that

$$\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F}) \cap \mathcal{G}^S \subseteq \mathcal{M}_{\mathfrak{M}'} \cap \mathrm{Forb}(\mathcal{F}') \subseteq \mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F}).$$

Finally note that $\mathfrak{M}'$ is countable (each distance from $\mathfrak{M}'$ corresponds to a finite $S$-edge-labelled graph) and has only finitely many blocks: They correspond to the sublattice of the block lattice of $\mathfrak{M}$ generated by blocks containing elements from $S$, because $\mathcal{F}$ synchronizes meets. □

## 4.3 Important and unimportant summands

This part is key for obtaining a locally finite description of $\mathcal{M}_{\mathfrak{M}}$ needed for Theorem 2.22. We show that from every non-$\mathfrak{M}$-metric cycle one can select a bounded number of important edges (bounded by a function of the set of distances used in the cycle) such that the cycle stays non-$\mathfrak{M}$-metric after replacing the unimportant edges by arbitrary distances from the same blocks. The $L^{\star}_{\mathfrak{M}}$-expansion is useful precisely because it can express "$d(u,v) \in \mathcal{B}'$ for some $\mathcal{B}' \preceq \mathcal{B}$".

Given a partially ordered commutative semigroup $\mathfrak{M} = (M, \oplus, \preceq)$ and $S \subseteq \mathfrak{M}$, we denote by $S^{\oplus}$ the set of all elements of $\mathfrak{M}$ which can be obtained as nonempty sums of values from $S$, i.e. the subsemigroup of $\mathfrak{M}$ generated by $S$.

Blocks of a semigroup may be infinite and may not contain a maximal element which would be useful in our arguments (these maximal elements are refered to as jump numbers in [HN16]). For a fixed finite $S \subseteq \mathfrak{M}$ we seek a sufficient approximation $\mathrm{mus}(\mathcal{B}, S)$ of $\max(\mathcal{B})$ given by the following lemma. Note that if $\mathfrak{M}$ is finite, this section could basically consist of the statement "Let $\mathrm{mus}(\mathcal{B}, S) = \max(\mathcal{B})$." The name mus means *maximum useful* distance (in $\mathcal{B}$ with respect to $S$).

**Lemma 4.23.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $S \subseteq \mathfrak{M}$ be a finite subset of $\mathfrak{M}$. Then for every block $\mathcal{B}$ of $\mathfrak{M}$ there is a distance $\mathrm{mus}(\mathcal{B}, S) \in \mathcal{B}$ such that for every $\ell \in S$ and $e \in S^{\oplus}$ one of the following holds:*

1. *$e \oplus \mathrm{mus}(\mathcal{B}, S) \succeq \ell$, or*
2. *$e \oplus b \not\succeq \ell$ for every $b \in \mathcal{B}$ (and thus also for every $b \in \mathcal{B}'$, where $\mathcal{B}' \preceq \mathcal{B}$).*

*Furthermore for every block such that $\mathcal{B} \cap S^{\oplus} \neq \emptyset$ we can pick $\mathrm{mus}(\mathcal{B}, S) \in S^{\oplus}$.*

**Example 4.24** (Running example — mus)**.** Consider the semigroup $\mathfrak{D}$ and let $S = \{(5,5,5), (7,1,0)\}$. Then one possible choice is $\mathrm{mus}(\mathcal{B}_{\{1,2,3\}}, S) = (7,5,5)$, $\mathrm{mus}(\mathcal{B}_{\{1,2\}}, S) = (7,5,0)$ and $\mathrm{mus}(\mathcal{B}_{\{3\}}, S) = (0,0,5)$.

We will prove Lemma 4.23 using the following claim:

**Claim 4.25.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup, $S \subseteq \mathfrak{M}$ a finite subset of $\mathfrak{M}$, $\mathcal{B}$ an arbitrary block of $\mathfrak{M}$ and $\ell \in S$ an arbitrary distance from $S$. There is a distance $d(\mathcal{B}, S, \ell) \in \mathcal{B}$ such that for every $e \in S^{\oplus}$ one of the following holds:*

1. *$e \oplus d(\mathcal{B}, S, \ell) \succeq \ell$, or*
2. *$e \oplus b \not\succeq \ell$ for every $b \in \mathcal{B}$.*

*Proof.* Suppose that the statement is not true. Then for every $d \in \mathcal{B}$ there is some $e \in S^{\oplus}$ and $b \in \mathcal{B}$ such that $e \oplus d \not\succeq \ell$, but $e \oplus b \succeq \ell$.

By induction we now define sequences $(d_i)_{i=0}^{\infty}$ and $(e_i)_{i=1}^{\infty}$. Let $d_0 \in \mathcal{B}$ be an arbitrary distance from $\mathcal{B}$.

Assume that both sequences are defined up to $i-1$. Take $e \in S^{\oplus}$ and $b \in \mathcal{B}$ such that $e \oplus d_{i-1} \not\succeq \ell$, but $e \oplus b \succeq \ell$ and set $d_i = d_{i-1} \oplus b$ and $e_i = e$. Note that $e_i \oplus d_i \succeq \ell$ and $d_j \preceq d_i$ for every $j \leq i$. We shall prove by contradiction that there are no indices $i < j$ with $e_i \preceq e_j$.

Suppose that there are such indices $i < j$ with $e_i \preceq e_j$. By definition we have $e_i \oplus d_i \succeq \ell$, but $d_{j-1} \succeq d_i$, so $e_j \oplus d_{j-1} \succeq e_i \oplus d_i \succeq \ell$, which is a contradiction with the definition of $e_j$.

Now observe that every distance $a \in S^{\oplus}$ corresponds to a function $f_a\colon S \to \mathbb{N}$ and vice versa by

$$a = \bigoplus_{s \in S} f_a(s) \times s.$$

Say that $f_a \trianglelefteq f_b$ if and only if $f_a(s) \leq f_b(s)$ for every $s \in S$ (i.e. $\trianglelefteq$ is the component-wise order of the vectors $\mathbb{N}^{|S|}$). Then clearly by monotonicity of $\oplus$ whenever $f_a \trianglelefteq f_b$, then also $a \preceq b$.

Take the sequence $(f_{e_i})_{i=1}^{\infty}$. By Dickson's lemma $\trianglelefteq$ is a well-quasi-order, hence there are indices $i < j$ with $f_{e_i} \trianglelefteq f_{e_j}$. But that implies $e_i \preceq e_j$, which is a contradiction and hence finishes the proof. □

*Proof of Lemma 4.23.* Using Claim 4.25 we can set $\mathrm{mus}(\mathcal{B}, S)$ to be an element of $\mathcal{B}$ greater than or equal to $d(\mathcal{B}, S, \ell)$ for every $\ell \in S$. As $S$ is finite, there is such an element (for example $\bigoplus_{\ell \in S} d(\mathcal{B}, S, \ell)$ for nonempty $S$). To satisfy the furthermore part it is enough to use archimedeanity. □

The following proposition (an easy consequence of Lemma 4.23) is the main result of this section which will be used in proving local finiteness.

**Proposition 4.26.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $S \subseteq \mathfrak{M}$ be a finite subset of $\mathfrak{M}$. There exists $n = n(S)$ such that for every $\ell \in S$ and every sequence $e_1, e_2, \ldots, e_k \in S$ with $\ell \not\preceq e_1 \oplus e_2 \oplus \cdots \oplus e_k$ there is a sequence $f_1, f_2, \ldots, f_m \in S$ satisfying the following properties:*

1. *$(f_i)$ is a subsequence of $(e_i)$;*
2. *$m \leq n$; and*
3. *if $(f_i) \subsetneq (e_i)$, $\mathcal{B}$ is the join $\bigvee\{\mathcal{B}(a); a \in (e_i) \setminus (f_i)\}$ and $b \in \mathcal{B}$ an arbitrary distance, then $\ell \not\preceq b \oplus f_1 \oplus f_2 \oplus \cdots \oplus f_m$.*

We will call the distances $(f_i)$ *important.*

*Proof.* Let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$ be the blocks of $\mathfrak{M}$ represented in $S$ by some distance (that is, for each $\mathcal{B}_i$ there is $a_i \in S$ with $a_i \in \mathcal{B}_i$). As $S$ is finite, there are only finitely many such blocks. For each $s \in S$ define $n_s$ to be the smallest integer such that $n_s \times s \succeq \mathrm{mus}(\mathcal{B}(s), S)$. Put

$$n = n(S) = \sum_{s \in S} n_s.$$

To simplify the argument, add a new element 0 to $\mathfrak{M}$, which is neutral with respect to addition.

Let $\ell, e_1, e_2, \ldots, e_k \in S$ be given with $\ell \not\preceq e_1 \oplus \cdots \oplus e_k$. Now we shall construct the sequence $(f_i)$ satisfying the properties from the statement. For each $\mathcal{B}_i$ create a variable $c_i$ which is initially set to 0. Now go through all $e_i$ and do the following:

1. Let $\mathcal{B}_j$ be the block containing $e_i$.
2. If $c_j \succeq \mathrm{mus}(\mathcal{B}_j, S)$, go to the next $e_i$.
3. Otherwise put $e_i$ into the $(f_i)$ sequence and increment $c_j \leftarrow c_j \oplus e_i$.

Now we check that $(f_i)$ satisfies all properties from the statement. The first two are trivial. It is enough to check the third for $b \in S^{\oplus}$. Let $I$ be the set such that $i \in I$ if and only if there is $a \in (e_j) \setminus (f_j)$ such that $a \in \mathcal{B}_i$. Clearly one can write $b$ as $b = \bigoplus_{i=1}^{p} b_i$ such that $b_i = 0$ if $i \notin I$ and $b_i \in \mathcal{B}_i \cap S^{\oplus}$ otherwise. Define

$$b'_i = \bigoplus_{j<i} b_i,$$
$$c_i = \bigoplus_{f_i \notin \mathcal{B}_i} f_i,$$
$$c'_i = \bigoplus_{f_i \in \mathcal{B}_i} f_i.$$

In particular, $b'_1 = 0$.

By induction we will prove that $\ell \not\preceq c_i \oplus c'_i \oplus b'_i$. This holds for $i = 1$. Suppose now that $\ell \not\preceq (c_i \oplus b'_i) \oplus c'_i$. We know that $c_i \oplus b'_i \in S^{\oplus}$. Either $i \notin I$ and we are done, or $i \in I$, but then $c'_i \succeq \mathrm{mus}(\mathcal{B}_i, S)$ and thus $\ell \not\preceq (c_i \oplus b'_i) \oplus (c'_i \oplus b_i)$ which is what we wanted. □

## 4.4 Completing structures with ball vertices

As was already mentioned, the problem of the class $\mathcal{M}^{\star}_{\mathfrak{M}}$ is that it is not hereditary. We defined $\mathcal{H}^{\star}_{\mathfrak{M}}$ as the smallest hereditary superclass of $\mathcal{M}^{\star}_{\mathfrak{M}}$ which corresponds to adding structures which contain some ball vertices to which no original vertex is linked. For applications, we need the class $\mathcal{H}^{\star}_{\mathfrak{M}}$ and we need it to have the strong amalgamation property.

In this section, we first axiomatize the class $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ of (unordered) $L^{\star}_{\mathfrak{M}}$-structures which will be rich enough to contain all free amalgams of structures from $\mathcal{H}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ as well as other structures important for the applications. Then we find a way to complete the structures from $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ to $\mathcal{M}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$, which will, as a corollary, give the strong amalgamation property for $\mathcal{H}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$, it is key in obtaining EPPA for $\mathcal{M}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ and is also one of the ingredients for Chapter 5.

The definition of $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ is unfortunately quite technical. The (worse) alternative was to do all the proofs three times, once for the free amalgams, then for EPPA in Section 4.5 and then for structures which Theorem 2.22 wants us to complete in Chapter 5.

**Definition 4.27** (The class $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$)**.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup with **finitely many blocks** where all meets of non-$\mathbf{0}$ blocks are defined and non-$\mathbf{0}$, let $\mathcal{F}$ be an $\mathfrak{M}$-omissible family of $\mathfrak{M}$-edge-labelled cycles containing all disobedient ones which synchronizes meets and let $S \subseteq \mathfrak{M}$ be a finite set of distances. Define $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ as the subclass of all $L^{\star}_{\mathfrak{M}}$-structures such that every $\mathbf{A} \in \mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ satisfies the following:

1. All the distance relations $R^a_{\mathbf{A}}, a \in \mathfrak{M}$ are symmetric and irreflexive;
2. each vertex is either *original* (that is, has all the functions $F^{\mathcal{B}}_{\mathbf{A}}$ defined, none of the functions $F^{\mathcal{B},\mathcal{B}'}_{\mathbf{A}}$ defined and is not in the range of any function) or it is a *ball vertex* (that is, is not in any distance relation, the functions in whose domain the vertex is are precisely all the $F^{\mathcal{B},\mathcal{B}'}_{\mathbf{A}}$ for some block $\mathcal{B}$ and is only in the range of functions $F^{\mathcal{B}}_{\mathbf{A}}$ and $F^{\mathcal{B}'',\mathcal{B}}_{\mathbf{A}}$);

3. the functions $F^{\mathcal{B}}_{\mathbf{A}}$ and $F^{\mathcal{B},\mathcal{B}'}_{\mathbf{A}}$ are *consistent*, that is, for every vertex $v \in A$, and every pair of blocks $\mathcal{B}, \mathcal{B}' \in I_{\mathfrak{M}} \setminus \{\mathbf{0}\}$ with $\mathcal{B} \prec \mathcal{B}'$ it holds that $F^{\mathcal{B}'}_{\mathbf{A}}(v) = F^{\mathcal{B},\mathcal{B}'}_{\mathbf{A}}(F^{\mathcal{B}}_{\mathbf{A}}(v))$;
4. all nonempty distance relations are from $S$;
5. if $u, v \in A$ have distance $a$, then $F^{\mathcal{B}}_{\mathbf{A}}(u) = F^{\mathcal{B}}_{\mathbf{A}}(u)$ if and only if $\mathcal{B} \succeq \mathcal{B}(a)$;
6. $\mathbf{A} \in \mathrm{Forb}(\mathcal{F})$; and
7. $\mathbf{A}$ contains no homomorphic images of *non-$\mathfrak{M}$-metric $\star$-cycles*.

To define what a *non-$\mathfrak{M}$-metric $\star$-cycle*, we first need to define the *common closure*. We say that vertices $u \neq v \in \mathbf{A}$ have *common closure for block $\mathcal{B}$* if there is $B \subseteq I_{\mathfrak{M}} \setminus \{\mathbf{0}\}$ such that $\bigwedge B = \mathcal{B}$ and for every $\mathcal{B}' \in B$ it holds that $F^{\mathcal{B}'}_{\mathbf{A}}(u) = F^{\mathcal{B}'}_{\mathbf{A}}(v)$.

Now we can define that an $L^{\star}_{\mathfrak{M}}$-structure $\mathbf{K}$ is a *non-$\mathfrak{M}$-metric $\star$-cycle* if it satisfies conditions 1–6, has original vertices $v_1, \dots, v_k$ such that the only edges (defined distances) are between some of the pairs $v_i, v_{i+1}$ (where we identify $v_1 = v_{k+1}$) and the cycle $\mathbf{K}'$, which we get from $\mathbf{K}$ by setting the distance of each non-edge $v_i, v_{i+1}$ to be $\mathrm{mus}(\mathcal{B}, S)$ where $\mathcal{B}$ is the smallest block for which $v_i$ and $v_{i+1}$ have common closure, is non-$\mathfrak{M}$-metric such that furthermore the longest edge is not one of the added $\mathrm{mus}(\mathcal{B}, S)$ edges.

While the definition might look scary, it is quite straightforward to check that the free amalgams of structures from $\mathcal{H}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ are from $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ for $S$ being the set of distances in the free amalgam. Indeed, all the conditions up to 5 are satisfied trivially, condition 6 follows from $\mathfrak{M}$-omissibility of $\mathcal{F}$ in the same way as it did in part 1 of Theorem 3.6 and condition 7 follows similarly.

The following observation is the reason why we introduced the $L^{\star}_{\mathfrak{M}}$-expansions the mus's:

**Observation 4.28.** *Let $\mathfrak{M}$, $S$ and $\mathcal{F}$ be as in Definition 4.27. Let $\mathbf{A}$ be an $L^{\star}_{\mathfrak{M}}$-structure satisfying all conditions of Definition 4.27 but condition 7. There exists $n = n(S)$ such that if $\mathbf{A}$ contains a non-$\mathfrak{M}$-metric $\star$-cycle, then it contains a non-$\mathfrak{M}$-metric $\star$-cycle with at most $n$ vertices.*

*Proof.* Suppose that there is a non-$\mathfrak{M}$-metric $\star$-cycle $\mathbf{K}$ in $\mathbf{A}$, denote its original vertices as $v_1, \dots, v_k$ and the lengths of its edges as $\ell, e_1, \dots, e_p$. $\mathbf{K}$ being non-$\mathfrak{M}$-metric means in particular that $\ell \not\preceq \bigoplus e_i$. By Proposition 4.26 we get a subset $f_1, \dots, f_q$ of $e_1, \dots, e_p$ of bounded size (by a function of $S$). And then it follows that the substructure of $\mathbf{K}$ induced by the endpoints of the edges $f_1, \dots, f_q$ is still a non-$\mathfrak{M}$-metric $\star$-cycle with a bounded number of vertices. □

We now show how to complete structures from $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ to $\mathcal{M}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. We will do it in several steps. For the next lemmas fix $\mathfrak{M}$, $S$ and $\mathcal{F}$ as in Definition 4.27.

