Extending partial automorphisms

Matěj Konečný

TU Dresden

Algebra Colloquium 2024

David Bradley-Williams, Peter J. Cameron, Jan Hubička, MK: EPPA numbers of graphs, Journal of Combinatorial Theory, Series B, Volume 170, 2025,

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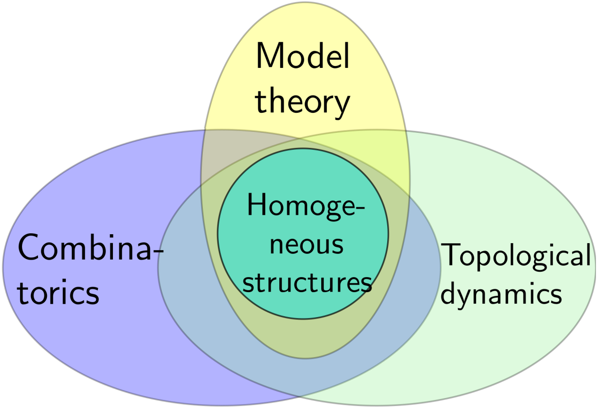
Model
theory
Combina-
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Homoge-
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structures
Topological
dynamics

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Definition

A structure **G** is homogeneous if every partial automorphism of **G** with finite domain extends to an automorphism of **G**.

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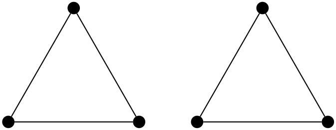

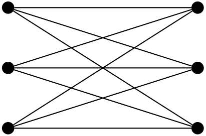

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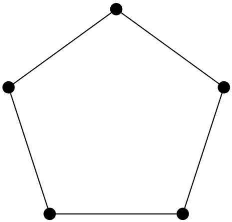

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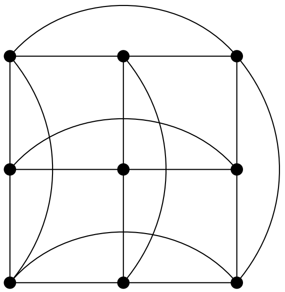

Definition (EPPA, extension property for partial automorphisms)

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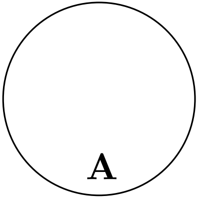

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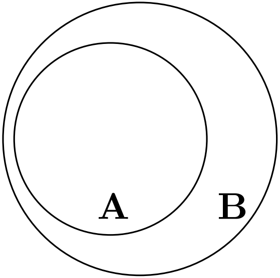

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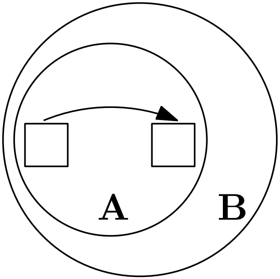

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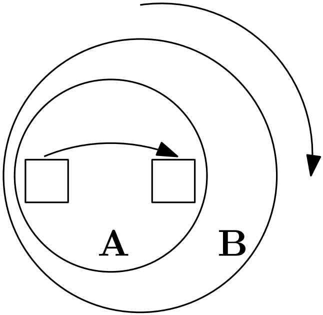

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2. Easy but nice combinatorics
3. Trivial observations
4. Many open problems

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A connection to model theory

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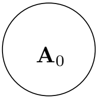

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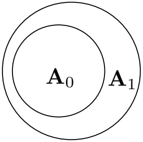

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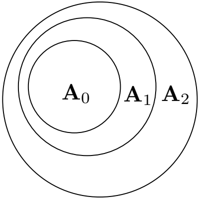

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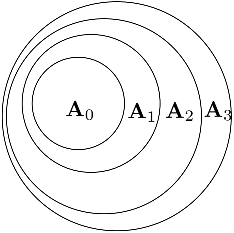

