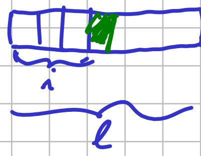
Network G with $c \in \{0 — C\}$ → Capacity Scaling



$O(nm + n\,\Delta f)$

$G_0, G_1 — G_\ell$
↓
$f_0, f_1 — f_\ell$

f_i is a max. flow in G_i

$\ell := \lfloor \log C \rfloor + 1$

G_i will be G with capacities
$c_i(e) := \lfloor c(e) / 2^{\ell - i} \rfloor$
↑ topmost i bits

$c_{i+1}(e) = \begin{cases} 2c_i(e) \\ 2c_i(e)+1 \end{cases}$

f_0 = const 0

$f_i \to f_{i+1}$, G_i, G_{i+1}

(1) $2f_i$ is a flow in G_{i+1}] $O(m)$

(2) improve $2f_i$ by Dinic's alg. to get f_{i+1}] $O(nm)$

$O(nm \cdot \log C)$
(↑ step, ↑ #steps)

we know: f_i is max. in G_i
→ ∃R min cut in G_i s.t. $|f_i| = c_i(R)$

use R in G_{i+1}:
$|f_{i+1}| \le c_{i+1}(R) \le 2c_i(R) + m = 2|f_i| + m$

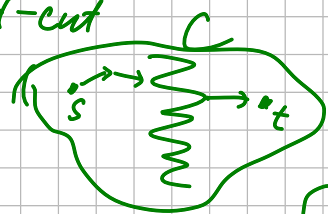
$\underbrace{|f_{i+1}| - 2|f_i|}_{\Delta f} \le m$

Cuts

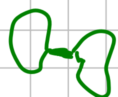
Df: In a (un)directed graph G with $s,t \in V$, $s \ne t$

- $C \subseteq E$ is a s,t-cut ≡ G − C contains no s,t-path.
- $C \subseteq E$ is a cut ≡ $\exists s,t \in V$, $s \ne t$: C is a s,t-cut
- G undirected is k-edge-connected ≡ all cuts have size at least k

s,t-cut



|C| = 1 : bridges



1-connected ⇔ connected
2-connected ⇔ connected & bridgeless

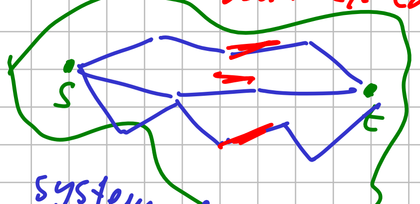
Thm: Max. # of edge-disjoint s,t-paths = size of min. s,t-cut.
(Menger theorems)

"flow cut"

E(A,B)

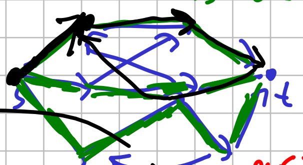
elementary cut C ≡ ∃ A,B partition of V s.t. C = E(A,B)

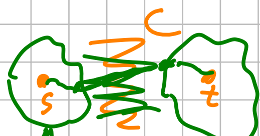
min. s,t-cut



system of edge-disjoint paths

Obs: every minimum cut is elementary





A, B, $E(A,B) \subseteq C$

G, s,t → network with c=1

by Dinitz $O(n^{2/3} \cdot m)$

integer max flow f ↔ system of |f| edge-disj. paths

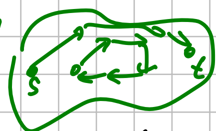
decompose f to edge-disjoint s,t-paths

"Global" greedy:
- find a path → remove → flow f′: |f′| = |f| − 1
- find a cycle → remove → flow f′: |f′| = |f|

can be done in O(m)

→ system of |f| edge disj. paths + some cycles

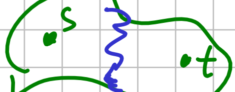
circulation



Thm: max. k : G is k-edge-connected
= max. k : ∀s,t ∃ system of k edge-disjoint s,t-paths

↳ algorithmic version: find such k (smallest cut in G)

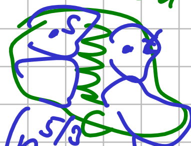
(1) try all pairs s,t — $O(n^2)$ choices



$O(n^{2+2/3} \cdot m)$

(2) fix s, try all t — O(n)

C is a min. cut



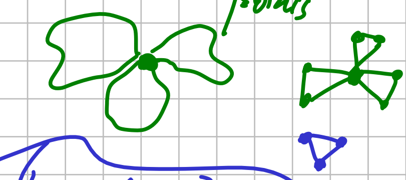
$O(n^{1+2/3} \cdot m) = O(n^{5/3} \cdot m)$

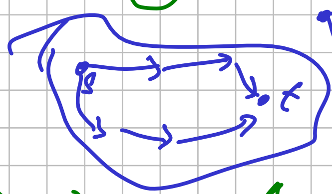
Nagamochi-Ibaraki: O(nm)
or Karger-Stein randomized