**Outline of the proof.** Given a structure from $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$, we first make sure that every ball vertex has an original vertex pointing to it (and that, in general, all the $F^{\mathcal{B}}$ and $F^{\mathcal{B},\mathcal{B}'}$ functions are as in $\mathcal{M}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$). Then we make sure that every pair of original vertices has common closure for $\mathcal{B}$ if and only if they are connected by a path whose length lies in some $\mathcal{B}' \preceq \mathcal{B}$ which we do by carefully adding new edges block by block. Then we can look only at the edge-labelled graph induced on the original vertices. If we were careful enough, it is $\mathfrak{M}$-metric and from $\mathrm{Forb}(\mathcal{F})$, hence we can take its shortest path completion and then check that it is consistent with the ball vertices (for this meet synchronization is needed).

**Lemma 4.29.** *For every $\mathbf{A} \in \mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ there is $\mathbf{B} \in \mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ such that $\mathbf{A} \subseteq \mathbf{B}$ and for every ball vertex $b$ of $\mathbf{B}$ there is an original vertex $o$ of $\mathbf{B}$ such that $F^{\mathcal{B}}_{\mathbf{B}}(o) = b$ for some block $\mathcal{B}$. Furthermore for every automorphism $\alpha$ of $\mathbf{A}$ there is an automorphism $\beta$ of $\mathbf{B}$ such that $\beta|_A = \alpha$.*

*Proof.* Enumerate all ball vertices of $\mathbf{A}$ having no original vertex pointing at them and for each such vertex $b$, which represents a ball of diameter $\mathcal{B}$, add vertices $b_{\mathcal{B}'}$ for every $\mathcal{B}' \in I_{\mathfrak{M}}$ such that $\mathcal{B}' \prec \mathcal{B}$ and an original vertex $o$ and let $F^{\mathcal{B}}_{\mathbf{B}}(o) = b$, $F^{\mathcal{B}'}_{\mathbf{B}}(o) = b_{\mathcal{B}'}$ for smaller blocks and analogously define all the other functions for $o$ and $b_{\mathcal{B}'}$. The only condition of $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ that could break is 7, but it is easy to see that if there was a *non-$\mathfrak{M}$-metric $\star$-cycle* in $\mathbf{B}$ containing some of the newly added vertices, then simply omitting them would give a *non-$\mathfrak{M}$-metric $\star$-cycle* in $\mathbf{A}$. The automorphism preservation follows from the canonicity of the construction. □

**Lemma 4.30.** *Let $\mathbf{A} \in \mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$. For every pair of original vertices $u, v \in \mathbf{A}$ and every block $\mathcal{B}$ of $\mathfrak{M}$ such that there exists a walk from $u$ to $v$ in $\mathbf{A}$ where every distance is at most some $a \in \mathcal{B}$ it holds that $u$ and $v$ have common closure for $\mathcal{B}$.*

*Proof.* Clearly it is enough to check for $\mathcal{B} \in I_{\mathfrak{M}}$. Let $u'$ and $w'$ be two neighbouring vertices in the walk. From Definition 4.27, condition 5 we have that $F^{\mathcal{B}}_{\mathbf{A}}(u') = F^{\mathcal{B}}_{\mathbf{A}}(w')$ and consequently all vertices of the walk have same value of $F^{\mathcal{B}}_{\mathbf{A}}$. □

**Proposition 4.31.** *Suppose that $S$ contains a distance from each block and that $\mathcal{F}$ is confined. Let $\mathbf{C}$ be a structure from $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$. There is $\mathbf{C}' \in \mathcal{M}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ which is a completion of $\mathbf{C}$. Furthermore, if $\mathbf{C}$ is connected, $\mathbf{C}'$ is its n automorphism-preserving completion.*

*Proof.* By Lemma 4.29 we can assume that each ball vertex of $\mathbf{C}$ has some original vertex which maps to it while preserving the automorphisms. Because $S$ contains a distance from each block we can further assume that $\mathrm{mus}(\mathcal{B}, S) \in S^{\oplus}$ for every block $\mathcal{B}$. If $\mathbf{C}$ is not connected, we can connect the components by their original vertices using a distance from $S$ which lies in the largest block. Note that this is not automorphism-preserving.

Denote by $\mathbf{G}$ the $\mathfrak{M}$-graph induced by $\mathbf{C}$ on the set of original vertices. We first prove that there exists an $\mathfrak{M}$-metric space $\mathbf{G}'$ satisfying the following:

1. $\mathbf{G}' \in \mathrm{Forb}(\mathcal{F})$;
2. $\mathbf{G}'$ is a strong completion of $\mathbf{G}$;
3. for every block $\mathcal{B}$ of $\mathfrak{M}$ and every pair $u, v$ of vertices of $\mathbf{G}$, whenever $u, v$ have a common closure for $\mathcal{B}$ in $\mathbf{C}$, then $u \sim_{\mathcal{B}} v$ in $\mathbf{G}'$; and
4. for every automorphism $\alpha$ of $\mathbf{G}$ it holds that $\alpha$ is also an automorphism of $\mathbf{G}'$.

The first two points follow directly from the definition of $\mathcal{G}^\star_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ and Theorem 3.6 (a non-$\mathfrak{M}$-metric cycle is also a non-$\mathfrak{M}$-metric $\star$-cycle). We need to ensure that the third point holds too.

We first construct $\mathbf{G}_0$ by adding edges to $\mathbf{G}$ such that for every pair of original vertices $u \neq v$ and block $\mathcal{B}$ of $\mathfrak{M}$ such that $u$ and $v$ have common closure for $\mathcal{B}$ (in $\mathbf{C}$) there exists a walk from $u$ to $v$ in $\mathbf{G}_0$ such that every pair of neighbouring vertices is $\sim_{\mathcal{B}}$-equivalent.

New edges to $\mathbf{G}_0$ are added by induction from the smallest blocks to the largest (ordered by $\preceq$, incomparable blocks ordered arbitrarily). For every $\mathcal{B}$ we then follow the following procedure:

Look at every pair of vertices $u$, $v$ having common closure for $\mathcal{B}$ with no walk in $\mathbf{G}_0$ such that every pair of neighbouring vertices is $\sim_{\mathcal{B}}$ equivalent and connect them in $\mathbf{G}_0$ by an edge of length $q \times \mathrm{mus}(\mathcal{B}, S)$ (cf. Lemma 4.23), where $q$ is the largest number of vertices of a cycle in $\mathcal{F}$ containing only edges from $S$ (here we are using the fact that $\mathcal{F}$ is confined, hence $S$-locally finite).

Note that as $\mathbf{C}$ is connected, $\mathbf{G}_0$ is a connected (even though $\mathbf{G}$ did not have to be).

It is easy to verify that this process does not introduce a cycle from $\mathcal{F}$. Indeed, suppose, to the contrary, that it does. Then from closedness of $\mathcal{F}$ to the inverse steps of the shortest path completion algorithm, we can replace each edge of length $q \times \mathrm{mus}(\mathcal{B}, S)$, which we added for some block $\mathcal{B}$, by $q$ edges of length $\mathrm{mus}(\mathcal{B}, S)$ and each of them can be further replaced by several edges from $S$ (because $\mathrm{mus}(\mathcal{B}, S) \in S^\oplus$) yielding a cycle from $\mathcal{F}$ containing only distances from $S$ with more that $q$ vertices, which is a contradiction.

Now we verify that this process does not introduce a non-$\mathfrak{M}$-metric cycle. Suppose, to the contrary, that $\mathcal{B}$ is the smallest block such that adding an edge of length $q \times \mathrm{mus}(\mathcal{B}, S)$ between vertices $u$ and $v$ created a non-$\mathfrak{M}$-metric cycle (call it $\mathbf{K}$) and let $\mathbf{K} = (\ell, q \times \mathrm{mus}(\mathcal{B}, S), e_1, e_2, \dots, e_k)$ such that $\ell \not\preceq q \times \mathrm{mus}(\mathcal{B}, S) \oplus e_1 \oplus e_2 \oplus \cdots \oplus e_k$. This implies that all the more so $\ell \not\preceq \mathrm{mus}(\mathcal{B}, S) \oplus e_1 \oplus e_2 \oplus \cdots \oplus e_k$.

Note that $\ell$ lies in a block larger than or incomparable to $\mathcal{B}$, as the reason for adding the edge was the lack of a $\mathcal{B}$-path between $u$ and $v$. (Hence also adding the edge $q \times \mathrm{mus}(\mathcal{B}, S)$ did not introduce any non-$\mathfrak{M}$-metric cycle where $q \times \mathrm{mus}(\mathcal{B}, S)$ would be the largest edge.) And thus the edge of length $\ell$ is an edge of $\mathbf{G}$. Also note that some $e_i$'s may have been added in this or previous steps of the construction of $\mathbf{G}_0$.

Let $(e'_i)$ be the subsequence of $e_i$ containing only edges from blocks larger than or incomparable to $\mathcal{B}$, so in particular it only contains edges from $\mathbf{G}$. Then also $\ell \not\preceq \mathrm{mus}(\mathcal{B}, S) \oplus \bigoplus_i e'_i$. And this is a contradiction with $\mathbf{C}$ containing no non-$\mathfrak{M}$-metric $\star$-cycles.

Now $\mathbf{G}'$ can be constructed as the shortest path completion of $\mathbf{G}_0$. Our way of adding edges preserved $\alpha$ being an automorphism of $\mathbf{G}_0$ and by part 2 of Theorem 3.6 it follows that $\alpha$ is also an automorphism of $\mathbf{G}'$.

Finally we put $\mathbf{C}' = L^\star(\mathbf{G}')$. Clearly $\mathbf{C}' \in \mathcal{M}^\star_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. We need to prove that $\mathbf{C}'$ is a completion of $\mathbf{C}$ and that it is automorphism-preserving. We know that $\mathbf{G}'$ is precisely the $\mathfrak{M}$-metric space induced by $\mathbf{C}'$ on the original vertices and that it is an automorphism-preserving completion of $\mathbf{G}$. Thus we only need to prove that the ball vertices in $\mathbf{C}'$ and $\mathbf{C}$ correspond to each other.

Look at the definition of $Ł^\star(\mathbf{G}')$ (Definition 4.15). By Lemma 4.30 and construction of $\mathbf{G}'$ we know that for every block $\mathcal{B}$ of $\mathfrak{M}$ with $\mathcal{B} = \bigwedge B$ for some $B \subseteq I_\mathfrak{M} \setminus \{\mathbf{0}\}$ and every two vertices of $\mathbf{G}'$ it holds that $u \sim_\mathcal{B} v$ in $\mathbf{G}'$ if and only if for every $\mathcal{B}' \in B$ it holds that $F^{\mathcal{B}'}_\mathbf{C}(u) = F^{\mathcal{B}'}_\mathbf{C}(v)$.

In particular, for every ball of diameter $\mathcal{B} \in I_\mathfrak{M} \setminus \{\mathbf{0}\}$ there is a ball vertex in $\mathbf{C}$. And thanks to Lemma 4.29, there are no other ball vertices. This means that $\mathbf{C}$ and $\mathbf{C}'$ have the same ball vertices and the same functions $F^\mathcal{B}$. It follows from the definition of $\mathcal{G}^\star_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ that $\mathbf{C}$ and $\mathbf{C}'$ also have the same functions $F^{\mathcal{B},\mathcal{B}'}$ for every $\mathcal{B} \prec \mathcal{B}'$ with $\mathcal{B}, \mathcal{B}' \in I_\mathfrak{M} \setminus \{\mathbf{0}\}$. Hence indeed $\mathbf{C}' = L^\star(\mathbf{G}')$ is an automorphism-preservng completion of $\mathbf{C}$. □

**Corollary 4.32.** *If $\mathcal{F}$ is confined, the class $\mathcal{H}^\star_\mathfrak{M} \cap \mathrm{Forb}(\mathcal{F})$ has the strong amalgamation property (see Definition 4.15).*

*Proof.* As we observed, the free amalgams of structures from $\mathcal{H}^\star_\mathfrak{M} \cap \mathrm{Forb}(\mathcal{F})$ are from $\mathcal{G}^\star_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$, strong amalgamation now follows by Proposition 4.31. □

## 4.5 EPPA

When proving the Ramsey property using Theorem 2.22, one usually proves a result like "This ordered class has the strong amalgamation property and here is a completion procedure as a certificate of local finiteness." The results of this section are a step towards a result of this type; in Chapter 5 we introduce ordering of $\mathcal{M}^\star_\mathfrak{M} \cap \mathrm{Forb}(\mathcal{F})$ and observe that it only depends on the ball structure, not the precise distances.

For EPPA, one usually proves something like "This class has the amalgamation property and here is an automorphism preserving completion procedure to certify that it is described by forbidding boundedly many homomorphisms." This roughly matches the results of this chapter. In this section we aim to prove Theorem 1.6.

We are going to use the same trick as [EHN17a], [EHN17b] or [ABWH+17c] for the antipodal structures. We first assume that $\mathfrak{M}$ has finitely many blocks. This way we can work with $\mathcal{M}^\star_\mathfrak{M} \cap \mathrm{Forb}(\mathcal{F})$ instead. For each $\mathbf{A} \in \mathcal{M}^\star_\mathfrak{M} \cap \mathrm{Forb}(\mathcal{F})$ we will find its EPPA-witness. Fix such $\mathbf{A}$ and let $S$ be the set of its distances. There are only finitely many minimal non-$\mathfrak{M}$-metric $\star$-cycles with distances from $S$ (cf. Observation 4.28) and only finitely many cycles from $\mathcal{F}$ with distances from $S$ (from confinedness), denote the set of all of these obstacles by $\mathcal{O}$. In particular, $\mathbf{A} \in \mathrm{Forb}(\mathcal{O})$.

We want to use the Herwig–Lascar theorem (Theorem 2.25), but it only works for relational languages. We thus replace the unary functions by "directed edges" (that is, asymmetric binary relations, one for each function) both in $\mathbf{A}$ and $\mathbf{O}$ and use Theorem 2.25 to get $\mathbf{B}$ which is an EPPA-witness for $\mathbf{A}$ (in particular, $\mathbf{A} \subseteq \mathbf{B}$) in the purely relational language. Note that we can assume that for every vertex and every pair of vertices in a relation in $\mathbf{B}$ there is an automorphism $\beta$ of $\mathbf{B}$ which sends the copy of $\mathbf{A}$ on the vertex/tuple. We will further ensure (by adding additional unary relations) that one can still divide the vertices of $\mathbf{B}$ into original and ball vertices (and determine which block each ball vertex belongs to).

Now it remains to un-glue $\mathbf{B}$ such that the directed edges are functions again (that is, the outdegrees are at most one). In order to do this, we will say that the *closure* of a vertex $v$ is the set of all vertices reachable from $v$ by directed paths. In particular, the closure of each vertex is a directed acyclic graph. Then we define $\mathbf{C}$ such that its vertices correspond to pairs $(u, \alpha)$, where $u$ is a vertex of $\mathbf{B}$ and $\alpha$ is an embedding of a "typical closure of this type" to the closure of $u$. It is possible to transfer all the relations from $\mathbf{B}$ to $\mathbf{C}$ such that $\mathbf{C}$ is still an EPPA-witness of $\mathbf{A}$, it is in $\mathrm{Forb}(\mathcal{O})$ and the directed relations are actually functions in $\mathbf{C}$, hence $\mathbf{C}$ is in $\mathcal{G}^\star_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ and has an automorphism-preserving completion in $\mathcal{M}^\star_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$.

The following definition says how to reduce an $L^\star_{\mathfrak{M}}$-structure to a relational structure rich enough so that we can carry out the rest of the proof.

**Definition 4.33.** Let $\mathfrak{M}$ be a partially ordered commutative semigroup. We denote by $L^-$ the language consisting of

- binary symmetric *distance* relations $R^x$ for every $x \in \mathfrak{M}$,
- unary relations $R^\mathcal{B}$ for each block $\mathcal{B} \in I_\mathfrak{M}$ (including $\mathbf{0}$), and
- binary asymmetric relations $R^F$ for each function $F \in L^\star_\mathfrak{M}$.

Given an $L^\star_\mathfrak{M}$-structure $\mathbf{A}$, we denote by $L^-(\mathbf{A})$ the $L^-$-structure $\mathbf{A}^-$ on the same vertices as $\mathbf{A}$ where all distance relations are copied from $\mathbf{A}$, $(u, v) \in R^F_{\mathbf{A}^-}$ if and only if $F_\mathbf{A}(u) = v$ and $u \in R^\mathcal{B}_{\mathbf{A}^-}$ if and only if either $\mathcal{B} = \mathbf{0}$ and $u$ is an original vertex of $\mathbf{A}$ (that is, is in the domain of functions $F^\mathcal{B}$ for every $\mathcal{B} \in I_\mathfrak{M} \setminus \{\mathbf{0}\}$), or $\mathcal{B} \neq \mathbf{0}$ and $u$ is in the range of the function $F^\mathcal{B}$.