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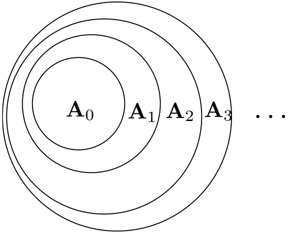

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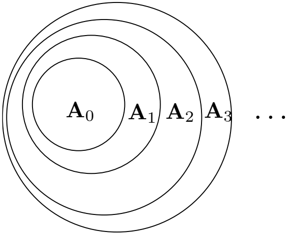

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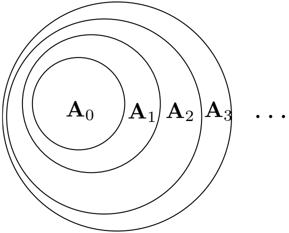

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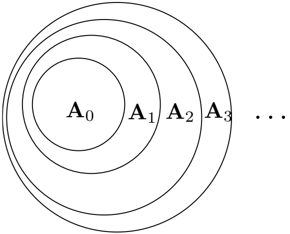

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Theorem [Kechris, Rosendal, 2007]: *If* $\mathbf{M}$ *is countable and homogeneous then* $\mathrm{Age}(\mathbf{M})$ *has EPPA if and only if* $\mathrm{Aut}(\mathbf{M})$ *can be written as the closure of a chain of compact subgroups.*

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Theorem [Hodges, Hodkinson, Lascar, Shelah, 1993 + Kechris, Rosendal, 2007] If $\mathbf{M}$ is countable and homogeneous and $\mathrm{Age}(\mathbf{M})$ has EPPA + some reasonable properties then every subgroup of $\mathrm{Aut}(\mathbf{M})$ of index $< 2^\omega$ is open.

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Problem (Hrushovski, 1992)

Improve the bounds.

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For every $\mathbf{G}$ *with* n *vertices,* m *edges and maximum degree* Δ *we have that* $\mathrm{eppa}(\mathbf{G}) \leq \binom{\Delta n - m}{\Delta} \in 2^{\mathcal{O}(n \log n)}$.
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Independently proved also by Andréka and Németi.

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If the maximum degree of $\mathbf{G}$ *is* Δ*, then it has an EPPA-witness on at most* $\binom{\Delta n}{\Delta}$ *vertices.*

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Proof.

1. Let $\mathbf{G} = (V, E)$ be a graph. Assume that $\mathbf{G}$ is Δ-regular.
2. Define $\mathbf{H}$ so that $V(\mathbf{H}) = \binom{E}{\Delta}$ and $XY \in E(\mathbf{H})$ if $X \cap Y \neq \emptyset$.
3. Embed $\psi \colon \mathbf{G} \to \mathbf{H}$ sending $v \mapsto \{e \in E : v \in e\}$.
4. A partial automorphism of $\mathbf{G}$ gives a partial permutation of E.
5. Extend it to a permutation of E respecting the partial automorphism.
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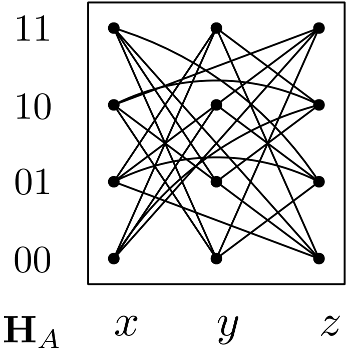
$$\mathrm{Sym}(E) \curvearrowright \binom{E}{\Delta}$$

An upper bound [Evans, Hubička, K, Nešetřil, 2021]

Given set A, define graph $\mathbf{H}_A$.
$H_A = \{(x,f) : x \in A, f\colon A \setminus \{x\} \to \{0,1\}\}$.
$\{(x,f),(y,g)\} \in E \iff x \neq y$ and $f(y) \neq g(x)$.

