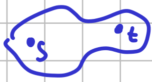
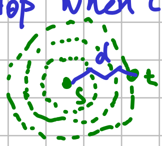
SSSP



PtP SP

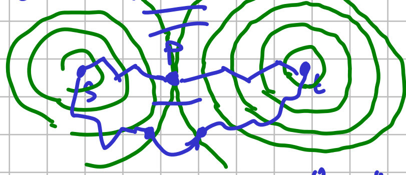
modified Dijkstra:

- stop when closing t



$\sim d^2$ vertices

- bi-directional D.



$\sim d^2$ vertices

Analysis of A*

recall: potential $\varphi : V \to \mathbb{R}$

$\ell_\varphi(uv) := \ell(uv) + \varphi(u) - \varphi(v)$

want: $\ell_\varphi(e) \geq 0$

A* with heuristics ψ

$\Leftrightarrow$ D. on graph reduced by potential $-\psi$

Theorem: If we run A* on G from s and D. on $G_{-\psi}$ from s, then in every step we have:

$\forall v \quad h(v) = h^*(v) - \psi(s) + \psi(v)$.

assuming ties are broken in the same way

Proof: A* minimizes $h^*(v) + \psi(v)$ } $\to$ same v

D. min. $h(v)$

at the beginning: $h^*(s) = 0$, $h^*(*) = +\infty$

$0 = h(s) = 0 - \psi(s) + \psi(s) = 0$

$h(*) = \infty$ ✓

In every step:

- both algs choose the same v

- A* algorithm: heuristic function $\psi : V \to \mathbb{R}$, lower bound on d. to target, $\psi(v) \leq d(v,t)$ } problem-specific

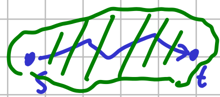
e.g. $\psi(v) := \varrho_{\text{euclid}}(v,t)$

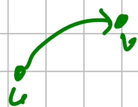
close vertex with min $h(v) + \psi(v)$

want: $\ell(uv) - \psi(u) + \psi(v) \geq 0$

$\underbrace{\ell(uv)}_{\geq d(u,v)} - d(u,t) + d(v,t) \geq 0$

Δ ineq. ✓





A*: set $h^*(v)$ to $h^*(u) + \ell(uv)$

D: set $h(v)$ to $h(u) + \ell_{-\psi}(uv)$

$h^*(u) - \psi(s) + \cancel{\psi(u)} +$

$\ell(uv) - \cancel{\psi(u)} + \psi(v)$

$h^*(v) - \psi(s) + \psi(v)$

new

All-Pairs Shortest Paths

- given L (matrix of edge lengths), compute D (matrix of distances) } APSP

- given A (adjacency matrix), compute $R = A^*$ (reachability matrix) } transitive closure (directed)

$A^*_{ij} = \begin{cases} 0 \\ 1 & \text{iff } \exists \text{ path } i \to j \end{cases}$

Repeated SSSP

A*: $n \times$ BFS $\quad O(nm)$

D: $n \times$ Dijkstra $\quad O(n(m + n\log n)) = O(nm + n^2 \log n)$

Best for sparse graphs

For dense ($m \sim n^2$): $O(n^3)$

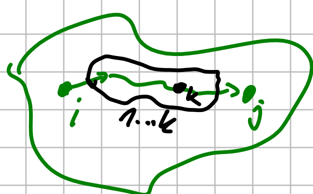
Floyd-Warshall alg.

Df: $D^k_{ij} :=$ min. length of a ij-walk with internal vertices in $\{1 \ldots k\}$

$D^0_{ij} = \ell(ij) \quad D^0_{ii} = 0 \quad \ldots \quad D^0 = L$ (with 0 at diagonal)

$D^n_{ij} = D$

Induction on k: $D^{k-1} \to D^k \quad O(n^2)$



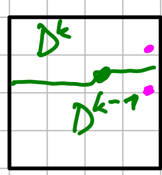
$$D^k_{ij} = \min \begin{cases} D^{k-1}_{ij} \\ D^{k-1}_{ik} + D^{k-1}_{kj} \end{cases}$$

$O(1)$ time

$D^0 \to D^1 \to D^2 \to \ldots \to D^n \quad O(n^3)$

reduce memory to $O(n^2)$

$D_{ij} \leftarrow \min(D_{ij}, D_{ik} + D_{kj})$



$D^k_{ik} = D^{k-1}_{ik}$

$D^k_{kj} = D^{k-1}_{kj}$

do everything in-place

F-W

```
for k = 1 to n
  for i = 1 to n
    for j = 1 to n
      Dij ← min(Dij, Dik + Dkj)
```

Walk Algebra

Df: uv-bundle := set of uv-walks

$\emptyset$ empty bundle

e_{uv} $\{uv\}$ for $uv \in E$, uv-bundle

ε_u single-vertex walk, uu-bundle

elementary walks

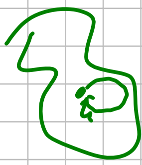
$x \cup y$ uv-bundle, uv-bundle $\to$ uv-bundle
set union

$\underset{uv}{x} \cdot \underset{vw}{y}$ uw-bundle — set of all concatenations of walk from x with walk from y

$uu \to x^*$ uu-bundle

$x^* := \varepsilon_u \cup x \cup x \cdot x \cup \underbrace{x \cdot x \cdot x}_{x^3} \cup \dots$

iteration



similar to regular expression

Distances

$$f(x) := \min_{p \in x} \ell(p)$$

x: uv-bundle

$f(\emptyset) = +\infty$

$f(e_{uv}) = \ell(uv)$

$f(\varepsilon_u) = 0$

$f(x \cup y) = \min(f(x), f(y))$

$f(x \cdot y) = f(x) + f(y)$

$f(x^*) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -\infty & \text{if } f(x) < 0 \end{cases}$

Widest path

$f(\text{path}) := \min_{e \in \text{path}} \text{width}(e)$

$f(\text{bundle}) := \max_{\text{walk} \in \text{bundle}} f(\text{walk})$

$f(\emptyset) = -\infty$

$f(e_{uv}) = \text{width}(uv)$

$f(\varepsilon_u) = +\infty$

$f(x \cup y) = \max(f(x), f(y))$

$f(x \cdot y) = \min(f(x), f(y))$

$f(x^*) = +\infty$

Generalized F-W

W^k_{ij} := bundle of all walks from i to j using 1–k

$W^0_{ij} := e_{ij}$ for $i \neq j$ if $ij \in E$, otherwise $W^0_{ij} := \emptyset$

$W^0_{ii} := \varepsilon_i \underset{\text{if exists}}{\cup e_{ii}}$

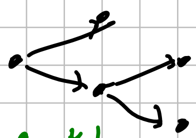
W^n … output

Single step: $W^{k-1} \to W^k$

$$W^k_{ij} = W^{k-1}_{ij} \cup W^{k-1}_{ik} \cdot (W^{k-1}_{kk})^* \cdot W^{k-1}_{kj}$$

Representation: walk expressions (exponentially large)

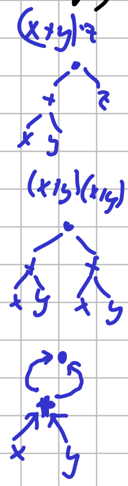
expression DAG of size $O(n^3)$



$(\emptyset, e_{uv}, \varepsilon_u, \cup, \cdot, *)$

$\downarrow$ hom.

$(f(\emptyset), f(e_{uv}), f(\varepsilon_u), \oplus, \odot, \circledast)$



$O(n^3)$ time