The structures which we encounter will have special structure, namely all their vertices will be reasonable:

**Definition 4.34.** Let $\mathfrak{M}$ be a partially ordered commutative semigroup, $\mathbf{A}$ be an $L^-$-structure and $u \in A$ a vertex of $\mathbf{A}$. We say that $u$ is *reasonable* if there is exactly one relation $R^{\mathcal{B}_0}$ with $u \in R^{\mathcal{B}_0}_\mathbf{A}$ and one of the following holds:

1. $\mathcal{B}_0 = \mathbf{0}$, for every function $F^\mathcal{B} \in L^\star_\mathfrak{M}$ and every vertex $v \in A$ it holds that if $(u, v) \in R^{F^\mathcal{B}}_\mathbf{A}$, then $v \in R^\mathcal{B}_\mathbf{A}$ and there is no vertex $v \in A$ and a function $F^{\mathcal{B},\mathcal{B}'} \in L^\star_\mathfrak{M}$ with $(u, v) \in R^{F^{\mathcal{B},\mathcal{B}'}}_\mathbf{A}$, or
2. $\mathcal{B}_0 \neq \mathbf{0}$, $v$ is in no distance relation, there is no vertex $v \in A$ and a function $F^\mathcal{B} \in L^\star_\mathfrak{M}$ with $(u, v) \in R^{F^\mathcal{B}}_\mathbf{A}$ and for every function $F^{\mathcal{B},\mathcal{B}'}$ and every vertex $v \in A$ it holds that if $(u, v) \in R^{F^{\mathcal{B},\mathcal{B}'}}_\mathbf{A}$, then $\mathcal{B} = \mathcal{B}_0$ and $v \in R^{\mathcal{B}'}_\mathbf{A}$, and if $(v, u) \in R^{F^{\mathcal{B},\mathcal{B}'}}_\mathbf{A}$, then $\mathcal{B}' = \mathcal{B}_0$ and $v \in R^\mathcal{B}_\mathbf{A}$.

For example if $\mathbf{A} \in \mathcal{G}^\star_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$, then $L^-(\mathbf{A})$ has all vertices reasonable. We first observe that if there is an EPPA-witness for $\mathbf{A}$, then there is an EPPA-witness for $\mathbf{A}$ of a special form.

**Lemma 4.35.** *Let $L$ be a relational language, $\mathbf{A}$ be a finite $L$-structure and $\mathbf{B}$ be a finite $L$-structure such that $\mathbf{A} \subseteq \mathbf{B}$ and every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$. Then there is a finite **connected** $L$-structure $\mathbf{C}$ and an injective homomorphism $f\colon \mathbf{C} \to \mathbf{B}$ such that $\mathbf{A} \subseteq \mathbf{C}$ and every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{C}$.*

*Furthermore it holds that for every tuple $\bar{x} \in C^n$ such that*

- *either $n = 1$ (hence $x$ is a vertex),*
- *or there is a relation $R \in L$ with $\bar{x} \in R_{\mathbf{C}}$*

*there is an automorphism $\gamma$ of $\mathbf{C}$ such that $\bar{x} \subseteq \gamma(A)$ and $A \cap \gamma(A) \neq \emptyset$.*

In other words, Lemma 4.35 says that the EPPA-witness can be chosen such that it only consists of copies of $\mathbf{A}$ glued over each other (cf. Definition 5.15).

*Proof.* The statement gives a recipe for $\mathbf{C}$. Let $\Gamma$ be the subgroup of $\mathrm{Aut}(\mathbf{B})$ generated by automorphisms $\gamma$ such that $A \cap \gamma(A) \neq \emptyset$. Note that every automorphism extending a partial automorphism of $\mathbf{A}$ is from $\Gamma$.

Define $\mathbf{C}$ such that the vertices of $\mathbf{C}$ are a subset of $B$ such that $v \in B$ is a vertex of $\mathbf{C}$ if and only if there is $\gamma \in \Gamma$ such that $v \in \gamma(A)$ and for every relation $R$ and every tuple $\bar{x} \in R_{\mathbf{B}}$ put $\bar{x} \in R_{\mathbf{C}}$ if and only if there is $\gamma \in \Gamma$ such that $\bar{x} \in \gamma(A)$.

Clearly the inclusion $C \subseteq B$ is an injective homomorphism, $\mathbf{A} \subseteq \mathbf{C}$, the "furthermore" part of the lemma is satisfied and $\mathbf{C}$ is connected (for connectedness we needed to introduce the subgroup $\Gamma$). It remains to check that $\mathbf{C}$ is an EPPA-witness for $\mathbf{A}$.

Note that $C$ is a union of orbits of $\Gamma$ acting on $B$ and hence for every $\gamma \in \Gamma$ it holds that $\gamma(C) = C$ and thus $\gamma|_C \in \mathrm{Aut}(\mathbf{C})$. Because $\mathbf{B}$ is an EPPA-witness for $\mathbf{A}$, there is an automorphism $\gamma \in \Gamma$ for every partial automorphism $\alpha$ such that $\alpha \subseteq \gamma$. And thus the automorphism $\gamma|_C$ of $\mathbf{C}$ has also $\alpha \subseteq \gamma$, which is what we wanted. □

**Corollary 4.36.** *If $\mathbf{A} = L^-(\mathbf{A}')$ where $\mathbf{A}'$ is from $\mathcal{M}^\star_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$, then all vertices of $\mathbf{C}$ are reasonable.*

Now we are almost ready to prove the main lemma needed for Theorem 1.6, we only need to define two more notions.

Let $\mathbf{A}$ be an $L^-$-structure and let $u \in A$ be a vertex. Then we call the *closure of $u$ (in $\mathbf{A}$)* and denote by $cl(u)$ the set of all vertices $v \in A$ such that there is a sequence $u = v_0, v_1, \ldots, v_k = v$ of vertices of $\mathbf{A}$ such that for each $i$ there is a function $F \in L^\star_{\mathfrak{M}}$ with $(v_i, v_{i+1}) \in R^F_{\mathbf{A}}$.

Let $\mathfrak{M}$ be a partially ordered commutative semigroup with finitely many blocks and $\mathbf{U}_0$ be the one-vertex structure from $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. Put $\mathbf{U} = L^-(L^\star(\mathbf{U}_0))$. In other words, $\mathbf{U}$ has $|I_{\mathfrak{M}}|$ vertices, each has a different unary mark $R^{\mathcal{B}}$, there are no distance relations and the relations $R^F$ for every function $F \in L^\star_{\mathfrak{M}}$ are "as they should be". Then for every block $\mathcal{B} \in I_{\mathfrak{M}}$ we denote by $\mathbf{U}(\mathcal{B})$ the substructure of $\mathbf{U}$ induced by the closure of the unique vertex of $\mathbf{U}$ in relation $R^{\mathcal{B}}$. For example $\mathbf{U}(\mathbf{0}) = \mathbf{U}$ and for a block $\mathcal{B}$ which is maximal in $I_{\mathfrak{M}}$, $\mathbf{U}(\mathcal{B})$ consists of a single vertex in the relation $R^{\mathcal{B}}$.

**Lemma 4.37.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathcal{F}$ be a family of $\mathfrak{M}$-edge-labelled cycles. Suppose that the following conditions hold:*

1. *$\mathcal{F}$ is $\mathfrak{M}$-omissible;*
2. *$\mathcal{F}$ contains all $\mathfrak{M}$-disobedient cycles;*
3. *$\mathcal{F}$ is confined;*

4. $\mathcal{F}$ *synchronizes meets; and*

5. $\mathfrak{M}$ *has finitely many blocks.*

*Then the class* $\mathcal{M}^\star_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ *has EPPA.*

*Proof.* Fix a structure $\mathbf{A} \in \mathcal{M}^\star_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ and let $S \subseteq \mathfrak{M}$ be the finite set of distances appearing in $\mathbf{A}$. Let $\mathcal{O}$ be the union of all minimal non-$\mathfrak{M}$-metric $\star$-cycles containing only distances from $S$ and all cycles from $\mathcal{F}$ containing only distances from $S$. By confinedness and Observation 4.28 $\mathcal{O}$ is finite.

Denote $\mathbf{A}^- = L^-(\mathbf{A})$ and $\mathcal{O}^- = \{L^-(\mathbf{O}) : \mathbf{O} \in \mathcal{O}\}$ and let $\mathbf{B}^- \in \mathrm{Forb}(\mathcal{O}^-)$ be the EPPA-witness for $\mathbf{A}^-$ and $\mathcal{O}^-$ given by Theorem 2.25. By Lemma 4.35 we can assume that $\mathbf{B}^-$ is connected and every vertex and every tuple in a relation of $\mathbf{B}^-$ is in the image of $\mathbf{A}^-$ under some automorphism and by Corollary 4.36 we know that every vertex of $\mathbf{B}^-$ is reasonable. This is the only thing we need the unary marks for.

We will now construct an $L^\star_{\mathfrak{M}}$-structure $\mathbf{C}$, prove that $\mathbf{C} \in \mathcal{G}^\star_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ and that it is an EPPA-witness for $\mathbf{A}$. We also need to ensure that $\mathbf{C}$ is connected. Then by Proposition 4.31 there is an EPPA-witness for $\mathbf{A}$ in $\mathcal{M}^\star_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$, which is what we want.

Let $u \in B^-$ be a vertex of $\mathbf{B}^-$ such that it has unary mark $R^{\mathcal{B}}$. By $\mathcal{E}_u$ we denote the set of all embeddings of $\mathbf{U}(\mathcal{B})$ to (the structure induced by $\mathbf{B}^-$ on) $cl(u)$. Put

$$C = \Big\{(u, \alpha) : u \in B^-, \alpha \in \mathcal{E}_u\Big\}.$$

Now we need to define the relations on $\mathbf{C}$. We put $(u, \alpha)$ and $(v, \beta)$ in distance $x$ in $\mathbf{C}$ if and only if $u$ and $v$ are in distance $x$ in $\mathbf{B}^-$. Let $F$ be a function from $L^\star_{\mathfrak{M}}$. We put $F_{\mathbf{C}}((u, \alpha)) = (v, \beta)$ if and only if $(u, v) \in R^F_{\mathbf{B}^-}$ and furthermore $\beta \subseteq \alpha$.

We first observe that $\mathbf{C}$ is well-defined, that is, for every $(u, \alpha) \in C$ there is at most one $(v, \beta) \in C$ with $F_{\mathbf{B}^-}(u) = v$ and $\beta \subseteq \alpha$. But this is easy, because given the embedding $\alpha \colon \mathbf{U}(\mathcal{B}) \to cl(u)$, it has exactly one restriction to $\mathbf{U}(\mathcal{B}')$ (assuming that $u \in R^{\mathcal{B}}_{\mathbf{B}^-}$, $v \in R^{\mathcal{B}'}_{\mathbf{B}^-}$ and $\mathcal{B} \prec \mathcal{B}'$).

Now we prove that $\mathbf{C}$ is an EPPA-witness for $\mathbf{A}$. For it we need an embedding $\varphi \colon \mathbf{A} \to \mathbf{C}$. We abuse the fact that $\mathbf{A}$ and $\mathbf{A}^-$ have the same vertex set and that $\mathbf{A}^- \subseteq \mathbf{B}^-$ and for each $u \in A$ put $\varphi(u) = (u, \alpha)$ where $\alpha$ is the unique embedding of the respective $\mathbf{U}(\mathcal{B})$ to $\mathbf{A}^-$ with $u$ being the root vertex.

Denote by $\pi \colon C \to B^-$ the *projection* assigning $(u, \alpha) \mapsto u$ and let $\psi$ be a partial automorphism of $\mathbf{A}$ and hence also a partial automorphism of $\mathbf{A}^-$. By the construction of $\mathbf{B}^-$, there is an automorphism $\bar{\psi}$ of $\mathbf{B}^-$ with $\psi \subseteq \bar{\psi}$. It follows that $\psi' \colon C \to C$ defined as $(u, \alpha) \mapsto (\bar{\psi}(u), \bar{\psi} \circ \alpha)$ is an automorphism of $\mathbf{C}$ extending $\psi$.

It remains to check that $\mathbf{C} \in \mathcal{G}^\star_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$. Conditions 1 to 5 of Definition 4.27 are satisfied trivially thanks to Lemma 4.35 and $\mathbf{A}$ being from $\mathcal{M}^\star_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. If there is $\mathbf{F} \in \mathcal{F}$ and a homomorphism $h \colon \mathbf{F} \to \mathbf{C}$, then by composing with the projection $\pi$ we get a homomorphism from $\mathbf{F}^-$ to $\mathbf{B}^-$, which is a contradiction as $\mathbf{B}^- \in \mathrm{Forb}(\mathcal{O}^-)$ and the same argument works also for non-$\mathfrak{M}$-metric $\star$-cycles. Hence $\mathbf{C}$ is indeed in $\mathcal{G}^\star_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$. If $\mathbf{C}$ is connected, we are done, otherwise it is enough to only take the substructure of $\mathbf{C}$ on the orbit(s) of the analogue of $\Gamma$ from Lemma 4.35 having a nonempty intersection with $\varphi(A)$. $\square$

Now we are ready to prove Theorem 1.6.

*Proof of Theorem 1.6.* From Lemma 4.37 and Proposition 4.17 we get that if $\mathfrak{M}$ has finitely many blocks, then $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ has EPPA. Using Theorem 4.22 we can extend this to semigroups with infinitely many blocks. □

*Remark.* There are several variations or strengthenings of the Herwig–Lascar theorem with stronger model-theoretical consequences (see for example [EHN17b] or [ABWH$^+$17c]). We only proved the minimum here, because the goal was to advocate the usefulness of the results of this chapter. It should be possible to carry out the methods which we use in full generality (even for the stronger variants of EPPA) and obtain a variant of the Herwig–Lascar theorem for classes with unary functions. And indeed, such a result has been announced, when it is published, it should be possible to simply plug Proposition 4.31 into it.

# 5. Convex order and the Ramsey property

In this chapter we aim to prove Theorem 1.5. We adapt Braunfeld's definition of convex ordering of $\Lambda$-ultrametric spaces for all semigroup-valued metric spaces. Then we show how to transfer this order between the $L_{\mathfrak{M}}$-structures and $L^{\star}_{\mathfrak{M}}$-structures, which enables us to use the machinery of Chapter 4.

Unless stated otherwise, in the whole chapter we fix a partially ordered commutative semigroup $\mathfrak{M} = (M, \oplus, \preceq)$ with **finitely many blocks** where the meet of every non-$\mathbf{0}$ pair of blocks is defined and non-$\mathbf{0}$.

## 5.1 Convex ordering

To obtain a Ramsey class we need to define a notion of ordering for classes $\mathcal{M}_{\mathfrak{M}}$. The convex ordering of a metric space was first used by Nguyen Van Thé [NVT09, NVT10]. In the case of ultrametric spaces it is possible to order vertices in such a way that every ball is a linear interval. Braunfeld [Bra17] generalised the concept of convex ordering to $\Lambda$-ultrametric spaces where this is no longer possible. We proceed analogously.

**Definition 5.1.** Let $L^{+}_{\mathfrak{M}}$ be the expansion of the language $L_{\mathfrak{M}}$ which adds a binary relation $\leq^{\mathcal{B}}$ for each $\mathcal{B} \in I_{\mathfrak{M}}$.

Given an $\mathfrak{M}$-metric space $\mathbf{A} = (A, (R^{s}_{\mathbf{A}})_{s\in\mathfrak{M}})$ its *convexly ordered expansion* is an $L^{+}_{\mathfrak{M}}$ expansion of $\mathbf{A}$ such that for every $\mathcal{B} \in I_{\mathfrak{M}}$ the relation $\leq^{\mathcal{B}}$ is a partial order satisfying the following

1. One of $u \leq^{\mathcal{B}} v$ and $v \leq^{\mathcal{B}} u$ is defined if and only if $u \not\sim_{\mathcal{B}} v$ and for every $\mathcal{B}' \succ \mathcal{B}$ it holds that $u \sim_{\mathcal{B}'} v$; and
2. for every $w \in A$ such that $u \sim_{\mathcal{B}} w$ we have $w \leq^{\mathcal{B}} v$ if and only if $u \leq^{\mathcal{B}} v$.

We will denote by $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ the class of all convexly ordered $\mathfrak{M}$-metric spaces.

As we shall see, the orders $\leq^{\mathcal{B}}$ are a concise description of orders of balls of diameter $\mathcal{B}$. In particular, if $\mathfrak{M}$ is archimedean then we only added $\leq^{\mathbf{0}}$ — a linear order of vertices of $\mathfrak{M}$.

**Example 5.2** (Running example — $\overrightarrow{\mathcal{M}}_{\mathfrak{D}}$)**.** In $\mathfrak{D}$ for $\mathcal{B}_{\{1,2\}}$, there is only one larger block, namely the largest one. Hence $u \leq^{\mathcal{B}_{\{1,2\}}} v$ or $v \leq^{\mathcal{B}_{\{1,2\}}} u$ is defined if and only if $u \nsim_{\mathcal{B}_{\{1,2\}}} v$, similarly for $\mathcal{B}_{\{1,3\}}$ and $\mathcal{B}_{\{2,3\}}$. On the other hand, $\mathcal{B}_{\emptyset}$ is the smallest block, hence $u \leq^{\mathcal{B}_{\emptyset}} v$ or $v \leq^{\mathcal{B}_{\emptyset}} u$ if and only $d(u,v) = (0,0,0)$.