1. For a permutation $\pi\colon A \to A$ define $\alpha_\pi\colon H_n \to H_n$ by $\alpha_\pi((x,f)) = (\pi(x), g)$, where $g(y) = f(\pi^{-1}(y))$.
2. $\alpha_\pi \in \mathrm{Aut}(\mathbf{H}_A)$.
3. For $x \neq y \in A$ define α_{xy} by $\alpha_{xy}((z,f)) = (z,g)$ where $g(w) = 1 - f(w)$ if $\{x,y\} = \{z,w\}$ and $g(w) = f(w)$ otherwise.
4. $\alpha_{xy} \in \mathrm{Aut}(\mathbf{H}_A)$.



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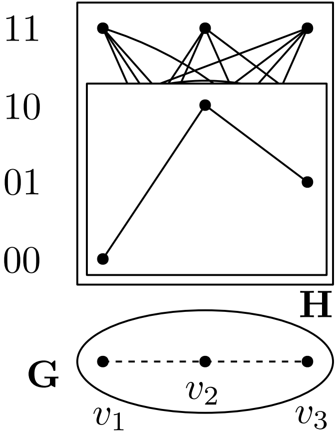
5. Fix a graph $\mathbf{G}$ and consider $\mathbf{H}_G$.
6. Embed $\mathbf{G}$ to $\mathbf{H}_G$ vertex-by-vertex, preserving projections.
7. Pick a partial automorphism f of $\mathbf{G}$, project it to G, and extend it to a permutation π of G.
8. Consider α_π. There is a canonical choice of $\alpha_{x_i y_i}$'s such that $\alpha_\pi \circ \alpha_{x_1 y_1} \circ \cdots \circ \alpha_{x_k y_k}$ extends f.



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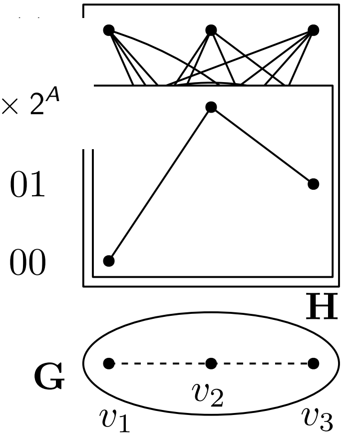

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2. Complements of Kneser graphs ($\mathcal{O}(n^\Delta)$ for constant Δ).
3. Valuation graphs ($n2^{n-1}$).

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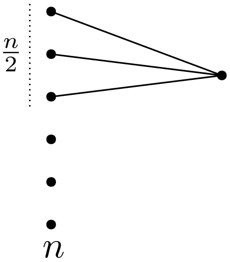

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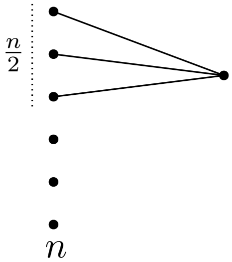

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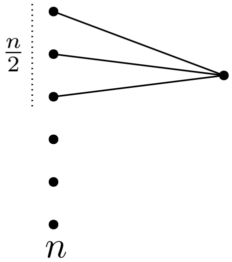

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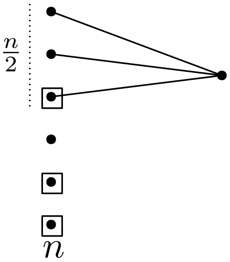

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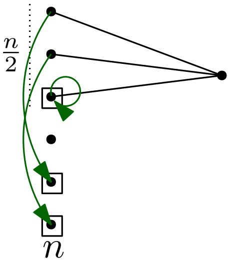

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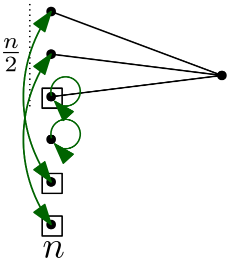

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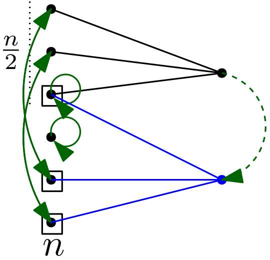

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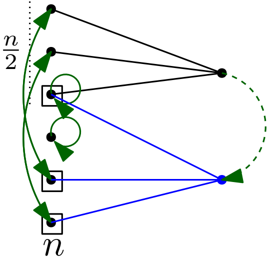

