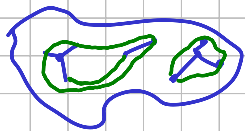
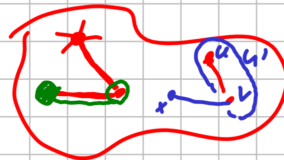
contractive
# Borůvka's alg.



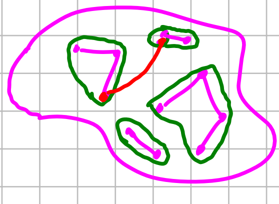
keep every tree contracted to a vertex

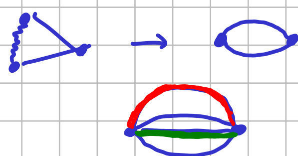



every edge keeps its id

1. $T \leftarrow \emptyset$
2. while $n > 1$:
3. Each vertex selects the lightest incident edge $\to L$. } $O(m)$
4. $T \leftarrow T \cup L$ } $O(n)$
5. Contract all edges in $L$. } $O(m)$
6. Filter out loops & parallel edges. } $O(m)$ — by bucket sorting
7. Return $T$.

$O(m_i)$ per step



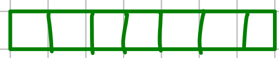


$G_i$ := graph at the end of phase $i$
(so $G_0$ is the input graph)

$n_i, m_i$

$\{u,v\} \to (\min(u,v), \max(u,v), id)$ — keeping vertices numbered from $1 \dots n$

Bucket Sort
2 passes, $n$ buckets
lex order



$n$ buckets

$m_i \le m$ $\qquad n_{i+1} \le n_i/2$, $\; n_i \le n/2^i$ } #steps $\le \log n$

$O(\underbrace{\#passes}_{2}(\underbrace{\#items}_{m} + \underbrace{\#buckets}_{n}))$

(1) $O(m \cdot \log n)$

(2) total time $O(\sum_i m_i) \le O(\sum_i \frac{n^2}{4^i}) \le O(n^2)$.
all $G_i$'s are simple $\Rightarrow m_i \le n_i^2 \le (\frac{n}{2^i})^2 = \frac{n^2}{4^i}$

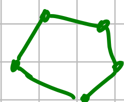
(3) On a planar graph: all $G_i$'s are planar,
so $m_i \le 3n_i \le 3n/2^i$
$\sum_i m_i \le 3n \cdot \sum_i 1/2^i \in O(n)$.

(4) For minor-closed class $\mathcal{C}$: $G_i \preccurlyeq G$, so $G_i \in \mathcal{C}$.
& non-trivial
$m_i \le \varrho \cdot n_i \Rightarrow \sum_i m_i \le n \cdot \varrho \cdot \sum_i 1/2^i \in O(n)$

**Df:** Edge density $\varrho(G) := m(G)/n(G)$
$\varrho(\mathcal{C}) := \sup_{G \in \mathcal{C}} \varrho(G)$ — for planar: $\varrho(\mathcal{C}) = 3$

**Thm:** For any non-trivial minor-closed class $\mathcal{C}$ of graphs: $\varrho(\mathcal{C})$ is finite.

**Df:** $H \preccurlyeq G$ ($H$ is a minor of $G$) $\equiv$
$H$ can be obtained from $G$ by v/e deletions & e contractions.

e.g. $\Delta \preccurlyeq$ 

**Df:** A class $\mathcal{C}$ of graphs is minor-closed $\equiv$
$\forall G \in \mathcal{C}, \forall H \preccurlyeq G : H \in \mathcal{C}$.

examples:
- empty } trivial
- all graphs } trivial
- planar
- drawn on a surface
- bounded tree width
- forests

**Df:** $\mathrm{Forb}(\mathcal{H})$ := class of all graphs $G$ s.t. $\nexists H \preccurlyeq G, H' \in \mathcal{H} : H \cong H'$.
($\mathcal{H}$: graph class)

e.g. $\mathrm{Forb}(\{K_2\})$ ... graphs with no edges
$\mathrm{Forb}(K_3)$ .... forests
$\mathrm{Forb}(K_5, K_{3,3})$ ... planar graphs
(Kuratowski thm.)

$\mathrm{Forb}(\mathcal{H})$ is minor-closed
if $G \in \mathrm{Forb}(\mathcal{H}), G' \preccurlyeq G \Rightarrow G' \in \mathrm{Forb}(\mathcal{H})$

if $G' \notin \mathrm{Forb}(\mathcal{H})$
then $H \preccurlyeq G'$ $H \in \mathcal{H}$
but $H \preccurlyeq G$, so $G \notin \mathrm{Forb}(\mathcal{H})$

**Thm:** For every MCC $\mathcal{C}$
$\exists \mathcal{H} : \mathcal{C} = \mathrm{Forb}(\mathcal{H})$.

↳ Robertson & Seymour:
$\exists \mathcal{H}$ finite

we can take $\mathcal{H} := \overline{\mathcal{C}}$

if $\mathcal{C}' \subseteq \mathcal{C}$, then $\varrho(\mathcal{C}') \le \varrho(\mathcal{C})$

take any $H \in \mathcal{H}$ $\qquad k := n(H)$

$\mathcal{C} = \mathrm{Forb}(\mathcal{H}) \subseteq \mathrm{Forb}(H) \subseteq \mathrm{Forb}(K_k)$