In order to prove the strong amalgamation property for the convexly ordered metric spaces, we need to strengthen our assumptions of $\mathfrak{M}$ and $\mathcal{F}$. In particular, we need the block lattice to be distributive and we need for every pair $\mathcal{B}$, $\mathcal{B}'$ of blocks of $\mathfrak{M}$ and every $a \in \mathcal{B}$, $b \in \mathcal{B}'$ which "we encounter in our metric spaces" that $\mathcal{B}(\inf(a,b)) = \mathcal{B} \wedge \mathcal{B}'$ (see Definition 4.18).

A lattice is *distributive* if it satisfies the following two equations

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Note that it is enough to check whether a lattice satisfies one of the equations, the other can then be derived syntactically.

Braunfeld [Bra17] proved that, for a distributive lattice $\Lambda = (\Lambda, \vee, \wedge, 0)$ where $0$ is meet-irreducible, the class of convexly ordered $\Lambda$-ultrametric spaces has the strong amalgamation property (see Remark 1.4). For completeness, we include the proof adapted to the setting of semigroup-valued metric spaces. Note that each block of $\Lambda$ other than $\mathbf{0}$ contains precisely one distance from $\Lambda$ and the block lattice without $\mathbf{0}$ is isomorphic to $\Lambda$.

**Theorem 5.3** (Braunfeld [Bra17])**.** *Let $\Lambda$ be a distributive lattice. The class $\overrightarrow{\mathcal{M}}_\Lambda$ has the strong amalgamation property.*

*Proof.* By Theorem 3.6 we know that the unordered reduct $\mathcal{M}_\Lambda$ has the strong amalgamation property. Hence it suffices to be able to complete the partial orders. The key step of the proof is noticing that if $\mathbf{C}_0$ is the free amalgam of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ then for a fixed block $\mathcal{B} \in I_\Lambda$ the relation $\leq^{\mathcal{B}}_{\mathbf{C}_0}$ can be extended to a linear order, that is, there are no vertices $u \in B_1 \setminus A$, $v \in B_2 \setminus A$ and $w \in A$ with, say, $u \leq^{\mathcal{B}}_{\mathbf{C}_0} w \leq^{\mathcal{B}}_{\mathbf{C}_0} v$ such that in the $\Lambda$-shortest path completion of $\mathbf{C}_0$ we have $u \sim_{\mathcal{B}} v$. For a contradiction, assume the existence of such $u, v, w$.

$\mathcal{B}$ being meet-irreducible and $u \sim_{\mathcal{B}} v$ in the shortest path completion of $\mathbf{C}_0$ means that there are vertices $w_1, \ldots, w_k \in A$ such that

$$\mathcal{B} \succeq \bigwedge\{\mathcal{B}(d_{\mathbf{C}_0}(u, w_i)) \vee \mathcal{B}(d_{\mathbf{C}_0}(w_i, v)); 1 \leq i \leq k\}.$$

Because $\Lambda$ (and thus also the block lattice which is isomorphic to $\Lambda$) is distributive, there is $i$ such that

$$\mathcal{B} \succeq \mathcal{B}(d_{\mathbf{C}_0}(u, w_i)) \vee \mathcal{B}(d_{\mathbf{C}_0}(w_i, v))$$

(indeed, if $\Xi$ is a distributive lattice, $a \in \Xi$ is meet-irreducible and there are $b_1, \ldots, b_k \in \Xi$ such that $a \geq \bigwedge b_i$, this means that $a = a \vee \bigwedge b_i$, or $a = \bigwedge(a \vee b_i)$ and from meet-irreducibility it follows that there is $i$ with $a \geq a \vee b_i$) and this means $u \sim_{\mathcal{B}} w_i$ in $\mathbf{B}_1$ and $v \sim_{\mathcal{B}} w_i$ in $\mathbf{B}_2$. But then $u \leq^{\mathcal{B}}_{\mathbf{B}_1} w$ if and only if $w_i \leq^{\mathcal{B}}_{\mathbf{B}_1} w$ and $v \leq^{\mathcal{B}}_{\mathbf{B}_2} w$ if and only if $w_i \leq^{\mathcal{B}}_{\mathbf{B}_2} w$. Both $w_i$ and $w$ are in $A$, thus $u \leq^{\mathcal{B}}_{\mathbf{C}_0} w$ if and only if $v \leq^{\mathcal{B}}_{\mathbf{C}_0} w$, which is a contradiction. □

We intend to reduce the strong amalgamation property for general convexly ordered $\mathfrak{M}$-metric spaces (satisfying some conditions) to the strong amalgamation property for convexly ordered $\Lambda$-ultrametric spaces. Again, we know that by Theorem 3.6 the classes $\mathcal{M}_\mathfrak{M} \cap \mathrm{Forb}(\mathcal{F})$ have the strong amalgamation property and thus it is sufficient to be able to complete the orders. And they only depend on the block structure, hence it is enough to look at the *reducts* of the structures where distances are replaced by their blocks. Note that these are not reducts in the sense of Definition 2.15, they however correspond to the more general model-theoretical definition of a reduct.

**Corollary 5.4.** *Assume that the block order of $\mathfrak{M}$ is a distributive lattice and let $\mathcal{F}$ be an omissible family of $\mathfrak{M}$-edge labelled cycles containing all disobedient ones which* ***synchronizes meets****. Then $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$, the subclass of $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ omitting homomorphic images of members of $\mathcal{F}$, is a strong amalgamation class.*

*Proof.* Let $\mathbf{C}_0$ be the free amalgam of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ where $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. We will show that $\mathbf{C}_0$ has a strong completion $\mathbf{C}' \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ which is then the strong amalgam of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$.

By Theorem 3.6 we know that if we forget the orders, $\mathbf{C}_0$ has a strong $\mathfrak{M}$-metric completion. Thus it remains to show that one can complete the orders.

Consider the reduct $\mathbf{C}_0^-$ of $\mathbf{C}_0$ where we replace every distance $a$ with $\mathcal{B}(a)$. From the definition of the block order and from the assumption that $\mathcal{F}$ synchronizes meets we get that $\mathbf{C}_0$ is a free amalgam of convexly ordered $\Lambda$-ultrametric spaces, that is, semigroup-valued metric spaces where the semigroup is $\Lambda = (\{\mathcal{B}(a); a \in \mathfrak{M}\}, \vee, \wedge)$. Furthermore the shortest path completion of the distances in $\mathbf{C}_0^-$ is the same as the reduct of the shortest path completion of the distances in $\mathbf{C}_0$.

By Theorem 5.3 we know that the class of convexly ordered $\Lambda$-ultrametric spaces has the strong amalgamation property and this gives us the completion of the partial orders which we needed. □

*Remark.* We essentially just used the fact that the *shortest path completion functor* and the *reduct to $\Lambda$-ultrametric spaces functor* commute. In order for them to commute, one needs $\mathcal{F}$ to synchronize meets.

## 5.2 Ordering the ball vertices

If $(X, \leq_X)$ and $(Y, \leq_Y)$ are two ordered sets, then the *lexicographic order* $\leq_{\mathrm{lex}}$ on $X \times Y$ is given by $(x, y) \leq_{\mathrm{lex}} (x', y')$ if and only if either $x <_X x'$ or $x = x'$ and $y \leq_Y y'$. This naturally generalizes to the product of several orders.

For every $\mathcal{B} \in I_{\mathfrak{M}}$, define $\mathcal{B}^+$ to be the unique block such that for every $\mathcal{B}' \succeq \mathcal{B}$ it holds that $\mathcal{B} \prec \mathcal{B}^+ \preceq \mathcal{B}'$ (its existence and uniqueness follow from the fact that there are only finitely many blocks and that all meets are defined).

We are going to show that from the partial orders $\leq^{\mathcal{B}}$, one can define linear orders of balls of every diameter. There are many possible ways to do it and we need to pick one. There is nothing special about the particular choices in the following definition.

**Definition 5.5.** Assume an arbirary but fixed choice of a linear order $\trianglelefteq$ of all blocks of $\mathfrak{M}$ such that $\mathcal{B}_i \trianglelefteq \mathcal{B}_j$ whenever $\mathcal{B}_i \succeq \mathcal{B}_j$. Then define the function $U \colon R_{\mathfrak{M}} \cup I_{\mathfrak{M}} \to (R_{\mathfrak{M}} \cup I_{\mathfrak{M}})^2$ as follows:

1. If $\mathcal{B} \in R_{\mathfrak{M}}$ then $U(\mathcal{B}) = (\mathcal{B}_1, \mathcal{B}_2)$, where $(\mathcal{B}_1, \mathcal{B}_2)$ is the $\trianglelefteq$-lexicographically smallest pair of blocks such that $\mathcal{B}_1, \mathcal{B}_2 \neq \mathcal{B}$ and $\mathcal{B} = \mathcal{B}_1 \wedge \mathcal{B}_2$.
2. If $\mathcal{B} \in I_{\mathfrak{M}}$ then $U(\mathcal{B}) = (\mathcal{B}, \mathcal{B})$.

**Definition 5.6.** Let $U$ be the function from Definition 5.5. Given $\mathbf{A} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}}$, define inductively for every block $\mathcal{B}$ of $\mathfrak{M}$ the relation $\ll^{\mathcal{B}}$ on balls of diameter $\mathcal{B}$ in $\mathbf{A}$.

If $\mathcal{B}$ is the largest block of $\mathfrak{M}$, then there is only one ball of diameter $\mathcal{B}$ and $\ll^{\mathcal{B}}$ is trivial. Otherwise $B_1 \ll^{\mathcal{B}} B_2$ if and only if one of the following holds:

1. $\mathcal{B} \in R_{\mathfrak{M}}$, $U(\mathcal{B}) = (\mathcal{B}_1, \mathcal{B}_2)$ and the unique pair $B_1^1, B_1^2$ of blocks of diameter $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively containing $B_1$ is lexicographically (in the orders $\ll^{\mathcal{B}_1}$ and $\ll^{\mathcal{B}_2}$) smaller than the unique pair $B_2^1, B_2^2$ of blocks of diameter $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively containing $B_2$.
2. $\mathcal{B} \in I_{\mathfrak{M}}$ and there exist $u \in B_1$, $v \in B_2$ such that $u \leq^{\mathcal{B}} v$.
3. $\mathcal{B} \in I_{\mathfrak{M}}$, $B_1'$ and $B_2'$ are the unique balls of diameter $\mathcal{B}^+$ containing $B_1$ and $B_2$ respectively, and $B_1' \ll^{\mathcal{B}^+} B_2'$.

**Lemma 5.7.** *All relations $\ll^{\mathcal{B}}$ given by Definition 5.6 are linear orders.*

*Proof.* Enumerate blocks of $\mathfrak{M}$ as $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n$ in non-increasing order of $\preceq$ (i.e. there is no $\mathcal{B}_i \prec \mathcal{B}_j$ for $1 \leq i \leq j \leq n$). Given $\mathbf{A} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ we verify by induction that for every block $\mathcal{B}_i$ the relation $\ll^{\mathcal{B}_i}$ is indeed a linear order of balls of diameter $\mathcal{B}_i$. As $\mathcal{B}_1$ is the largest block, the statement is trivial for $\mathcal{B}_1$.

Now the induction step. If $\mathcal{B}_k \in R_{\mathfrak{M}}$, $U(\mathcal{B}) = (\mathcal{B}(a), \mathcal{B}(b))$ then by the induction hypothesis we know that $\ll^{\mathcal{B}(a)}$ and $\ll^{\mathcal{B}(b)}$ are linear orders of balls of diameter $\mathcal{B}(a)$ and $\mathcal{B}(b)$. Because every ball of diameter $\mathcal{B}_k$ lies in a unique intersection of $\mathcal{B}(a)$ and $\mathcal{B}(b)$, these two orders uniquely define lexicographically the linear order $\ll^{\mathcal{B}_k}$.

Otherwise $\mathcal{B}_k \in I_{\mathfrak{M}}$. Now for every pair $B_1$, $B_2$ of distinct balls of diameter $\mathcal{B}_k$ we have that $\leq^{\mathcal{B}_k}$ is defined for an arbirary choice of $u \in B_1$, $v \in B_2$ if and only if both blocks $B_1$ and $B_2$ belong to the same ball of diameter $\mathcal{B}_k^+$. In this case we use rule 2, otherwise the order is defined by rule 3. □

Note that, as a special case of Lemma 5.7, we get that $\ll^{\mathbf{0}}$ is a definable linear order on vertices of $\mathbf{A}$.

Now we need to transfer the convex order to the $L^\star$-expansions, which will make it possible to apply Theorem 2.22.

**Definition 5.8.** Denote by $L_{\mathfrak{M}}^{\star,\leq}$ the expansion of $L_{\mathfrak{M}}^{\star}$ adding the order $\leq$.

For a given convexly ordered metric space $\mathbf{A} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}}$, denote by $L^{\star,\leq}(\mathbf{A})$ the $L_{\mathfrak{M}}^{\star,\leq}$-structure $\mathbf{A}^{\star,\leq}$ created by the following procedure:

1. Start with $\mathbf{A}^{\star,\leq} = L^\star(\mathbf{A}^-)$ given by Definition 4.15, where $\mathbf{A}^-$ is the $L_{\mathfrak{M}}$-reduct of $\mathbf{A}$ (that is, we forget the orders).
2. Define the linear order $\leq_{\mathbf{A}^{\star,\leq}}$ as follows:
   (a) Order vertices of $\mathbf{A}$ according to $\ll^{\mathbf{0}}$ and let them form an initial segment of $\leq_{\mathbf{A}^{\star,\leq}}$.
   (b) For every pair of ball vertices $v_{\mathcal{B}}^i$ and $v_{\mathcal{B}'}^j$ (see Definition 4.15) put $v_{\mathcal{B}}^i \leq_{\mathbf{A}^{\star,\leq}} v_{\mathcal{B}'}^j$ if and only if $\mathcal{B} \lhd \mathcal{B}'$ or $\mathcal{B} = \mathcal{B}'$ and $E_{\mathcal{B}}^i \ll^{\mathcal{B}} E_{\mathcal{B}}^j$ (see Definitions 5.5 and 5.6).

Denote by $\mathcal{M}_{\mathfrak{M}}^{\star,\leq}$ the class of all $L^{\star,\leq}(\mathbf{A})$ for $\mathbf{A} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ and by $\mathcal{H}_{\mathfrak{M}}^{\star,\leq}$ the smallest hereditary superclass of $\mathcal{M}_{\mathfrak{M}}^{\star,\leq}$.

**Lemma 5.9.** *The categories $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ and $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ are isomorphic. In other words, there is a bijection between $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ and $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ which preserves embeddings and their compositions.*

*Proof.* Given $\mathbf{A}^{\star,\leq} \in \mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ we want to construct $\mathbf{A} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ such that $L^{\star,\leq}(\mathbf{A}) = \mathbf{A}^{\star,\leq}$. For this it is enough to reconstruct the partial orders $\leq^{\mathcal{B}}$. This can be done by putting $u \leq^{\mathcal{B}} v$ if and only if $u \nsim_{\mathcal{B}} v$, $F^{\mathcal{B}}_{\mathbf{A}}(u) \leq_{\mathbf{A}} F^{\mathcal{B}}_{\mathbf{A}}(v)$ and $u \sim_{\mathcal{B}'} v$ for every $\mathcal{B}' \succeq \mathcal{B}$.

It is then straightforward to check that indeed this is the inverse of $L^{\star,\leq}$ and that they give an isomorphism of categories. □

**Corollary 5.10.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup with finitely many blocks such that the block order is a distributive lattice and let $\mathcal{F}$ be an omissible family of $\mathfrak{M}$-edge labelled cycles containing all disobedient ones which synchronizes meets. Then $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ is a strong amalgamation class.*

*Proof.* By the category isomorphism we immediately get the amalgamation property. Strong amalgamation property follows from meet synchronization as in Corollary 4.19. □

*Remark.* Again, the class $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ is not hereditary. One gets $\mathcal{H}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ by adding structures containing some ball vertices which are not linked to any original vertices. Later we shall observe that this still is a strong amalgamation class. Note that, thanks to having a linear order, $\mathcal{H}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ consists of irreducible structures, which is another requirement of Theorem 2.22.

## 5.3 Completing the order

For the Ramsey property we need order. This is why we introduced the $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ and $\mathcal{H}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ classes (see Definition 5.8). In this section we show that the structures from $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$ with an additional relation $\leq$ which is "tame enough" have a completion in $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. This will imply both the strong amalgamation property and the Ramsey property of $\mathcal{H}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ and also the Ramsey property of $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$, which is what we aim for. But first we observe that if $\mathcal{F}$ is confined while containing all disobedient cycles (and synchronizing meets), it has some consequences for the block lattice.

