Notation:

G ... current graph

V ... set of vertices, $V \neq \emptyset$

E ... set of edges

$uv \in E$ $\quad \{u,v\}$ $\quad (u,v)$

$n := |V(G)|$

$m := |E(G)|$

$\mathcal{O}(n+m) \to \mathcal{O}(m)$

WLOG no isolated vertices

$\hookrightarrow$ m is $\Omega(n)$

H, $V(H)$, $E(H)$, $n(H)$, $m(H)$

$G_1, G_2, \dots$

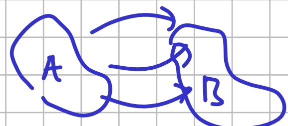
$V_1, V_2, \dots$

$m_1, m_2, \dots$

Alternatively: $|f| = -f^\Delta(s)$

$$f^\Delta(s) + f^\Delta(t) \overset{!}{=} \sum_v f^\Delta(v) = 0$$

Cuts



Df: For A, B disjoint subsets of V:
$E(A,B) := \{ab \in E \mid a \in A \text{ \& } b \in B\}$

Df: A cut (an elementary st-cut) is $E(A,\bar{A})$ for any $A \subseteq V$ s.t. $s \in A$, $t \notin A$

Df: $f(A,B) := \sum_{e \in E(A,B)} f(e)$

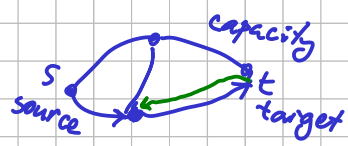
$f(A,\bar{A})$

$f^\Delta(A,\bar{A}) := f(A,\bar{A}) - f(\bar{A},A)$ net flow

$f^\Delta(v) := f^\Delta(\overline{\{v\}}, \{v\})$

Capacity of cut (directed): $c(A,\bar{A})$

Network Flows



Network

$G = (V,E)$ directed graph

$s,t \in V$ source, target $(s \neq t)$

$c: E \to [0,+\infty)$ capacities

Flow $f: E \to [0,+\infty)$

Such that ① $f \leq c$

② $\forall v \neq s,t: f^\Delta(v) = 0$

Kirchhoff's Law

Flow size

$|f| := f^\Delta(t)$

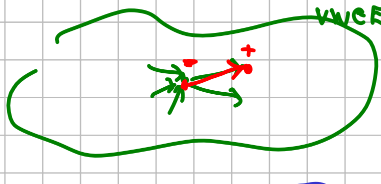
Goal: Find a flow of max. size

for $v \in V$: $f^+(v) := \sum_{uv \in E} f(uv)$ inflow

$f^-(v) := \sum_{vw \in E} f(vw)$ outflow

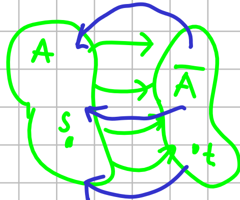
excess

$f^\Delta(v) := f^+(v) - f^-(v)$



Lemma: For every $E(A,\bar{A})$ cut:

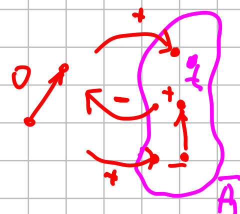
$f^\Delta(A,\bar{A}) = |f|$



Proof: $\underset{|f|}{\overset{\text{kir. law}}{f^\Delta(t)}} = \sum_{v \in \bar{A}} f^\Delta(v) = \underbrace{f(A,\bar{A})}_{\leq c(A,\bar{A})} - \underbrace{f(\bar{A},A)}_{\leq 0} = f^\Delta(A,\bar{A})$

Cor.: $|f| \leq c(A,\bar{A})$

Size of every flow $\leq$ capacity of every cut

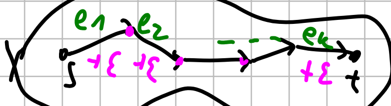


Self-certifying algorithm

! if $|f| = c(A,\bar{A})$, then

① f is maximum $|f|$

② $\underbrace{E(A,\bar{A})}_{\text{cut}}$ is minimum $c(A,\bar{A})$



Naively: augmenting paths

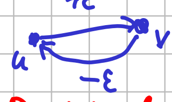
Find st-path $e_1 - e_k$

$\forall i \; f(e_i) < c(e_i)$

$\Rightarrow$ we can "push" $\varepsilon := \min_i c(e_i) - f(e_i)$

$\varepsilon > 0$

increases $|f|$ by ε



push ε

Df: Residual capacity of $uv \in E$:

$r(uv) := \underbrace{c(uv) - f(uv)}_{\text{inc. } u \to v} + \underbrace{f(vu)}_{\text{dec. } v \to u}$

Real alg. (Ford-Fulkerson alg.)

Repeat: Find st-path $e_1 - e_k$ s.t. $\forall i \; r(e_i) > 0$ — BFS, $\mathcal{O}(m)$

$\varepsilon := \min_i r(e_i)$

Push ε through the path

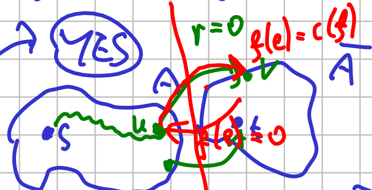
Q1 Does FF alg. stop? ! BROKEN !

Q2 Does it produce a max. flow? YES

Thm: For every max. flow f there is a cut $E(A,\bar{A})$ s.t. $|f| = c(A,\bar{A})$.

$|f| \leq c^+(s)$

$|f| = c(A,\bar{A})$



$A := \{v \mid \exists \text{ an augmenting sv-path}\}$

Does F-F always terminate?
Generally: No!

1. If capacities are integers ⇒ YES → but it can be slow
   ↳ but not for small int. c's

Thm: In a network with int. capacities, at least one max. flow is integer.

$c \le L \cdot n$

1. $c \in \{0,1\}$ $|f| \le n$ ⇒ # iterations $\le n$
   $O(nm)$ time
2. $c \in \{0 \ldots L\}$ $|f| \le n \cdot L$
   $O(Lnm)$ time

2. If caps are rational ⇒ YES

F-F invariant to scaling: if we multiply capacities by $\alpha > 0$, all results also multiply by α

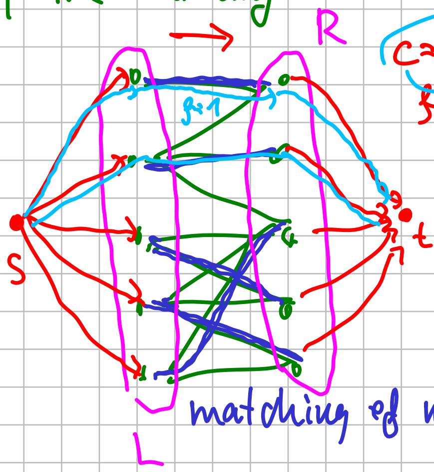
F-F with shortest aug. path
→ Edmonds-Karp alg.
→ runs in time $O(m^2 n)$. (even for real capacities)

Next: Dinitz's alg. .... $O(n^2 m)$

• Data Structures 2 (summer semester)

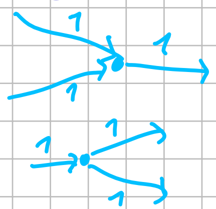
APPLICATIONS

Bipartite Matching



$c = 1$

find max. integer flow ≅ matchings



matching of max size

$O(\sqrt{n} \cdot m)$ ← next week