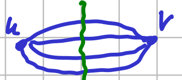
$$\boxed{\begin{array}{l} \Pr[\text{min cut survives}] \geq \dfrac{\ell(\ell-1)}{n(n-1)} \\ \text{Contract}(G,\ell) \text{ runs in } O(n^2) \end{array}}$$

Min. cut C survives
- the 1st step $\frac{n-2}{n} = 1 - \frac{2}{n}$
- the final step $\frac{\ell-1}{\ell+1} = \frac{1}{3}$

$$\ell = \lceil n/\sqrt{2} + 1 \rceil$$

$$\Pr[C \text{ survives}] \geq \frac{(n/\sqrt{2}+1)\cdot \cancel{n/\sqrt{2}}}{\cancel{\sqrt{2}}\,\cancel{n}(n-1)} = \frac{n+\sqrt{2}}{2(n-1)} \geq \frac{n}{2n} = \frac{1}{2}.$$

Silly attempt: $\ell := 2$



$$\Pr[\text{this cut is min}] \geq \frac{2}{n(n-1)} \geq \frac{c}{n^2} \quad \text{for almost all } n$$

$\hookrightarrow$ run the alg. k times, use min of found cuts

$$\Pr\left[\begin{array}{l}\text{min}\\ \text{is wrong}\end{array}\right] \leq \left(1 - \frac{c}{n^2}\right)^k \leq e^{-\frac{ck}{n^2}}$$

$\left(1 - \frac{c}{n^2}\right) \leq e^{-c/n^2}$

$e^x \geq 1 + x$

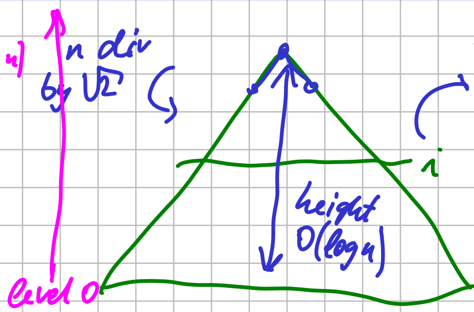
$e^{-x} \geq 1 - x$

- $\to k \approx n^2$: $\Pr \leq \text{const}$
- $\to k \approx n^3$: $\Pr \leq e^{-c' \cdot n}$
- $\to k \approx n^2 \log n$: $\Pr \leq e^{-c' \cdot \log n} = \frac{1}{\text{poly}(n)}$ — with high probability of success

for $k \approx n^2 \log n$: runs in time $O(n^2 \cdot n^2 \log n) = O(n^4 \log n)$ ☹

Karger-Stein alg.

Min Cut (G):
1. If $n \leq 7$: use brute force
2. $\ell \leftarrow \lceil n/\sqrt{2} + 1 \rceil$
3. $C_1 \leftarrow \text{MinCut}(\text{Contract}(G,\ell))$
4. $C_2 \leftarrow$ ——— " ———
5. Return $\min(C_1, C_2)$.



2^i problems of size $n/2^{i/2}$

time per problem: $O(n^2/2^i)$

time per level: $O(n^2)$

$\downarrow$

total time: $O(n^2 \cdot \log n)$

$P_i := \Pr[\text{we find min cut at level } i]$

$P_0 = 1$

$$P_i \geq 1 - \left(1 - \tfrac{1}{2} P_{i-1}\right)^2$$

$g_0 = 1$

$$g_i = 1 - \big(\underbrace{1 - \tfrac{1}{2} g_{i-1}}\big)^2$$

$1 - g_{i-1} + \frac{1}{4} g_{i-1}^2$

$P_i \geq g_i$

$$P_i \in \Omega\left(\frac{1}{\log n}\right) \geq \frac{c}{\log n}$$

$$g_i = g_{i-1} - \frac{g_{i-1}^2}{4}$$

$z_i = \frac{4}{g_i} - 1 \quad \dots \quad g_i = \frac{4}{z_i + 1}$

$z_0 = 3$

$$\frac{\cancel{4}}{z_i+1} = \underbrace{\frac{\cancel{4}}{z_{i-1}+1} - \left(\frac{4}{z_{i-1}+1}\right)^2\Big/4}_{} \qquad \frac{\cancel{4}}{(z_{i-1}+1)^2}$$

$$\frac{z_{i-1} \cancel{+1} \cancel{-1}}{(z_{i-1}+1)^2} \qquad z_{i-1}^2 + 2z_{i-1} + 1$$

$$\frac{1}{z_i+1} = \frac{z_{i-1}}{z_{i-1}^2 + 2z_{i-1} + 1}$$

$$z_i + 1 = \frac{z_{i-1}^2 + 2z_{i-1} + 1}{z_{i-1}} \qquad z_{i-1} + 2 + \frac{1}{z_{i-1}}$$

$$z_i \cancel{+1} = z_{i-1} + \cancel{2} + \underbrace{\frac{1}{z_{i-1}}}_{\leq 1} \qquad z_i \geq 1$$

$z_i \leq z_{i-1} + 2$

$z_i \leq 3 + 2i \quad \to \quad g_i \geq \frac{4}{4 + 2i}$

Iterate MinCut k times:

time $O(n^2 \log n \cdot k)$

$$\Pr[\text{fail}] \leq \left(1 - \frac{c}{\log n}\right)^k \approx e^{-\frac{ck}{\log n}}$$