**Lemma 5.11.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathcal{F}$ be an $\mathfrak{M}$-omissible, **confined** family of $\mathfrak{M}$-labelled cycles containing all disobedient ones.*

1. *Suppose that $\mathfrak{M}$ has finitely many blocks. Then for every two blocks $\mathcal{B}_1$ and $\mathcal{B}_2$ of $\mathfrak{M}$ their meet $\mathcal{B}_1 \wedge \mathcal{B}_2$ is defined, that is, the order of blocks is a lattice. Furthermore $\mathbf{0}$ is meet-irreducible.*
2. *Suppose that $\mathcal{F}$ synchronizes meets. Then the order of blocks is a distributive lattice. Furthermore $\mathbf{0}$ is meet-irreducible.*

On the first sight, it might seem surprising that having a family $\mathcal{F}$ can have some implications for the block structure of $\mathfrak{M}$. But on the second sight, the existence of a **confined** family $\mathcal{F}$ with the desired properties is quite a strong condition.

*Proof.* If $\mathcal{B}_1 \preceq \mathcal{B}_2$ or vice-versa, the smaller of them is their meet. So now suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are not comparable and thus different from $\mathbf{0}$.

Take arbitrary $a \in \mathcal{B}_1$ and $b \in \mathcal{B}_2$ and define the sequence $(c_n)_{n=1}^{\infty}$ by putting $c_n = \inf(n \times a, n \times b)$.

Observe that there are only finitely many $c_n$'s undefined: If $\inf(n \times a, n \times b)$ is undefined for $n \geq 2$, then the cycle $\mathbf{C}_n = (a, \ldots, a, b, \ldots, b)$ with $n$ edges of length $a$ and $n$ edges of length $b$ is disobedient and hence $\mathbf{C}_n \in \mathcal{F}$. But as $\mathcal{F}$ is confined, it is in particular $\{a, b\}$-locally finite and therefore there can only be finitely many such cycles in $\mathcal{F}$.

Now note that whenever $m \leq n$ and both $c_m$ and $c_n$ are defined, then $c_m \preceq c_n$ and thus $\mathcal{B}(c_m) \preceq \mathcal{B}(c_n)$.

Put

$$\mathcal{B} = \lim_{n\to\infty} \mathcal{B}(c_n),$$

where we ignore all the $n$'s with $c_n$ undefined. Note that if this limit is defined, then it certainly is different from $\mathbf{0}$. Thus if we prove that $\mathcal{B}$ is the meet of $\mathcal{B}_1$ and $\mathcal{B}_2$, we in particular get that $\mathbf{0}$ is not the meet of any two non-$\mathbf{0}$ blocks and hence is meet-irreducible.

For part 1 note that the sequence of blocks $(\mathcal{B}_{c_n})$ is non-decreasing and $\mathfrak{M}$ has only finitely many blocks. Thus the limit is well-defined.

For part 2 we may have infinitely many blocks, but we now have the assumption that $\mathcal{F}$ synchronizes meets. This means that whenever $\mathcal{B}(c_n) \neq \mathcal{B}(c_m)$, then one of the cycles $\mathbf{C}_n$, $\mathbf{C}_m$ needs to be in $\mathcal{F}$ (indeed, if this wasn't so, then their disjoint union would have a completion in $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ by Theorem 3.6 and this would contradict Definition 4.18). As $\mathcal{F}$ is $\{a, b\}$-locally finite, there can only be finitely many such forbidden $\mathbf{C}_n$'s, hence the limit is again well-defined.

Blocks are archimedean, hence for every $a' \in \mathcal{B}_1$ there are $m$ and $n$ with $m \times a' \succeq a$ and $n \times a \succeq a'$, the same holds also for $\mathcal{B}_2$. This implies that $\mathcal{B}$ doesn't depend on the choice of $a$ and $b$.

We shall prove that $\mathcal{B}$ is the meet of $\mathcal{B}_1$ and $\mathcal{B}_2$. The fact that $\mathcal{B} \preceq \mathcal{B}_1, \mathcal{B}_2$ follows straight from the definition of $\mathcal{B}$. Take some $x \in \mathfrak{M}$ such that there are $a \in \mathcal{B}_1$ and $b \in \mathcal{B}_2$ such that $a, b \succeq x$. Then also $n \times a, n \times b \succeq x$ for every $n$. Let $n$ be an arbitrary integer such that $\inf(n \times a, n \times b)$ is defined (by $\{a, b\}$-local finiteness of $\mathcal{F}$ there is such $n$). Clearly $\inf(n \times a, n \times b) \succeq x$, but also there is $c \in \mathcal{B}$ such that $c \succeq \inf(n \times a, n \times b)$, hence $c \succeq x$.

To prove distributivity for part 2 it suffices to check

$$\mathcal{B}_1 \vee (\mathcal{B}_2 \wedge \mathcal{B}_3) = (\mathcal{B}_1 \vee \mathcal{B}_2) \wedge (\mathcal{B}_1 \vee \mathcal{B}_3).$$

Take arbitrary $a \in \mathcal{B}_1, b \in \mathcal{B}_2, c \in \mathcal{B}_3$ and let $m$ be larger than the number of vertices of any cycle in $\mathcal{F}$ which contains only distances from $\{a, b, c\}$ ($\mathcal{F}$ is confined, hence $\{a, b, c\}$-locally finite and thus there is such a finite $m$). From archimedeanity we get $a' = m \times a \in \mathcal{B}_1$, $b' = m \times b \in \mathcal{B}_2$ and $c' = m \times c \in \mathcal{B}_3$. First note that

$$a' \oplus \inf(b', c') = \inf(a' \oplus b', a' \oplus c'),$$

because $\mathcal{F}$ contains all disobedient cycles and cycles with $2m$ edges cannot hence be forbidden. Clearly

$$\mathcal{B}(a' \oplus b') = \mathcal{B}_1 \vee \mathcal{B}_2$$

and

$$\mathcal{B}(a' \oplus c') = \mathcal{B}_1 \vee \mathcal{B}_3.$$

But $\mathcal{F}$ also synchronizes meets, this means that

$$\mathcal{B}(\inf(b', c')) = \mathcal{B}_2 \wedge \mathcal{B}_3$$

and

$$\mathcal{B}(\inf(a' \oplus b', a' \oplus c')) = (\mathcal{B}_1 \vee \mathcal{B}_2) \wedge (\mathcal{B}_1 \vee \mathcal{B}_3).$$

Finally

$$\mathcal{B}(a' \oplus \inf(b', c')) = \mathcal{B}_1 \vee (\mathcal{B}_2 \wedge \mathcal{B}_3),$$

hence indeed

$$\mathcal{B}_1 \vee (\mathcal{B}_2 \wedge \mathcal{B}_3) = (\mathcal{B}_1 \vee \mathcal{B}_2) \wedge (\mathcal{B}_1 \vee \mathcal{B}_3).$$

□

We want to prove the following proposition.

**Proposition 5.12.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathcal{F}$ be a family of $\mathfrak{M}$-labelled cycles. Suppose that the following conditions hold:*

1. *$\mathcal{F}$ is $\mathfrak{M}$-omissible;*
2. *$\mathcal{F}$ contains all $\mathfrak{M}$-disobedient cycles;*
3. *$\mathcal{F}$ is confined;*
4. *$\mathcal{F}$ synchronizes meets; and*
5. *$\mathfrak{M}$ has finitely many blocks.*

*Then the class $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ of all finite $L^{\star,\leq}_{\mathfrak{M}}$-expansions of $\mathfrak{M}$-metric spaces omitting $\mathcal{F}$ has the Ramsey property.*

In order to do that, we first need some auxilliary lemmas. For the rest of this section fix $\mathfrak{M}$ and $\mathcal{F}$ as in Proposition 5.12.

The following lemma is, similarly to Definition 4.27 and the subsequent lemmas, stated in a more abstract setting in order to avoid having to repeat the proof twice.

**Lemma 5.13.** *Let $\mathbf{C}_0$ be an $L^{\star,\leq}_{\mathfrak{M}}$-structure such that its $L^{\star}_{\mathfrak{M}}$ reduct is from $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap \mathrm{Forb}(\mathcal{F})$, there is a linear extension $\leq_0$ of $\leq_{\mathbf{C}_0}$ and there is a finite set $\mathcal{E}$ of embeddings of structures from $\mathcal{H}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ to $\mathbf{C}_0$ such that $u \leq_{\mathbf{C}_0} v$ or $v \leq_{\mathbf{C}_0} u$ is defined if and only if there is $\epsilon \in \mathcal{E}$ such that both $u$ and $v$ are in the range of $\epsilon$. Then $\mathbf{C}_0$ has a completion in $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$.*

*Proof.* By Lemma 5.11 we know that meets of all blocks are defined in $\mathfrak{M}$-metric spaces which do not contains cycles from $\mathcal{F}$, that the block lattice is distributive and that $\mathbf{0}$ is meet-irreducible.

By Proposition 4.31 we know that if we forget the order, $\mathbf{C}_0$ has a completion in $\mathcal{M}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. Suppose that $\mathbf{C}$ is this completed structure (that is, all functions and distances are defined and each ball vertex is pointed to by an original vertex). It remains to complete the order. We can assume that $\leq_0$ is a linear order of vertices of $\mathbf{C}$ (because the vertices added to $\mathbf{C}$ are not in any $\leq$ relation, hence one can add them to the order arbitrarily).

Let $\trianglelefteq$ be the order of blocks of $\mathfrak{M}$ from Definition 5.5 and enumerate $I_{\mathfrak{M}} \setminus \{\mathbf{0}\}$ as $\mathcal{B}_1 \trianglerighteq \mathcal{B}_2 \trianglerighteq \cdots \trianglerighteq \mathcal{B}_p$ (recall that this is a $\preceq$-non-decreasing order of blocks). Now define $\leq_{\mathbf{C}}$ on vertices of $\mathbf{C}$ as follows:

1. For every pair $u, v$ of original vertices of $\mathbf{C}$ put $u \leq_{\mathbf{C}} v$ if the sequence of vertices $(F^{\mathcal{B}_i}_{\mathbf{C}}(u))_i$ is in the order $\leq_0$ lexicographically before $(F^{\mathcal{B}_i}_{\mathbf{C}}(v))_i$ or they are equal and $u \leq_0 v$.
2. For every pair $u, v$ of ball vertices such that $u$ corresponds to block $\mathcal{B}_i$ and $v$ to block $\mathcal{B}_j$ put $u \leq_{\mathbf{C}} v$ if one of the following holds:
   (a) $i < j$,
   (b) $i = j$ and the sequence $(F^{\mathcal{B}_i,\mathcal{B}_{i'}}_{\mathbf{C}}(u))_{1\leq i' \leq i, \mathcal{B}_i \prec \mathcal{B}_{i'}}$ is in the order $\leq_0$ lexicographically before $(F^{\mathcal{B}_i,\mathcal{B}_{i'}}_{\mathbf{C}}(v))_{1\leq i' \leq i, \mathcal{B}_i \prec \mathcal{B}_{i'}}$,
   (c) $i = j$, $(F^{\mathcal{B}_i,\mathcal{B}_{i'}}_{\mathbf{C}}(u))_{1\leq i' \leq i, \mathcal{B}_i \prec \mathcal{B}_{i'}} = (F^{\mathcal{B}_i,\mathcal{B}_{i'}}_{\mathbf{C}}(v))_{1\leq i' \leq i, \mathcal{B}_i \prec \mathcal{B}_{i'}}$ and $u \leq_0 v$.
3. Finally put $u \leq_{\mathbf{C}} v$ if $u$ is an original vertex and $v$ is a ball vertex.

$\leq_{\mathbf{C}}$ is clearly a linear oder and by comparing the construction with Definition 4.15 it can be verified that $\mathbf{C}$ (with the order) is in $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ and that $\leq_{\mathbf{C}}$ and $\leq_{\mathbf{C}_0}$ agree on the images of every $\epsilon \in \mathcal{E}$ because the convex ordering coincides with the lexicographic ordering according of balls. Hence $\mathbf{C}$ is a completion of $\mathbf{C}_0$ in $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. □

**Corollary 5.14.** *The class $\mathcal{H}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ has the strong amalgamation property.*

*Proof.* Cf. Corollary 4.32; it is enough to check that every free amalgam of structures in $\mathcal{H}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ satisfies the conditions of Lemma 5.13. □

Now we have a completion procedure for certain $L^{\star,\leq}_{\mathfrak{M}}$-structures. The obvious next step is to show that it implies that $\mathcal{H}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ is a locally finite subclass of $\mathcal{R}_{\mathfrak{M}}$, the class of all $L^{\star,\leq}_{\mathfrak{M}}$-structures where $\leq$ is a linear order, which is Ramsey by Theorem 2.23. For it, we need to give a completion with respect to copies of $\mathbf{B}$ of certain special $L^{\star,\leq}_{\mathfrak{M}}$-structures. We will show that in that case the conditions of Lemma 5.13 are satisfied and it thus gives desired completion.

Theorem 2.22, given structures $\mathbf{A}, \mathbf{B} \in \mathcal{H}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ and a constant $n = n(\mathbf{B})$ provides us with an $L^{\star,\leq}_{\mathfrak{M}}$-structure $\mathbf{C}$ such that it has a homomorphism-embedding to some $\mathbf{C}_0 \in \mathcal{R}_{\mathfrak{M}}$ and all its substructures on at most $n$ vertices have a completion to $\mathcal{H}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ and we need to complete $\mathbf{C}$ to $\mathcal{H}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$.

Remember that for Theorem 2.22 it is enough to give a completion with respect to copies of $\mathbf{B}$ (see Definition 2.20). It motivates the following definition:

**Definition 5.15** (Multiamalgam of copies of **B**)**.** Let $L$ be a language and let $\mathbf{B}$ and $\mathbf{C}$ be $L$-structures. We say that $\mathbf{C}$ is a *multiamalgam of copies of* $\mathbf{B}$ if the following holds:

1. For every vertex $v \in \mathbf{C}$ there is an embedding $f \colon \mathbf{B} \to \mathbf{C}$ such that $v \in \mathrm{Range}(f)$;
2. for every relation $R \in L$ and every tuple $(v_1, \ldots, v_{a(R)}) \in R_{\mathbf{C}}$ there is an embedding $f \colon \mathbf{B} \to \mathbf{C}$ such that $v_1, \ldots, v_{a(R)} \in \mathrm{Range}(f)$; and
3. for every function $F \in L$, and every sequence of vertices $v_1, \ldots, v_{a(F)}, w \in C$ such that $F_{\mathbf{C}}(v_1, \ldots, v_{a(F)}) = w$ there is an embedding $f \colon \mathbf{B} \to \mathbf{C}$ such that $v_1, \ldots, v_{a(F)}, w \in \mathrm{Range}(f)$.

We can assume that the $\mathbf{C}$ which we need to complete is a multiamalgam of copies of $\mathbf{B}$, that is, every vertex, every relation and every function of $\mathbf{C}$ is part of some copy of $\mathbf{B}$; if it was not the case, one can simply drop the excess vertices, relations and functions. This clearly preserves the existence of a homomorphism-embedding to $\mathbf{C}_0$.

From $\mathbf{C}$ being a multiamalgam of copies of $\mathbf{B}$ it follows that it only contains the same distances as $\mathbf{B}$ and thus the first five conditions from Definition 4.27 are satisfied. The existence of a homomorphism-embedding to a linearly ordered $\mathbf{C}_0$ ensures that $\leq_{\mathbf{C}}$ is acyclic, that is, it has a linear extension $\leq_0$. And clearly $\leq_{\mathbf{C}}$ is defined precisely in the copies of $\mathbf{B}$. So of all conditions of Lemma 5.13 it remains to check that $\mathbf{C}$ is in $\mathrm{Forb}(\mathcal{F})$ and that it contains no homomorphic images of non-$\mathfrak{M}$-metric $\star$-cycles. And here the $n = n(\mathbf{B})$ parameter comes to play.

Let $S$ be a finite set of distances which contains all distances from $\mathbf{B}$ and at least one distance from every block. Because $\mathcal{F}$ is confined and hence $S$-locally finite, there is $m = m(\mathcal{F}, S)$ such that every cycle from $\mathcal{F}$ containing only distances from $S$ has at most $m$ vertices. Therefore if we set $n \geq m$, we get that $\mathbf{C} \in \mathrm{Forb}(\mathcal{F})$.

**Claim 5.16.** *For large enough $n$ it holds that $\mathbf{C}$ contains no non-$\mathfrak{M}$-metric $\star$-cycles.*

*Proof.* By Observation 4.28 we get a bound on the size on minimal non-$\mathfrak{M}$-metric $\star$-cycles with distances from $S$.

To finish the proof we need to observe that non-$\mathfrak{M}$-metric $\star$-cycles have no completion on $\mathcal{H}^{\star}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. But this follows from the definition of non-$\mathfrak{M}$-metric $\star$-cycles, the properties of $\mathrm{mus}(\mathcal{B}, S)$ and the fact that as $S$ contains at least one distance from every block, we can take $\mathrm{mus}(\mathcal{B}, S) \in S^{\oplus}$ for every block $\mathcal{B}$. □

*Proof of Proposition 5.12.* Using Lemma 5.13 we know that $\mathcal{H}^{\star;\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ is a locally finite subclass of $\mathcal{R}_{\mathfrak{M}}$, the class of all $L^{\star;\leq}_{\mathfrak{M}}$-structures where $\leq$ is a linear order, which is Ramsey by Theorem 2.23. We also know that $\mathcal{H}^{\star;\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ is a strong amalgamation class (Corollary 5.14) and it is hereditary, therefore by Theorem 2.22 we know that $\mathcal{H}^{\star;\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ is Ramsey.