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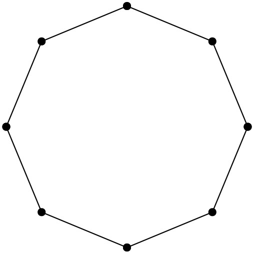

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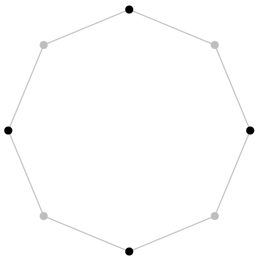

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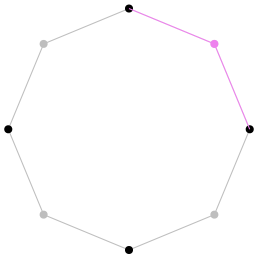

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$\mathrm{eppa}(G(n, 1/2))$ *is superpolynomial.*

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(Note that there are only $2^{O(n \log n)}$ partial automorphisms of any n-vertex structure.)

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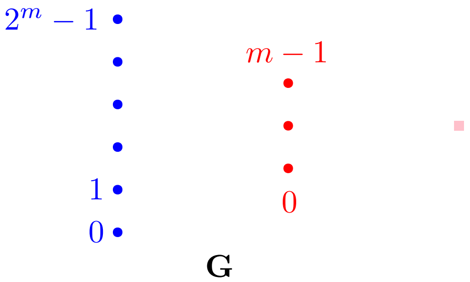

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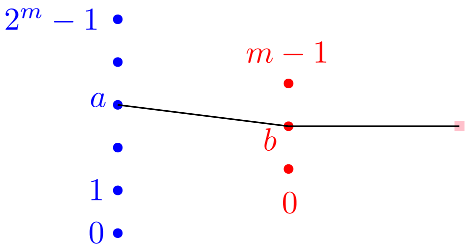

G

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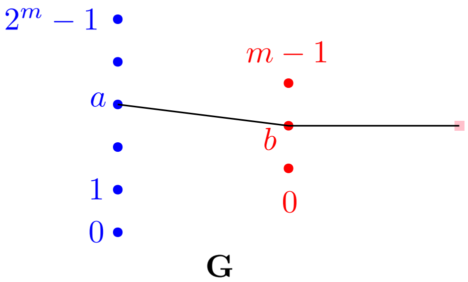

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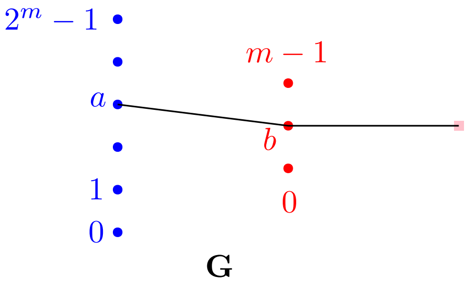

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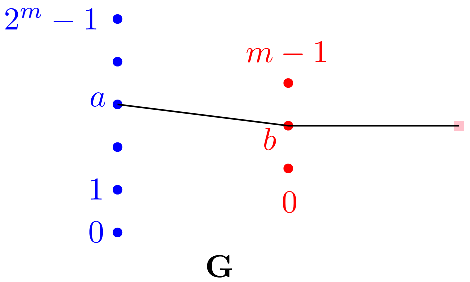

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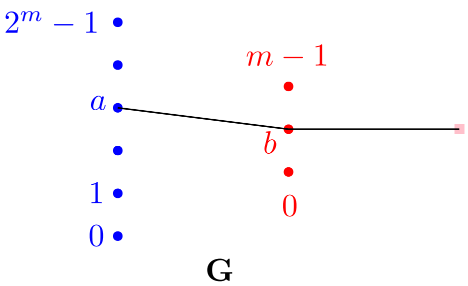

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- Consequently, $|H| \geq (2^m)! \in 2^{\Omega(n \log n)}$.