

Heilbronn's Triangle Problem

An elementary probabilistic lower bound

A proof using geometric probability, the law of total probability, and deletion

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Theorem. For every integer $n \geq 3$, there is a set S of n points in the unit square $U = [0, 1]^2$ such that every triangle with vertices in S has area at least

$$\frac{1}{100n^2}.$$

Equivalently, if $T(S)$ is the smallest area of a triangle determined by three points of S , then $T(S) \geq 1/(100n^2)$.

The plan

First choose three points independently and uniformly in U . We will prove that the probability of obtaining a triangle of area at most ε is *less than* 60ε . Then we will choose $2n$ random points and remove one point from each triangle with small area. The key trick is to divide the possible distances between two points into dyadic intervals.

1. A geometric estimate for one triangle

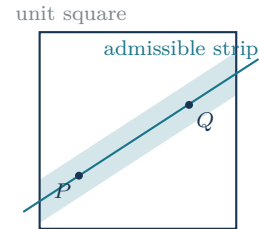
Let P, Q, R be independent uniform points in U , and put

$$E = \{\text{area}(PQR) \leq \varepsilon\}, \quad b = |P - Q|.$$

Fix two distinct points P, Q at distance $b > 0$, and consider only the random choice of R . Since the area of a triangle is half its base times its altitude, E can occur only if the distance from R to the line PQ is at most $2\varepsilon/b$.

Thus R must lie in a strip of width $4\varepsilon/b$ around the line PQ . The intersection of this strip with the unit square is contained in a (possibly rotated) rectangle of length at most $\sqrt{2}$ and width $4\varepsilon/b$. As the area of U is 1,

$$\mathbb{P}_R(E) \leq \frac{4\sqrt{2}\varepsilon}{b} \quad (\text{for fixed } P, Q). \quad (1)$$



The bound in (1) is good when b is large, but gets worse as b approaches zero. Fortunately, two random points are unlikely to be very close.

2. Partition according to the distance $|P - Q|$

Consider the following events:

$$A_0 = \{|P - Q| \geq 1\}, \quad A_k = \{2^{-k} \leq |P - Q| < 2^{1-k}\} \quad (k \geq 1).$$

These events partition all possibilities except $P = Q$, which has probability zero. By the **law of total probability**,

$$\mathbb{P}(E) = \sum_{k=0}^{\infty} \mathbb{P}(A_k) \mathbb{P}(E | A_k). \quad (2)$$

The case $k = 0$. On A_0 we have $b \geq 1$. Thus (1) gives

$$\mathbb{P}(A_0)\mathbb{P}(E | A_0) \leq 4\sqrt{2}\varepsilon. \quad (3)$$

The cases $k \geq 1$. There are two competing estimates:

- *How likely is this distance?* For fixed P , the point Q must lie in an annulus centered at P with inner radius 2^{-k} and outer radius 2^{1-k} . Its area is

$$\pi((2^{1-k})^2 - (2^{-k})^2) = 3\pi 4^{-k},$$

$$\text{so } \mathbb{P}(A_k) \leq 3\pi 4^{-k}.$$

- *How likely is a small triangle at this distance?* On A_k we have $b \geq 2^{-k}$, so (1) gives

$$\mathbb{P}(E \mid A_k) \leq 4\sqrt{2}\varepsilon 2^k.$$

Multiplying, the contribution of the k th distance scale is bounded by

$$\mathbb{P}(A_k)\mathbb{P}(E \mid A_k) \leq 12\sqrt{2}\pi \varepsilon 2^{-k}. \quad (4)$$

Combining (2)–(4) and using the geometric series $\sum_{k=1}^{\infty} 2^{-k} = 1$, we get

$$\begin{aligned} \mathbb{P}(E) &\leq 4\sqrt{2}\varepsilon + \sum_{k=1}^{\infty} 12\sqrt{2}\pi \varepsilon 2^{-k} \\ &= 4\sqrt{2}(1 + 3\pi)\varepsilon < \boxed{60\varepsilon}. \end{aligned} \quad (5)$$

3. From three random points to n well-spaced points

Choose $2n$ points $X_1, \dots, X_{2n}$ independently and uniformly in U . Call a triple of these points *bad* if its triangle has area at most

$$\varepsilon = \frac{1}{100n^2}.$$

Let B be the number of bad triples. By **linearity of expectation** and (5),

$$\begin{aligned} \mathbb{E}[B] &= \sum_{1 \leq i < j < k \leq 2n} \mathbb{P}(\text{area}(X_i X_j X_k) \leq \varepsilon) \\ &< \binom{2n}{3} \cdot \frac{60}{100n^2} < \frac{4}{5}n < n. \end{aligned}$$

Consequently, there is a choice of the $2n$ points with *fewer than* n bad triples. (The points are distinct with probability one.) For each bad triple, remove one of its vertices, unless a vertex of that triple was already removed. This deletes fewer than n points, leaving at least n points and no bad triples. Keep any n of the remaining points.

Every triangle among these n points has area greater than $1/(100n^2)$, proving the theorem. $\square$

Key idea. At distance scale 2^{-k} , the probability of landing in that scale is $O(4^{-k})$, while the conditional probability of a small triangle is $O(\varepsilon 2^k)$. Their product is $O(\varepsilon 2^{-k})$, which can be summed using only a geometric series. No probability densities or integration are needed.