We, however, need to show that $\mathcal{M}^{\star;\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ is Ramsey. Because $\mathcal{M}^{\star;\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F}) \subseteq \mathcal{H}^{\star;\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$, it follows that for every $\mathbf{A}, \mathbf{B} \in \mathcal{M}^{\star;\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ there is $\mathbf{C} \in \mathcal{H}^{\star;\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. By Lemma 5.13 there is

$\mathbf{C}' \in \mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ with $\mathbf{C} \subseteq \mathbf{C}'$. And thus $\mathbf{C}' \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$ and $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ indeed is Ramsey. □

## 5.4 Proof of Theorem 1.5 and the expansion property

Now we are ready to prove the main theorem of this thesis.

*Proof of Theorem 1.5.* Theorem 3.6 gives us the strong amalgamation property. Proposition 5.12 shows that $\mathcal{M}^{\star,\leq}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ is Ramsey and thus by Lemma 5.9 we then get that $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ is Ramsey when $\mathfrak{M}$ has finitely many blocks.

Suppose now that $\mathfrak{M}$ has infinitely many blocks. Take $\mathbf{A} \subseteq \mathbf{B} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. We want to find $\mathbf{C} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}\,(\mathcal{F})$ such that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. Let $S$ be the (finite) set of distances in $\mathbf{B}$. By Theorem 4.22 we get $\mathfrak{M}'$ and $\mathcal{F}'$ such that $\mathcal{M}_{\mathfrak{M}'} \cap \mathrm{Forb}(\mathcal{F}') \subseteq \mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. Therefore $\overrightarrow{\mathcal{M}}_{\mathfrak{M}'} \cap \mathrm{Forb}(\mathcal{F}')$ has the Ramsey property by the previous paragraphs.

Clearly, in $\mathbf{A}$ and $\mathbf{B}$ the only relations $\leq_{\mathcal{B}}$ which are not empty are those where $\mathcal{B}$ is also a block of $\mathfrak{M}'$. We can thus consider $\mathbf{A}$ and $\mathbf{B}$ to be from $\overrightarrow{\mathcal{M}}_{\mathfrak{M}'} \cap \mathrm{Forb}(\mathcal{F}')$ and from the Ramsey property we get $\mathbf{C} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}'} \cap \mathrm{Forb}(\mathcal{F}')$ which is the Ramsey witness for $\mathbf{A}$ and $\mathbf{B}$. Then it is enough to again add empty orders to $\mathbf{C}$ to get a structure from $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. □

**Theorem 5.17.** *Let $\mathfrak{M}$ and $\mathcal{F}$ be as in Theorem 1.5. Then $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ has the expansion property with respect to $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$.*

The proof is a generalization of Braunfeld's proof [Bra17, Lemma 7.10] which is in turn an adaptation of the standard argument. However, it is quite technically challenging and thus, as an introduction, we first sketch the proof of the case when $\mathfrak{M}$ is archimedean. Then there are no ball vertices, no unary functions and the expansion is just all linear orders. Note that $\mathrm{mus}(\mathfrak{M}, S)$ is a well-defined distance for a finite set $S \subseteq \mathfrak{M}$.

Given $\mathbf{A} \in \mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ we need to find $\mathbf{B} \in \mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ such that every ordering of $\mathbf{B}$ contains every ordering of $\mathbf{A}$. Denote by $S$ the set of distances in $\mathbf{A}$. Let $\overrightarrow{\mathbf{A}} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ be a fixed ordering of $\mathbf{A}$. Enumerate all pairs of vertices $u < v \in A$ as $(u_1, v_1), \ldots, (u_k, v_k)$ and define a sequence $\overrightarrow{\mathbf{A}} = \overrightarrow{\mathbf{A}}^0 \subseteq \overrightarrow{\mathbf{A}}^1 \subseteq \ldots \subseteq \overrightarrow{\mathbf{A}}^k$ of structures from $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ by induction such that $\overrightarrow{\mathbf{A}}^{i+1}$ is the strong amalgam of $\overrightarrow{\mathbf{A}}^i$ and the ordered triangle $\mathbf{T}_i$ with vertices $u_i, v_i, x_i$ where $d_{\mathbf{T}}(u_i, v_i) = d_{\overrightarrow{\mathbf{A}}}(u_i, v_i)$, $d_{\mathbf{T}}(u_i, x_i) = d_{\mathbf{T}}(v_i, x_i) = \mathrm{mus}(\mathfrak{M}, S)$ and $u_i \leq_{\mathbf{T}} x_i \leq_{\mathbf{T}} v_i$ over the edge $u_i, v_i$.

Now let $\overrightarrow{\mathbf{D}}$ be the joint embedding of the $\overrightarrow{\mathbf{A}}^k$'s for all linear orderings $\overrightarrow{\mathbf{A}}$ of $\mathbf{A}$. By the Ramsey property there exists $\overrightarrow{\mathbf{B}} \longrightarrow (\overrightarrow{\mathbf{D}})^{\overrightarrow{\mathbf{H}}}_2$, where $\overrightarrow{\mathbf{H}}$ is the ordered pair of vertices in distance $\mathrm{mus}(\mathfrak{M}, S)$. Let $\leq_0$ be the linear order on vertices of $\overrightarrow{\mathbf{B}}$ and finally let $\mathbf{B}$ be the reduct of $\overrightarrow{\mathbf{B}}$ by forgetting the order.

We claim that $\mathbf{B}$ is the desired expansion property witness for $\mathbf{A}$. Indeed, let $\leq_1$ be an arbitrary linear ordering of $\mathbf{B}$. Define the colouring $c\colon \binom{\overrightarrow{\mathbf{B}}}{\overrightarrow{\mathbf{H}}} \to \{0,1\}$ by putting $c(\alpha) = 0$ if the orders $\leq_0$ and $\leq_1$ agree on $\alpha(\overrightarrow{\mathbf{H}})$ and 1 otherwise.

By the Ramsey property we get a copy $\overrightarrow{\mathbf{D}} \subseteq \overrightarrow{\mathbf{B}}$ such that $\leq_0$ and $\leq_1$ are on the copy either the same or opposite. In both cases $\overrightarrow{\mathbf{D}}$ contains all possible orderings of $\mathbf{A}$.

*Proof of Theorem 5.17.* The complication now is that there are multiple (partial) orders which depend on the distances. There is no hope to find a universal distance (like $\mathrm{mus}(\mathfrak{M}, S)$ in the archimedean case) which would allow us to place auxilliary vertices between every pair of vertices where we want to fix the order. Instead, we are going to do that separately for every meet-irreducible block (and thus every $\leq^{\mathcal{B}}$). This will hence need an iterated use of the Ramsey property.

Let $\mathbf{A} \in \mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ be given and let $S$ be the set of distances in $\mathbf{A}$. Enumerate the non-maximal meet-irreducible blocks of $\mathfrak{M}$ (including $\mathbf{0}$) which "nontrivially appear in $\mathbf{A}$" as $\mathcal{B}_1, \ldots, \mathcal{B}_b$, that is, $\mathcal{B} \in I_{\mathfrak{M}}$ is in the sequence if and only if $\leq^{\mathcal{B}}_{\overrightarrow{\mathbf{A}}}$ is nonempty in some convex ordering $\overrightarrow{\mathbf{A}}$ of $\mathbf{A}$. Also define a sequence of distances $m_1, \ldots, m_b$ where $m_i = \mathrm{mus}(\mathcal{B}_i, S \cup \{m_j; j < i\})$. By induction we define $\mathbf{A} = \mathbf{B}_0 \subseteq \mathbf{B}_1 \subseteq \cdots \subseteq \mathbf{B}_b = \mathbf{B}$ as follows.

Let $\mathbf{B}_i$ be defined. We now do a very similar construction as in the archimedean case with $\mathbf{B}_i$ playing the role of $\mathbf{A}$ and $\mathbf{B}_{i+1}$ being the expansion property witness. Fix an expansion $\overrightarrow{\mathbf{B}}_i$ of $\mathbf{B}_i$ in $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. Enumerate the balls of diameter $\mathcal{B}_{i+1}$ in $\overrightarrow{\mathbf{B}}_i$ as $E_1, \ldots, E_\ell$ and pick a representant $w_j \in E_j$ for each ball $E_j$ arbitrarily. Finally enumerate by $(u_1, v_1), \ldots, (u_k, v_k)$ all pairs of the representants such that $u_j <^{\mathcal{B}_{i+1}}_{\overrightarrow{\mathbf{B}}_i} v_j$.

Now define a sequence $\overrightarrow{\mathbf{B}}_i = \overrightarrow{\mathbf{B}}^0_i \subseteq \overrightarrow{\mathbf{B}}^1_i \subseteq \ldots \subseteq \overrightarrow{\mathbf{B}}^k_i$ by induction such that $\overrightarrow{\mathbf{B}}^{j+1}_i$ is the strong amalgam of $\overrightarrow{\mathbf{B}}^j_i$ and the ordered triangle $\mathbf{T}$ over the edge $u_j, v_j$, where $\mathbf{T}$ is the triangle with vertices $u_j, v_j, x_j$ with $u_j <^{\mathcal{B}_{i+1}}_{\mathbf{T}} x_j <^{\mathcal{B}_{i+1}}_{\mathbf{T}} v_j$, $d_{\mathbf{T}}(u_j, v_j) = d_{\overrightarrow{\mathbf{B}}_i}(u_j, v_j)$ and $d_{\mathbf{T}}(u_j, x_j) = d_{\mathbf{T}}(v_j, x_j) = m_{i+1}$.

Put $\overrightarrow{\mathbf{D}}_{i+1}$ be the joint embedding of the $\overrightarrow{\mathbf{B}}^k_i$'s for all expansions $\overrightarrow{\mathbf{B}}_i$ of $\mathbf{B}_i$ in $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$. By the Ramsey property there exists $\overrightarrow{\mathbf{B}}_{i+1} \longrightarrow (\overrightarrow{\mathbf{D}}_{i+1})^{\overrightarrow{\mathbf{H}}_{i+1}}_2$, where $\overrightarrow{\mathbf{H}}_{i+1}$ is the ordered pair of vertices in distance $m_{i+1}$. Let $\leq^{\mathcal{B}_{i+1}}_0$ be the partial order for $\mathcal{B}_{i+1}$ of $\overrightarrow{\mathbf{B}}_{i+1}$ and finally let $\mathbf{B}_{i+1}$ be the reduct of $\overrightarrow{\mathbf{B}}_{i+1}$ forgetting all the orders.

By an analogous argument as in the archimedean case (after remembering that if $u \sim^{\mathcal{B}} v$, then $u \leq^{\mathcal{B}} w$ if and only if $v \leq^{\mathcal{B}} w$) we get that in every expansion of $\mathbf{B}_{i+1}$ in $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ there are all possible orders $\leq^{\mathcal{B}_{i+1}}$ of $\mathbf{B}_i$.

It follows that in every expansion of $\mathbf{B} = \mathbf{B}_b$ there are all possible expansions of $\mathbf{A}$ as desired. □

# 6. Applications

It is trivial to check that the (partially) ordered commutative semigroups connected to the $S$-metric spaces or $\Lambda$-ultrametric spaces satisfy the axioms of Theorem 1.5. Similarly, we also get Ramsey expansions and EPPA for the multiplicative metric spaces (which has not been done before). Another original corollary of our results is EPPA for Sauer's $S$-metric spaces, Braunfeld's $\Lambda$-ultrametric spaces (our methods would work even for the ones with meet-reducible identity) and non-semi-archimedean Conant's generalised metric spaces.

## 6.1 Primitive metrically homogeneous graphs

A metrically homogeneous graph is a countable connected graph which gives rise to a homogeneous metric space when one computes the distances between all vertices. Cherlin [Che11, Che16] recently gave a catalogue of the known homogeneous metric spaces of this type. A key role in the catalogue is played by the so-called *primitive 3-constrained cases*. They can be described as $\{1, \ldots, \delta\}$-edge-labelled graphs with some triangles forbidden. And these triangles can be described in terms of five parameters $(\delta, K_1, K_2, C_0, C_1)$.

Ramsey expansions and other combinatorial properties of the classes from Cherlin's catalogue have been found by Aranda, Bradley-Williams, Hubička, Karamanlis, Kompatscher, Pawliuk and the author [ABWH⁺17c, ABWH⁺17a, ABWH⁺17b] and were also studied by others [Cou17, Sok17]. In [ABWH⁺17c] the key ingredient is an explicit completion procedure quite similar to the shortest path completion.

To interpret every primitive 3-constrained case as a semigroup-valued metric space would take a lot of mechanical work and inequality checking. In this thesis, we only sketch the proof for part of the primitive 3-constrained cases (namely we omit the cases when $|C_0 - C_1| > 1$) and therefore only give the minimum necessary introduction and definitions. For a broader overview see [ABWH⁺17c] or the author's Bachelor thesis [Kon18].

**Definition 6.1.** We say that a sequence of integers $(\delta, K_1, K_2, C)$ is *relevant* if the following conditions hold:

- $3 \leq \delta < \infty$;
- $1 \leq K_1 \leq K_2 \leq \delta$;
- $2\delta + 2 \leq C \leq 3\delta + 2$;
- one of the following holds:
  - (II) $C \leq 2\delta + K_1$, and:
    - $C = 2K_1 + 2K_2 + 1$;
    - $K_1 + K_2 \geq \delta$;
    - $K_1 + 2K_2 \leq 2\delta - 1$,
  - (III) $C \geq 2\delta + K_1 + 1$, and:
    - $K_1 + 2K_2 \geq 2\delta - 1$ and $3K_2 \geq 2\delta$;

– If $K_1 + 2K_2 = 2\delta - 1$ then $C \geq 2\delta + K_1 + 2$.

This definition is a subset of the union of two of Cherlin's definitions (*acceptability* and *admissibility*) and is tailored so that it only contains the classes interesting for our purpouses, which is also why Case I is missing.

Coulson, Hubička, Kompatscher and the author [CHKK18] characterized all the $\{1, \ldots, \delta\}$-edge-labelled graphs which have a completion into one of Cherlin's classes with relevant parameters as $\mathrm{Forb}(\mathcal{F}^\delta_{K_1,K_2,C})$ for $\mathcal{F}^\delta_{K_1,K_2,C}$ being a family of the following cycles, where we say that a cycle has distances $a_1, \ldots a_k$ if it has $k$ edges labelled by $a_1, \ldots, a_k$ in some order. A perimeter of a cycle is the sum of its distances.

**$C$-cycles:** Cycles with distances $d_0, d_1, \ldots, d_{2n}, x_1, \ldots, x_k$ for some $n \geq 0$ such that
$$\sum_{i=0}^{2n} d_i > n(C-1) + \sum_{i=1}^{k} x_i.$$

**$K_1$-cycles:** Non-$C$-cycles of odd perimeter with distances $x_1, \ldots, x_k$ such that
$$2K_1 > \sum_{i=1}^{k} x_i.$$

**$K_2$-cycles:** Non-$C$-cycles of odd perimeter with distances $d_1, \ldots, d_{2n+2}, x_1, \ldots, x_k$ such that
$$\sum_{i=1}^{2n+2} d_i > 2K_2 + n(C-1) + \sum_{i=1}^{k} x_i.$$

Notice that a non-metric cycle is a $C$-cycle with $n = 0$. Cherlin's class $\mathcal{A}^\delta_{K_1,K_2,C}$ can then be defined as the subclass of $\mathrm{Forb}(\mathcal{F}^\delta_{K_1,K_2,C})$ containing only complete graphs. It is then already determined by forbidding the three-vertex cycles from $\mathcal{F}^\delta_{K_1,K_2,C}$. Cherlin proved that all the $\mathcal{A}^\delta_{K_1,K_2,C}$'s are strong amalgamation classes.

In [ABWH+17c] we proved that the classes $\mathcal{A}^\delta_{K_1,K_2,C}$ are locally finite subclasses of the class of all $\{1, \ldots, \delta\}$-edge-labelled complete graphs using a special *magic completion algorithm* for which the following definition was key (it was stated in a different way).

**Definition 6.2** (Magic semigroup)**.** Let $3 \leq \delta < \infty$, $2\delta + 2 \leq C \leq 3\delta + 1$ and $\lceil \frac{\delta}{2} \rceil \leq M \leq \frac{C-\delta-1}{2}$ be integers. Then the operation $\oplus\colon \{1, \ldots, \delta\}^2 \to \{1, \ldots, \delta\}$ is defined as follows
$$x \oplus y = \begin{cases} |x-y| & \text{if } |x-y| > M \\ \min(x+y, C-1-x-y) & \text{if } \min(\ldots) < M \\ M & \text{otherwise.} \end{cases}$$

The magic completion is a refinement of the shortest path completion algorithm which step-by-step adds edges according to the $\oplus$ operation. Observe that in particular $M \oplus x = M$ for every $x$.

It is straightforward to check that $\oplus$ is associative and commutative, hence $(\{1, \ldots, \delta\}, \oplus)$ is indeed a commutative semigroup for every valid choice of $\delta$, $C$ and $M$.