$\varrho(\mathcal{C}) = \varrho(\mathrm{Forb}(\mathcal{H})) \le \varrho(\mathrm{Forb}(H)) \le \varrho(\mathrm{Forb}(K_k))$

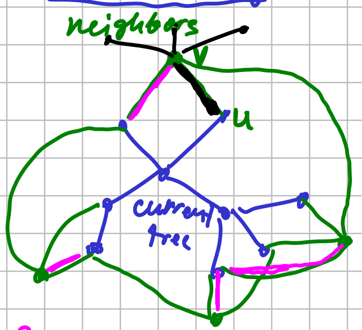
**Thm (Mader):** $\forall k \; \exists h$ ($2^{k^2}$) $\forall G$
if $\mathrm{avg\,deg}(G) \ge h$,
then $G$ contains a sub-division of $K_k$.

(see Diestel: Graph Theory)

$\varrho(\mathrm{Forb}(K_k))$ is finite

• Kostochka & Thomason: if $\varrho \in \Omega(k \cdot \sqrt{\log k})$, then $K_k \preccurlyeq G$

# Jarník's alg – improved version (Dijkstra)




for each neighbor: the lightest edge going to the tree ] value of neighbor

keep values in a heap

in every step:

1. select neighbor $v$ with the smallest value → edge $uv$, $u \in T$, $v \notin T$ ] $n\times$ Extract Min
2. Add $uv$ to $T$
3. Update values:
   - new neighbors ] $n\times$ Insert
   - change active edges of existing neighbors ] Decrease $m\times$

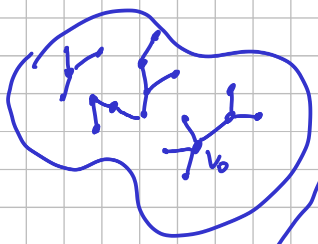
Fibonacci heap: $O(m + n\log n)$ ← $O(m)$ if $\varrho \geq \log n$

**Idea:** identify $F \subseteq MST$, add $F$ to current tree, contract $F$

**Toy example**

- $O(\log\log n)$ Borůvka steps ↓ $O(m\log\log n)$ time
  $G'$: $m' \leq m$, $n' \leq n/2^{\log\log n} = n/\log n$
- Jarník/Dijkstra $O(\underbrace{m'}_{m} + \underbrace{n' \cdot \log n'}_{n}) = O(m)$

# Fredman–Tarjan alg.




$t$: limit on heap size

$t := 2^{\lceil 2m_0/n \rceil}$ (orig. $m_0$, current $n$)

↓ Jarník/Dijkstra

forest of blue trees $F$

Stop if:
- heap became empty
- $|\text{heap}| > t$
- join with an existing tree

↓

add $F$ to $T$ & contract $F$

phase

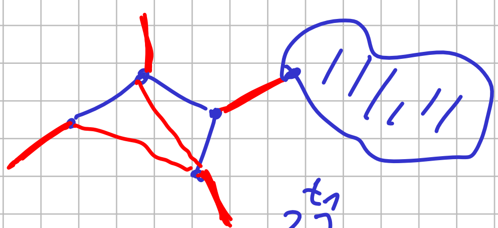
① one phase runs in $O(m_0)$ time

$O(m_i + n_i \cdot \underbrace{\log t}_{\frac{2m_0}{n_i}})$

$O(\underbrace{m_i}_{\leq m_0} + \cancel{n_i} \cdot \frac{2m_0}{\cancel{n_i}}) = O(m_0)$

② If a phase is not final, every tree of $F_i$ is incident with at least $t_i$ edges.

$G_i :=$ graph at the start of phase $i$



③ If not final...

$n_{i+1} = \#\text{trees of } F_i \leq \frac{2m_0}{t_i}$

④ $t_{i+1} = 2^{\lceil 2m_0/n_{i+1} \rceil} \geq 2^{2m_0/n_{i+1}} \geq 2^{2m_0 / \frac{2m_0}{t_i}} = 2^{t_i}$

So: $t_i \geq 2^{2^{2^{\cdot^{\cdot^{2^{t_1}}}}}}$ } if $i \geq \log^* n$

if $t_i \geq n$, the alg. stops.

So #phases $\leq \log^* n$ ⟶ whole alg. runs in $O(m \cdot \log^* n)$ time.

But $t_1 \sim m/n$ #phases ~ min. $i$: $2^{2^{\cdot^{\cdot^{2^{m/n}}}}} \geq n$

if $\varrho \geq \log^{(k)} n$, then #phases $\in O(k)$ ⟹ time $O(k \cdot m)$.

Open: $O(m)$ for all graphs? But we have: $O(m)$ expected time, $O(m)$ w.c. time for integers | optimal alg.