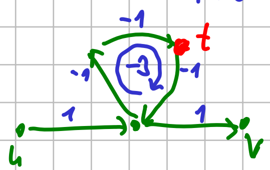
- for $k \sim \log n$: $\Pr[\text{fail}] \leq \text{const}$
- for $k \sim \log^2 n$: $\Pr[\text{fail}] \leq \frac{1}{\text{poly}(n)}$ — time $O(n^2 \cdot \log^3 n)$
- for $k \sim n \log n$: $\Pr[\text{fail}] \leq \frac{1}{\exp(n)}$

Shortest Paths

G ... directed graph
$\ell : E \to \mathbb{R}$ ... length of edges
$d : V^2 \to \mathbb{R}$ ... distance

$d(u,v) :=$ min. length of u,v-path
$+\infty$ if no path exists

(simple) path
walk



simple: $d(u,v) = 2$
walk: $d(u,v) = -\infty$
but: $d(u,t) = -1$
$d(t,v) = 1$
$\underbrace{d(u,t) + d(t,v)}_{0} \geq d(u,v)$
$-1 \quad 1 \quad\quad 1$

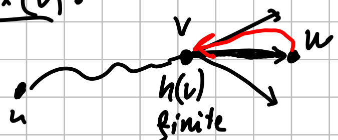
For G with neg. cycles & simple path: shortest path is NP-hard!

Kind of SP problems

- P2PSP (point-to-point): given u,v — $d(u,v)$ path
- ⊙ SSSP (single-source): given u, all v — $d(u,-)$ shortest path tree $O(n)$
- ⊙ APSP (all-pairs): all u,v — distance matrix $d(-,-)$; collection of SP trees for all sources $O(n^2)$

Relaxation Scheme (SSSP, u = source)

- maintaining values $h(v)$ of vertices
  idea: $h(v) = \begin{cases} +\infty \\ \text{length of some } u,v\text{-walk} \end{cases}$
- relax(v):



For each $vw \in E$:
$h(w) \leftarrow \min(h(w), h(v) + \ell(vw))$

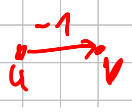
- state(v): $\begin{cases} \text{unseen} & \text{not reached yet, } h(v) = +\infty \\ \text{open} & \text{reached, need to relax} \\ \text{closed} & \text{reached, no need to relax} \end{cases}$

Thm: 0. $h(-)$ never increases
1. $h(v)$ is a length of some u,v-walk $(h(v) \geq d(u,v))$
2. NNC $\Rightarrow$ $h(v)$ is ... of some u,v-path

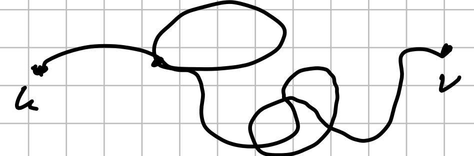
if $h(v)$ is finite

NNC: no negative cycles

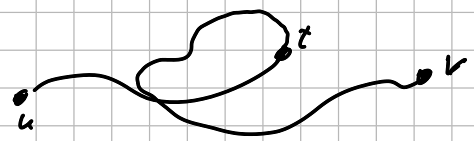
3. NNC $\Rightarrow$ The alg. always stops.
4. STOP $\Rightarrow$ state(v) = closed $\Leftrightarrow$ v reachable from u
5. STOP $\Rightarrow$ v reachable $\Leftrightarrow$ $h(v)$ is finite
6. STOP $\Rightarrow$ $h(v) = d(u,v)$



For G with no negative cycles: $\geq$ one of shortest walks is a path.

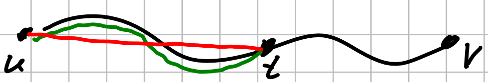


$d(u,v) \leq d(u,t) + d(t,v)$

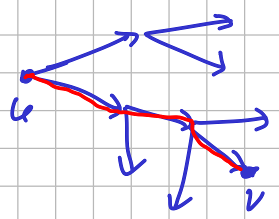


Prefix property

A prefix of a shortest path is also a shortest path.



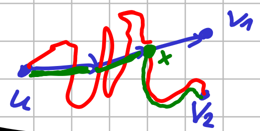
Df: Shortest Path Tree from $u \in V$:



oriented tree T_u from u outwards
$\forall v$ the path $u \to v$ in T_u is a shortest u,v-path in G.

Lemma: SP tree exists

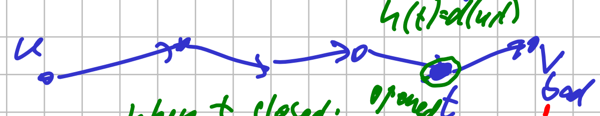
Proof: Build iteratively:



until all vertices are covered

1. $h(*) \leftarrow +\infty$, $h(u) \leftarrow 0$
2. state($*$) $\leftarrow$ unseen,
   State(u) $\leftarrow$ open
3. While $\exists v$: state(v) = open:
4. state(v) $\leftarrow$ closed
5. relax v
6. If $h(w)$ got changed,
   state(w) $\leftarrow$ open
   pred(w) $\leftarrow v$

by contradiction:
v is bad $\equiv h(v) > d(u,v)$
if $\exists v$ bad ... consider v bad s.t. shortest uv-path has min # edges



$v \neq u$ $\quad h(v) \leq h(t) + \ell(tv) = d(u,v)$ — not bad
when t closed; $h(t) = d(u,t)$