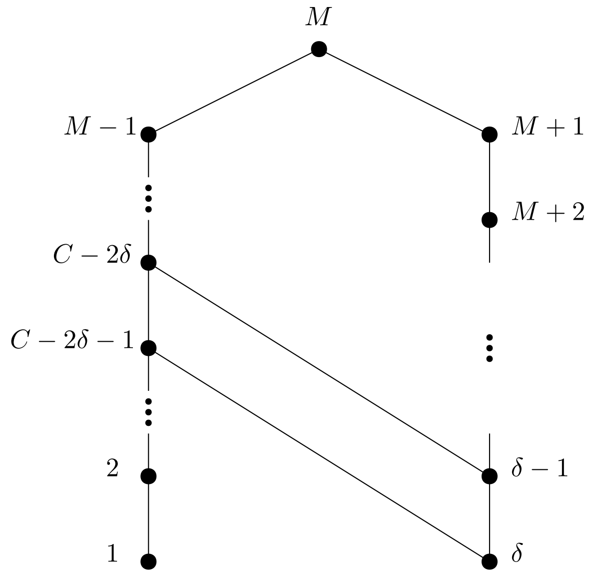



Figure 6.1: The Hasse diagram of the magic order $\preceq$.

Fix $\delta$, $C$ and $M$ from Definition 6.2. The *natural partial order* $\preceq$ on $\{1,\dots,\delta\}$ is defined by $a \preceq b$ if and only if $a = b$ or there is $1 \le c \le \delta$ such that $b = a \oplus c$. It is again straightforward to check that $\preceq$ is a partial order and that the *ordered magic semigroup* $\mathfrak{M}^\delta_{M,C} = (\{1,\dots,\delta\}, \oplus, \preceq)$ is a partially ordered commutative semigroup. Note that if $C = 3\delta + 1$ and $M = \delta$ then $\mathfrak{M}^\delta_{M,C}$ is just $\{1,\dots,\delta\}$ with the standard order and addition capped by $\delta$.

**Observation 6.3.** *In $\mathfrak{M}^\delta_{M,C}$ we have $a \preceq b$ if and only if one of the following holds (see Figure 6.1):*

1. *$a \le b \le M$;*
2. *$a \ge b \ge M$; or*
3. *$a \ge M$ and $C - 1 - \delta - a \le b \le M$.*

*In other words, $a$ is $\preceq$-incomparable with $b$ (without loss of generality we can assume $a \ge b$) if and only if $a > M$, $b < M$ and $C - 1 - \delta - a > b$.*

Clearly $\mathfrak{M}^\delta_{M,C}$ is archimedean. In order to plug it into the machinery developed here, we need to understand the $\preceq$-infima, their distributivity with $\oplus$ and what the non-$\mathfrak{M}^\delta_{M,C}$-metric cycles are.

**Proposition 6.4.** *Let $x_1,\dots,x_k$ and $d_1,\dots,d_n$ be two sequences of integers such that $1 \le x_i < M$ and $M < d_i \le \delta$ for every $i$. Denote $S = x_1 \oplus \cdots \oplus x_k \oplus d_1 \oplus \cdots \oplus d_n$. Then one of the following happens:*

1. *$S = M$;*
2. *$S < M$, $n$ is even and*

$$S = \frac{n}{2}(C-1) + \sum_{i=1}^{k} x_i - \sum_{i=1}^{n} d_n;$$

*3. $S > M$, $n$ is odd and*

$$S = \sum_{i=1}^{n} d_n - \frac{n-1}{2}(C-1) - \sum_{i=1}^{k} x_i.$$

*Proof.* By induction. Clearly holds for $k + n \leq 2$, and the induction step is just straightforward high-school algebra. □

**Corollary 6.5.** *Let $1 \leq a, b \leq \delta$ be $\preceq$-incomparable and $\mathbf{K}$ a cycle with distances $a_1, \dots, a_k, b_1, \dots, b_\ell$ such that $k + \ell \geq 3$, $a = a_1 \oplus \cdots \oplus a_k$ and $b = b_1 \oplus \cdots \oplus b_\ell$. Then $\mathbf{K}$ is a $C$-cycle.*

*Proof.* We can without loss of generality (cf. Observation 6.3) assume $a < M$ and $b > M$.

By Proposition 6.4 we get $a = \frac{n_a}{2}(C-1) + \sum_{i=1}^{m_a} x_i^a - \sum_{i=1}^{n_a} d_i^a$ and $b = \sum_{i=1}^{n_b} d_i^b - \frac{n_b - 1}{2}(C-1) - \sum_{i=1}^{m_b} x_i^b$ for $\{d_i^a; 1 \leq i \leq n_a\} \cup \{x_i^a; 1 \leq i \leq m_a\} = \{a_i; 1 \leq i \leq k\}$ with the union being disjoint, analogously for $b$. As $b > M > a$, we get

$$\sum_{i=1}^{n_b} d_i^b - \frac{n_b - 1}{2}(C-1) - \sum_{i=1}^{m_b} x_i^b > \frac{n_a}{2}(C-1) + \sum_{i=1}^{m_a} x_i^a - \sum_{i=1}^{n_a} d_i^a,$$

or

$$\sum_{i=1}^{n_b} d_i^b + \sum_{i=1}^{n_a} d_i^a > \frac{n_a + n_b - 1}{2}(C-1) + \sum_{i=1}^{m_b} x_i^b + \sum_{i=1}^{m_a} x_i^a,$$

which is just the $C$-inequality. □

This corollary implies that the family of $C$-cycles contains all disobedient ones: Indeed, whenever there are two paths between two vertices in a $\mathfrak{M}^\delta_{M,C}$-metric space omitting $C$-cycles, then their $\mathfrak{M}^\delta_{M,C}$-lengths have to be comparable, therefore there are no non-trivial infima. And trivial infima (that is, minima) distribute with $\oplus$.

**Lemma 6.6.** *The family $\mathcal{F}^-$ of all $\mathfrak{M}^\delta_{M,C}$-metric $C$-cycles is $\mathfrak{M}^\delta_{M,C}$-omissible.*

*Proof.* We need to check that $\mathcal{F}^-$ contains no geodesic cycles, that it is downwards closed and that it is closed under inverse steps of the shortest path completion. Closedness under the inverse steps is straightforward: There are no non-trivial infima involved, so any edge gets replaced by a path whose $\mathfrak{M}^\delta_{M,C}$-length is the length of the edge and it is easy to check that this preserves the $C$-inequality and $\mathfrak{M}^\delta_{M,C}$-metricity.

From Proposition 6.4 it also follows that $\mathcal{F}^-$ contains no geodesic cycles (if it did, then we use the correspondence from Proposition 6.4 to get a contradiction with the $C$-inequality being strict).

It remains to check downwards closedness. Take any contiguous segment of edges of a cycle $\mathbf{K} \in \mathcal{F}^-$ and enumerate the lengths of its edges as $e_1, \dots, e_n$, $y_1, \dots, y_k$ where $e_i > M$ and $y_i < M$ (there is no edge of length $M$ in $\mathbf{K}$ because otherwise it would be non-$\mathfrak{M}^\delta_{M,C}$-metric or geodesic). The parity of $n$ determines whether $\bigoplus e_i \oplus \bigoplus y_i$ is greater than or smaller than $M$ and based on this and Proposition 6.4 and some high-school algebra we get the desired result. □

Again by an analysis of different reasons for $a \preceq b$ it follows that every non-$\mathfrak{M}^\delta_{M,C}$-metric cycle is a $C$-cycle. This in particular implies that all $K_1$- and $K_2$-cycles are $\mathfrak{M}^\delta_{M,C}$-metric. It then again takes some checking (in a similar fashion as before) that if further $K_1 \leq M \leq K_2$, the $K_1$- and $K_2$-cycles are not geodesic and that their union is closed both downwards and on the inverse steps of the shortest path completion. Here one needs to use the fact that the parameters are relevant. But in the end we get the following:

**Theorem 6.7.** *Let $(\delta, K_1, K_2, C)$ be relevant parameters and define $\mathcal{F}$ as the union of $\mathcal{F}^-$ from Lemma 6.6 and all $K_1$- and $K_2$-cycles. Then $\mathcal{F}$ is $\mathfrak{M}^\delta_{M,C}$-omissible, contains all $\mathfrak{M}^\delta_{M,C}$-disobedient cycles and is finite. Therefore $\mathfrak{M}^\delta_{M,C}$ and $\mathcal{F}$ satisfy the assumptions of Theorem 1.5.*

*Furthermore, the class $\mathfrak{M}^\delta_{M,C} \cap \mathrm{Forb}(\mathcal{F})$ is precisely $\mathcal{A}^\delta_{K_1,K_2,C}$.*

In order to get an analogue of Theorem 6.7 for all primitive 3-constrained metrically homogeneous graphs one only needs to do more rather mechanical inequality checking (cf. [CHKK18]).

It is worth stressing out that Theorem 6.7 depends heavily on previous results [ABWH$^+$17c, CHKK18] and the author does not see a way to bypass this which would be simpler then redoing all the proofs in a different language.

## 6.2 Henson constraints

In the catalogue of metrically homogeneous graphs, Cherlin also allows to forbid arbitrarily large metric spaces containing only distances 1 and $\delta$ (when certain conditions are satisfied) and he calls them *Henson constraints*. It motivates the following paragraphs.

**Definition 6.8.** Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup. We say that $a \in \mathfrak{M}$ is *reducible* if there are $b, c \in \mathfrak{M}$ such that either $a = b \oplus c$, or $a = \inf(b, c)$ and $a \neq b, c$. Otherwise we call *a irreducible*.

Let further $\mathcal{F}$ be a $\mathfrak{M}$-omissible family containing all $\mathfrak{M}$-disobedient cycles. We say that $a \in \mathfrak{M}$ is *irreducible with respect to* $\mathcal{F}$ if whenever $\mathbf{A}$ is an $\mathfrak{M}$-edge-labelled $\mathfrak{M}$-metric graph from $\mathrm{Forb}(\mathcal{F})$, $\mathbf{A}'$ its shortest path completion and $u, v \in A$ vertices such that $d_{\mathbf{A}'}(u, v) = a$, then $d_{\mathbf{A}}(u, v) = a$. Otherwise $a$ is *reducible with respect to* $\mathcal{F}$

**Example 6.9.** If $\mathfrak{M}$ is a monoid with neutral element 0, then all elements are reducible (as $a = a \oplus 0$).

**Example 6.10.** In the magic semigroup $\mathfrak{M}^\delta_{M,C}$, unless $M = \delta$ or $C = 2\delta + 2$, the irreducible elements with respect to the union of $K_1$-, $K_2$- and $\mathfrak{M}^\delta_{M,C}$-metric $C$-cycles are precisely 1 and $\delta$ (whereas if $C > 2\delta + 3$ and $\delta$ is large enough then $\inf(2, \delta - 1) = 1$, so 1 is reducible).

Clearly if an element is irreducible, then it is irreducible with respect to any family $\mathcal{F}$. We say that $\mathcal{H}$ is a *family of Henson constraints* if $\mathcal{H}$ consists of $\mathfrak{M}$-metric spaces only using distances irreducible with respect to $\mathcal{F}$.

Oce can then prove that both the classes $\mathcal{M}_\mathfrak{M} \cap \mathrm{Forb}(\mathcal{F}) \cap \mathrm{Forb}(\mathcal{H})$ and $\overrightarrow{\mathcal{M}}_\mathfrak{M} \cap \mathrm{Forb}(\mathcal{F}) \cap \mathrm{Forb}(\mathcal{H})$ have the strong amalgamation property and that the latter is Ramsey. The strong amalgamation property is absolutely straightforward, if there are no cliques from $\mathcal{H}$ in either of the amalgamated structures and the shortest path completion does not produce any distances used in $\mathcal{H}$, then there are no cliques from $\mathcal{H}$ in the shortest path completion of the amalgam. And $\mathcal{H}$ does not interfere with the orders in any way.

For Ramseyness, one would needs to redo the proofs in Chapters 4 and 5 and add checks for $\mathcal{H}$. They are, however, also quite straightforward. From the existence of a homomorphism-embedding to a (complete) structure $\mathbf{C}_0$ we get a bound on the number of vertices of the largest clique in $\mathbf{C}$ (that is, the number of vertices of $\mathbf{C}$) and thus we can require that there are no cliques from $\mathcal{H}$ of at most that size in $\mathbf{C}$. And then, again, the shortest path completion does not produce any members of $\mathcal{H}$ and we are done.

We chose to only mention the Henson constraints here in order not to add more technical complications into the proofs.

It is also possible to prove EPPA for the Henson-constrained classes, it is however more complicated because one needs to use the Hodkinson–Otto theorem [HO03] and essentially combine the techniques from this thesis with the techniques from a paper by Evans, Hubička and Nešetřil [EHN17b].

*Remark.* While the Hubička–Nešetřil theorem works for languages with functions of arbitrary arity, EPPA has so far only been proved for unary functions.

## 6.3 Stationary independence relations

In 2012 Tent and Ziegler [TZ13b] proved that the automorphism group of the Urysohn space has one proper normal subgroup consisting of the *bounded automorphisms* (that is, automorphisms $\alpha$ such that there is a distance $a = a(\alpha)$ such that for every vertex $x$ it holds that $d(x, \alpha(x)) \leq a$). In the paper, they defined the *(local) stationary independence relation* (or *SIR*), which is a ternary relation on a homogeneous structure satisfying some axioms (see for example [ABWH+17c]).

As observed in [ABWH+17c], (local) stationary independence relations correspond to (local) canonical amalgamations on Fraïssé classes, and because of their generality, combinatorial simplicity and convenience we will only present what a canonical amalgamation operator on a strong amalgamation class is.

**Definition 6.11.** Let $\mathcal{C}$ be a strong amalgamation class. We say that $\oplus$ is an *amalgamation operator*, if it assigns to every triple of structures $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{C}$ with embeddings $e_1 \colon \mathbf{A} \to \mathbf{B}_1$ and $e_2 \colon \mathbf{A} \to \mathbf{B}_2$ a unique amalgam, i.e. a structure $\mathbf{D} \in \mathcal{C}$ and embeddings $f_1 \colon \mathbf{B}_1 \to \mathbf{D}$, $f_2 \colon \mathbf{B}_2 \to \mathbf{D}$, such that $f_1 \circ e_1 = f_2 \circ e_2$. For short let us then write $\mathbf{D} = \mathbf{B}_1 \oplus_\mathbf{A} \mathbf{B}_2$. We call $\oplus$ a *local* amalgamation operator if it is only defined for non-empty $\mathbf{A}$. Finally, we say that $\oplus$ is *canonical* on $\mathcal{C}$ if additionally the following hold:

1. $\oplus$ only depends on the isomorphism type, that is, if $\mathbf{D} = \mathbf{B}_1 \oplus_\mathbf{A} \mathbf{B}_2$ with embeddings $e_1 \colon \mathbf{A} \to \mathbf{B}_1$, $e_2 \colon \mathbf{A} \to \mathbf{B}_2$, $f_1 \colon \mathbf{B}_1 \to \mathbf{D}$, $f_2 \colon \mathbf{B}_2 \to \mathbf{D}$ and there are $\mathbf{B}'_1, \mathbf{B}'_2, \mathbf{A}' \in \mathcal{C}$ with isomorphisms $\alpha \colon \mathbf{A}' \to \mathbf{A}$, $\beta_1 \colon \mathbf{B}_1 \to \mathbf{B}'_1$ and $\beta_2 \colon \mathbf{B}_2 \to \mathbf{B}'_2$, then if $\mathbf{D}' = \mathbf{B}'_1 \oplus_{\mathbf{A}'} \mathbf{B}'_2$ with embeddings $\beta_1 \circ e_1 \circ \alpha \colon \mathbf{A}' \to \mathbf{B}'_1$, $\beta_2 \circ e_2 \circ \alpha \colon \mathbf{A}' \to \mathbf{B}'_2$, $f'_1 \colon \mathbf{B}'_1 \to \mathbf{D}'$, $f'_2 \colon \mathbf{B}'_2 \to \mathbf{D}'$, then there is an isomorphism $\delta \colon \mathbf{D}' \to \mathbf{D}$ with $\delta \circ f'_i \circ \beta_i = f_i$ for $i \in \{1, 2\}$.

2. The vertices of $\mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2$ are precisely the union of $f_1(B_1)$ and $f_2(B_2)$ and $f_1(B_1)$ intersects with $f_2(B_2)$ only in $f_1(e_1(A))$.
3. Monotonicity: If $\mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2 = \langle f_1(B_1) \cup f_2(B_2)\rangle$, then $\mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B'_2} = \langle f_1(B_1) \cup f_2(B'_2)\rangle$ for all structures $e_2(\mathbf{A}) \subseteq \mathbf{B'_2} \subseteq \mathbf{B}_2$, where by $\langle X\rangle$ we mean the structure induced (by the canonical amalgam) on the set $X$.
4. Associativity: $(\mathbf{A} \oplus_{\mathbf{C_1}} \mathbf{B}) \oplus_{\mathbf{C_2}} \mathbf{D} = \mathbf{A} \oplus_{\mathbf{C_1}} (\mathbf{B} \oplus_{\mathbf{C_2}} \mathbf{D})$.