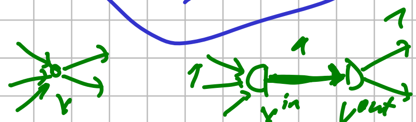
Df: for G, s,t:

- $U \subseteq V$ is a <u>s,t-separator</u> $\equiv$ $G-U$ contains no s,t-path & $s,t \notin U$
- $U \subseteq V$ is a <u>separator</u> $\equiv \exists s,t : U$ is s,t-sep.
- G undirected is <u>k-(vertex-)connected</u> $\equiv$ all separators have size at least k & $n > k$.
- system of internally vertex-disjoint s,t-paths

1-v-connected = connected
2-v-connected = connected & no articulation points



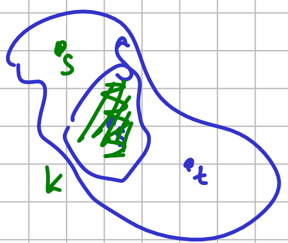




Mengerian theorems ... min. size of s,t-sep. = max. # of IVD s,t-paths

Algorithmically: Finding min s,t-sep. using flows: $O(n^{1/2} \cdot m)$ by Dinitz's alg. $st \in E$

Globally: min. sep.
① all pairs s,t $\quad O(n^2 \cdot n^{1/2} \cdot m) = O(n^{2.5} \cdot m)$
② fix s, try all t broken!



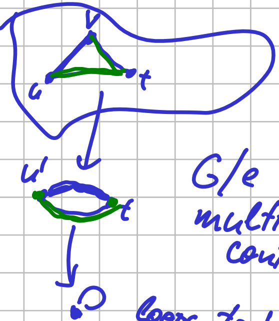
③ • try all s
 • try all t
 • if $|sep.| <$ #s we tried: stop

$O(k \cdot n \cdot n^{1/2} \cdot m) = O(k \cdot n^{1.5} \cdot m)$ (size of min. sep.; #t)

<u>Randomized Alg. for Min Cut</u> : G is undirected multigraphs (goal)

<u>Contract</u> (G_0, ℓ):
1. $G \leftarrow G_0$
2. while $n > \ell$:
3. Pick $e \in E$ uniformly at random.
4. $G \leftarrow G/e$, remove loops
5. Return G.

$G_0 \rightsquigarrow G$ with ℓ vertices



G/e multigraph contraction

loops to be removed

$\text{mincut}(G_0) \le \text{mincut}(G)$

We fix a specific min. cut C in G_0,

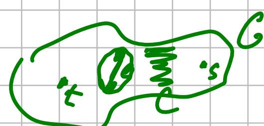
$p \le \Pr[C \text{ survives Contract}] \le \Pr\left[\begin{matrix}\text{mincut}\\ \text{is correct}\end{matrix}\right]$

C is a cut in $G/e \Rightarrow \exists C'$ cut in G s.t. $|C'| = |C|$

corresp. between edges of $G-e$ $\leftrightarrow$ edges of G/e

$k := |C|$

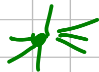
$G_i :=$ graph G before i-th contraction



We can lose a cut C if we pick $e \in C$.

$n_i = n - i + 1 \qquad$ Assuming that C survived to G_i

$m_i \ge k n_i / 2 \quad \leftarrow \quad \forall v\ \deg(v) \ge k$

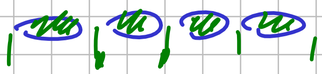


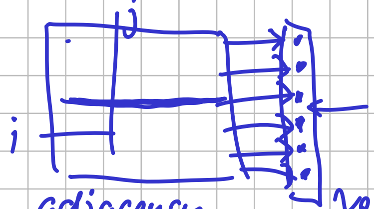
$\Pr[\text{we select } e \in C] = \frac{k}{m_i} \le \frac{k}{k \cdot n_i/2} = \frac{2}{n_i} = \frac{2}{n-i+1}$

$\Pr[C \text{ survives the } i\text{-th step}] \ge 1 - \frac{2}{n-i+1} = \frac{n-i-1}{n-i+1}$

$\Pr\left[\begin{matrix}C \text{ survives}\\ \text{all steps}\end{matrix}\right] \ge \prod_{i=1}^{n-\ell} \frac{n-i-1}{n-i+1} = \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdot \ldots \cdot \frac{\ell}{\ell+2} \cdot \frac{\ell-1}{\ell+1} = \frac{\ell(\ell-1)}{n(n-1)} \sim \frac{\ell^2}{n^2}$

Implementation:



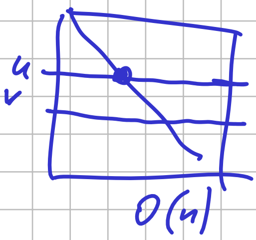


adjacency matrix

vector of degrees

A_{ij} = # parallel edges between i, j

1. Pick e uniformly at random in $O(n)$ time
2. Contract e in $O(n)$ time



1 step: $O(n)$ time

total: $O((n-2)\cdot n)$ time $\leq O(n^2)$