**Theorem 6.12.** *Let $\mathfrak{M} = (M, \oplus, \preceq)$ be a partially ordered commutative semigroup and $\mathcal{F}$ be an $\mathfrak{M}$-omissible family of $\mathfrak{M}$-edge-labelled cycles containing all disobedient ones which synchronizes meets. Then the shortest path completion for $\mathcal{M}_{\mathfrak{M}} \cap \operatorname{Forb} \mathcal{F}$ gives a local canonical amalgamation operator. It gives a non-local canonical amalgamation operator if $\mathfrak{M}$ contains a maximum element.*

*Proof.* Put $\mathbf{B}_1 \oplus_{\mathbf{A}} \mathbf{B}_2$ to be the shortest path completion of the free amalgam of $\mathbf{B}_2$ and $\mathbf{B}_2$ over $\mathbf{A}$. (If $\mathbf{A} = \emptyset$ and $\mathfrak{M}$ has a maximum element $m$, then $\inf(\emptyset) = m$ and the shortest path completion works as well.) It is straightforward to check that this satisfies the conditions of Definition 6.11. □

Let $L$ be a set and $\mathcal{C}$ be a class of complete $L$-edge-labelled graphs (or, in other words, $L$ is a binary symmetric relational language and $\mathcal{C}$ is a class of $L$-structures such that every pair of vertices is in exactly one relation). We say that $\mathcal{C}$ is *triangle constrained* if there exists a family $\mathcal{T}$ of $L$-edge-labelled triangles such that $\mathcal{C} = \operatorname{Forb}(\mathcal{T})$, that is, $\mathcal{C}$ is precisely the class of all complete $L$-edge-labelled graphs which contain no triangle from $\mathcal{T}$.

Let $a, b, c \in L$. Then by $\bullet$ we mean the one-vertex structure, by $\mathbf{E}(a)$ we mean the two-vertex structure with one edge with label $a$ and by $\mathbf{T}(a, b, c)$ we mean the triangle with edges labelled $a$, $b$ and $c$.

**Definition 6.13** (Canonical amalgamation to semigroup)**.** Let $L$ be a binary symmetric relational language and $\mathcal{C}$ be a triangle constrained strong amalgamation class of $L$-structures with a canonical amalgamation operator $\oplus$. We define an operation $\oplus\colon L^2 \to L$ by saying that $a \oplus b = c$ if and only if $\mathbf{T}(a, b, c) = \mathbf{E}(a) \oplus_{\bullet} \mathbf{E}(b)$.

For $\mathcal{C} = \mathcal{M}_{\mathfrak{M}} \cap \operatorname{Forb} \mathcal{F}$ and the canonical amalgamation operator given by the shortest path completion, the operation $\oplus$ from Definition 6.13 is precisely the semigroup operation we started with and it is possible to define an order (using only the canonical amalgamation operator) which agrees with the semigroup order "for all practical purposes" (that is, all situations where they do not agree are non-$\mathfrak{M}$-metric or from $\mathcal{F}$). In other words, if the canonical amalgamation operator is given by the shortest path completion, it is possible to essentially reconstruct the partially ordered commutative semigroup.

In general, the $\oplus$ from Definition 6.13 is always a commutative and associative operation, however, it might not be monotone in any order (see for example the bipartite classes in [ABWH$^+$17c]). This motivates the following question:

**Question 4.** Under which circumstances does a triangle constrained strong amalgamation class with a canonical amalgamation operator admit an interpretation as a semigroup-valued metric space? Is it possible to explicitly define the order using the amalgamation operator? How to get the corresponding omissible family $\mathcal{F}$?

As we already mentioned, the answer to this question is not “always” with the finite-diameter bipartite metrically homogeneous graphs being a counterexample (cf. [ABWH+17c]).[1] A computer search suggests that for primitive classes (that is, with no definable equivalence relations) in finite binary symmetric language it might always be possible (something even stronger might hold and that is that one could perhaps drop the assumption on the existence of a canonical amalgamation operator), but we were only able to handle languages with up to five relations, which is at best a weak evidence.

[1]For infinite-diameter we do not get a confined family $\mathcal{F}$, but the other conditions hold.

# 7. Conclusion and open problems

We found a common generalisation of results of Hubička and Nešetřil [HN16] on Ramsey expansions of $S$-metric spaces (with, possibly, certain forbidden families of short odd cycles), results of Hubička, Nešetřil and the author [HKN17] on Ramsey expansions of metric spaces with values from a linearly ordered monoid and Braunfeld's [Bra17] results on Ramsey expansions of $\Lambda$-ultrametric spaces with strong amalgamation. The three mentioned papers include also classes studied earlier by Solecki [Sol05], Nešetřil [Neš07], Vershik [Ver08], Nguyen Van Thé [NVT10], Conant [Con15], Mašulović [Maš17] and others.

The unifying property of all the aforementioned classes of structures is that they admit some form of the shortest path completion. In this paper we tried to extract some properties which ensure that the shortest path completion works.

Cherlin in the appendix of [Che98] gave a list of all 26 classes of complete $\{A, B, C, D\}$-edge-labelled graphs determined by forbidden triangles which form a non-free strong amalgamation class whose Fraïssé limit is *primitive* (no non-trivial definable equivalences). Li [Li18] studied simplicity of the automorphism groups of these classes and in the process found partially ordered commutative semigroups that, as it turns out, can be plugged into our machinery and give Ramsey expansions and prove EPPA for these spaces[1], although this was not Li's goal. This is yet another evidence supporting the importance of studying completions.

It only remains to guide the reader through what has not been done yet.

**Normal subgroups of the automorphism group** As mentioned in Section 6.3, Tent and Ziegler [TZ13b] classified the normal subgroups of the automorphism group of the Urysohn space. Later [TZ13a] they also proved that the automorphism group of the bounded Urysohn space with distances from the closed interval $[0, 1]$ is simple. In both proofs, the properties of the shortest path completion played a key role. It is hence natural to ask for a generalisation of their results for semigroup-valued metric spaces.

**Question 5.** Let $\mathfrak{M}$ and $\mathcal{F}$ be as in Theorem 3.6 and assume that $\mathcal{M}_\mathfrak{M} \cap \operatorname{Forb} \mathcal{F}$ has a Fraïssé limit $\mathbf{M}$. What are the normal subgroups of $\operatorname{Aut}(\mathbf{M})$?

For unbounded semigroups (that is, with no maximum element), $\operatorname{Aut}(\mathbf{M})$ is not simple for the same reason that the automorphism group of the Urysohn space is not simple (see Section 6.3). When $\mathfrak{M}$ is not archimedean, then the automorphisms which only permute balls of certain diameters also form a normal subgroup. But in particular, when the semigroup is finite and archimedean (such as, for example, the magic semigroups), it is quite possible that the automorphism group is simple. We only dare to explicitly conjecture this for the primitive metrically homogeneous graphs.

**Conjecture 1.** The automorphism groups of the the Fraïssé limits of the primitive metrically homogeneous graphs of finite diameter are simple.

[1] It would however be straightforward to find their Ramsey expansions directly using Li's completions and Theorem 2.22.

It is an ongoing work of Evans, Hubička, Li and the author to try to answer Question 5.

While we do have some partial examples, it is not clear to which extent the conditions on the semigroup $\mathfrak{M}$ and family $\mathcal{F}$, which we presented here, are necessary.

**Question 6.** Let $\mathfrak{M}$ be a partially ordered commutative semigroup and $\mathcal{F}$ a family of $\mathfrak{M}$-edge-labelled cycles. What are the necessary and sufficient conditions for $\mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$ admitting a shortest path completion as in this paper? What are the conditions for it having a (precompact) Ramsey expansion? What do these expansions look like?

**Bipartiteness.** There are two extremal variants of the 3-constrained metrically homogeneous graphs which our machinery does not cover, namely the bipartite ones and the antipodal ones. It is reasonable to ask whether one can get similar (or even more general) properties also for the semigroup-valued metric spaces, for example by considering an infinite semigroup and then identifying some distances after doing the shortest path completion.

The question on bipartiteness can be generalised. The author has a computer program which can find all amalgamation classes of complete $\{A, B, C, D, E\}$-edge-labelled graphs which are given by forbidding triangles (that is, a generalisation of [Che98, Appendix]). If we only consider the strong amalgamation ones, it seems that the structure of definable equivalences is not too wild. In particular, it gives rise to the following conjecture and question.

**Conjecture 2.** Let $L$ be a finite set and let $\mathcal{C}$ be a strong amalgamation class of $L$-edge-labelled graphs such that $\mathcal{C} = \mathrm{Forb}(\mathcal{T})$ for $\mathcal{T}$ a family of $L$-edge-labelled triangles. (In other words, $\mathcal{C}$ is a triangle constrained strong amalgamation class with finitely many 2-types.) Then $\mathcal{C}$, when expanded by quotients by 0-definable equivalence relations on vertices, has *weak elimination of imaginaries* (its Fraïssé limit does).

Informally, Conjecture 2 says that for finite symmetric binary relational languages, the "only definable equivalences one needs to understand" are 0-definable equivalences on vertices. There are, of course, other definable equivalences such as the ones with finite equivalence classes or, say, definable equivalences on tuples which are products of equivalences on vertices, or also restrictions of 0-definable equivalences to some $A$-types. These do not seem to be important for the applications which we have in mind.

It is easy to see that the zero-definable equivalences form a lattice (meets correspond to conjunctions, joins are their transitive closure (here we are using the fact that there are only finitely many 2-types). Braunfeld proved [Bra16, Lemma 4.5] that if the equivalences satisfy the *infinite index property*, the lattice is distributive, and found an example of a non-distributive lattice of definable equivalences [Bra18b, Example 7] when there is no infiniteness condition. It is still worth asking if one can say anything more about the lattice in the setting of Conjecture 2.[2]

[2] The author would like to thank Sam Braunfeld for pointing this out to me.

We dare to state the following question because the computer search on classes with five relations does not seem to give an counterexample (although it still needs to be thoroughly checked for mistakes).

**Question 7.** Assume the setting of Conjecture 2 and further assume that all the definable equivalences have the *infinite index property* (every equivalence has infinitely many[3] equivalence classes even when restricted to an equivalence class of a coarser equivalence). Is then $\mathcal{C}$ necessarily a semigroup-valued metric space, that is, is there a partially ordered commutative semigroup $\mathfrak{M}$ and an $\mathfrak{M}$-omissible family $\mathcal{F}$ of $\mathfrak{M}$-edge labelled cycles containing all disobedient ones such that $\mathcal{C} = \mathcal{M}_{\mathfrak{M}} \cap \mathrm{Forb}(\mathcal{F})$? Can one require $\mathcal{F}$ to synchronize meets and to be confined?

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[3]Infinitely many in the potential Fraïssé limit, for the finite structures there should be no bound on the number of the equivalence classes.

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# A. Vocabulary

**$L$-edge-labelled graph**

A graph such that each edge has a label from the set $L$. Can also be understood as the pair $(V, \ell)$, where $V$ is the vertex set and $\ell\colon \binom{V}{2} \to L$ is a partial function. See Definition 3.1.

**Shortest path completion**

Given an $\mathfrak{M}$-edge-labelled graph $\mathbf{G}$ for a partially ordered commutative semigroup $\mathfrak{M}$, the shortest path completion sets each non-edge to be the infimum of the $\mathfrak{M}$-lengths of all paths connecting the two vertices. A family of $\mathfrak{M}$-edge-labelled cycles $\mathcal{F}$ is then introduced to ensure that this completion exists and that it has nice properties. See Definition 3.2.

**Block**

A maximal archimedean subsemigroup of $\mathfrak{M}$, or $\mathbf{0}$. Corresponds to the definable equivalences on the $\mathfrak{M}$-valued metric spaces. See Definition 4.5.

**Omissible family**

A family of $\mathfrak{M}$-edge-labelled cycles $\mathcal{F}$ which behaves nicely with respect to the shortest path completion (that is, the shortest path completion of a graph $\mathbf{G}$ is in $\mathrm{Forb}(\mathcal{F})$ if and only if $\mathbf{G}$ is, furthermore $\mathcal{F}$ contains only non-geodesic $\mathfrak{M}$-metric cycles). See Definition 3.4.

**Disobedient cycles**

A family $\mathcal{F}$ contains all disobedient cycles if, in the graphs from $\mathrm{Forb}(\mathcal{F})$, all infima for the shortest path completion are defined and furthermore distribute with $\oplus$. See Definition 3.3.

**Confined family**

A family $\mathcal{F}$ is confined if it is $S$-locally finite for every finite $S \subseteq \mathfrak{M}$, which means that there are only finitely many $S$-edge-labelled cycles in $\mathcal{F}$. See Definition 4.21.

**Meet synchronization**

A family $\mathcal{F}$ synchronizes meets if, in the situations one encounters in graphs from $\mathrm{Forb}(\mathcal{F})$, the block in which the infimum of some set of distances lies is the meet of the blocks in which the distances lie. See Definition 4.18.

**mus**

An approximation of the (nonexistent) maximum of a block relative to a finite set of distances $S$. Implies that if we have a non-$\mathfrak{M}$-metric cycle with distances from $S$, then only a bounded number of distances from each block are important, the rest of them only represent the block. See Section 4.3.

**Original and ball vertices**

If $\mathbf{A}$ is an $L^\star$ expansion of a $\mathfrak{M}$-valued metric space, it contains two kinds of vertices: The original vertices which are in the distance relations and newly added ball vertices which represent the balls of irreducible diameters. See Definition 4.15.

# B. List of classes

**Forb($\mathcal{F}$)**

If $\mathcal{F}$ is a family of finite $L$-structures, then Forb($\mathcal{F}$) is the class of all finite $L$-structures (or, if it is clear from the context, $L^+ \supset L$-structures) $\mathbf{A}$ such that there is no $\mathbf{F} \in \mathcal{F}$ with a homomorphism $\mathbf{F} \to \mathbf{A}$. See the beginning of Chapter 2.

$\boldsymbol{\mathcal{M}_{\mathfrak{M}}}$

The class of all finite $\mathfrak{M}$-valued metric spaces. See Definition 1.2.

$\boldsymbol{\mathcal{M}_{\mathfrak{M}} \cap}$ **Forb($\mathcal{F}$)**

Expectably, the class of all finite $\mathfrak{M}$-valued metric spaces containing no homomorphic images of members of $\mathcal{F}$.

$\boldsymbol{\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap}$ **Forb($\mathcal{F}$)**

The class of all convexly ordered $\mathfrak{M}$-valued metric spaces omitting homomorphic images from $\mathcal{F}$, see Definition 5.1. They differ from $\mathcal{M}_{\mathfrak{M}}$ by adding a partial orders $\leq^{\mathcal{B}}$ for each non-maximal meet-irreducible block of $\mathfrak{M}$, from these it is possible to define linear orders of balls of every diameter.

$\boldsymbol{\mathcal{M}^{\star}_{\mathfrak{M}} \cap}$ **Forb($\mathcal{F}$)**

See Definition 4.15. This is a class of all $L^{\star}$ lifts of members of $\mathcal{M}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$). This means that we add ball vertices for each block in $I_{\mathfrak{M}}$ and link the original and the ball vertices by several different unary functions, in particular to each original vertex we link the ball vertices representing the balls it lies in. This makes it possible to describe the non-metric cycles using structures of bounded size. Remember that $\mathcal{M}_{\mathfrak{M}} \cap$Forb($\mathcal{F}$) and $\mathcal{M}^{\star}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$) are isomorphic as categories (Proposition 4.17).

$\boldsymbol{\mathcal{H}^{\star}_{\mathfrak{M}} \cap}$ **Forb($\mathcal{F}$)**

The "hereditary closure" of $\mathcal{M}^{\star}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$) (see Definition 4.15). Namely, we allow for the structures in $\mathcal{H}^{\star}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$) to contain ball vertices to which no original vertex is linked. We need this class because Theorem 2.22 requires a hereditary class, but clearly each member of $\mathcal{H}^{\star}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$) can be completed to $\mathcal{M}^{\star}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$).

$\boldsymbol{\mathcal{G}^{\star}_{\mathfrak{M},S} \cap}$ **Forb($\mathcal{F}$)**

A class of **incomplete** $L^{\star}_{\mathfrak{M}}$-structures. All members of $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap$Forb($\mathcal{F}$) have a completion to $\mathcal{M}^{\star}_{\mathfrak{M}} \cap$Forb($\mathcal{F}$), on the other hand $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap$Forb($\mathcal{F}$) contains all free amalgams of structures from $\mathcal{H}^{\star}_{\mathfrak{M}} \cap$Forb($\mathcal{F}$) (with distances from $S$) as well as structures which we need to complete for the Ramsey property and EPPA. This is also the reason why the definition of $\mathcal{G}^{\star}_{\mathfrak{M},S} \cap$ Forb($\mathcal{F}$) (Definition 4.27) is rather technical and very explicit.

$\boldsymbol{\mathcal{M}^{\star;\leq}_{\mathfrak{M}} \cap}$ **Forb($\mathcal{F}$)**

This class is to $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$) what $\mathcal{M}^{\star}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$) is to $\mathcal{M}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$). We added the ball vertices and also a linear order which corresponds to the partial orders $\leq^{\mathcal{B}}$ in $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$). Again, $\overrightarrow{\mathcal{M}}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$) and $\mathcal{M}^{\star;\leq}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$) are isomorphic as categories. See Definition 5.8.

$\boldsymbol{\mathcal{H}^{\star;\leq}_{\mathfrak{M}} \cap}$ **Forb($\mathcal{F}$)**

The "hereditary closure" of $\mathcal{M}^{\star;\leq}_{\mathfrak{M}} \cap$ Forb($\mathcal{F}$), see Definition 5.8.