A short proof of the existence of the Jordan normal form of a matrix

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Theorem 1 *Let V be an n-dimensional vector space and $\Phi : V \to V$ be a linear mapping of V into itself. Then there is a basis of V such that the matrix representing Φ with respect to the basis is*

$$\begin{pmatrix} J_1 & & & & \\ & J_2 & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & J_k \end{pmatrix}$$

where empty space is filled by 0's and $J_1, \ldots, J_k$ are square matrices, called Jordan blocks, of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & \cdots & \cdots & & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{pmatrix}$$

for $i = 1, \ldots, k$, where $\lambda_1, \ldots, \lambda_k$ are complex numbers and empty space is filled by 0's.

Conclusion 1 (Jordan's normal form of a matrix) *Let $\mathbf{A}$ be a square matrix; there is a regular matrix $\mathbf{P}$ such that the matrix $\mathbf{P^{-1}AP}$ has the form described in the preceeding theorem.*

The matrix form shown in the theorem is called Jordan canonical form or Jordan normal form.

Remark: The numbers $\lambda_1, \ldots, \lambda_k$ of the theorem need not be distinct. E.g., the unit matrix is a matrix is a matrix in Jordan canonical form, where Jordan blocks are matrices of size 1×1 equal to (3), i.e. with $\lambda_1 = \cdots = \lambda_k = 1$.

We need one definition

Definition 1 *We say that a vector space V is a direct sum of its subspaces $V_1, \ldots, V_m$, if for each vector $v \in V$ there is the* unique *sequence of vectors $v_1, \ldots, v_m$ such that $v_i \in V_i$ for $i = 1, \ldots, m$ and $v = v_1 + \cdots + v_m$. In such a case we write $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$.*

Uniqueness in the definition means that it must be $V_i \cap V_j = \mathbf{0}$ for any two different i and j in the range $1 \leq i, j \leq m$, because if a non-zero vector v was a member of both V_i and V_j then the uniqueness of the sequence $v_1, \ldots, v_m$ is corupted: it would be possible to choose $v_i = v$ and the other vector equal to $\mathbf{0}$, of $v_j = v$ and others vectors equal to the null vector.
Thus, $\dim(V) = \dim(V_1) + \cdots + \dim(V_m)$.

The proof of the theorem is based of the following two lemmae:

Lemma 1 *Let V be an n-dimensional vector space and $\Phi : V \to V$ be a linear mapping of V into itself. Let $\lambda_1, \ldots, \lambda_r$ be different eigenvalues of Φ. Then there are integer $s_1, \ldots, s_r$ such that*

$$V = Ker(\Phi - \lambda_1 I)^{s_1} \oplus \cdots \oplus Ker(\Phi - \lambda_r I)^{s_r}.$$

Proof Choose first one of the eigenvalues of Φ and denote it by λ.

Part 1
Define $W_i = \mathrm{Ker}(\Phi - \lambda I)^i$ for each natural number i. It is clear that

$$W_1 \subset W_2 \subset W_3 \subset \ldots \subset W_i \subset \ldots$$

Since we suppose that V has finite dimension, the sequence could not be strictly increasing forever, but there must be a number t such that $W_t = W_{t+1}$. Assume that t is the smallest among such numbers. It is almost obvious that this would imply $W_{t+1} = W_{t+2} = W_{t+3} = \cdots$.

Part 2
We will prove that $\mathrm{Ker}(\Phi - \lambda I)^t \cap \mathrm{Im}(\Phi - \lambda I)^t = \mathbf{0}$.
Assume that a non-zero vector v belongs to $\mathrm{Ker}(\Phi - \lambda I)^t \cap \mathrm{Im}(\Phi - \lambda I)^t$.
This implies that
there exists $w \in V$ such that $v = (\Phi - \lambda I)^t(w)$ (because $v \in \mathrm{Im}(\Phi - \lambda I)^t$)
and also $(\Phi - \lambda I)^t(v) = 0$ (because $v \in \mathrm{Ker}(\Phi - \lambda I)^t$).

Thus, $(\Phi - \lambda I)^{2t}(w) = (\Phi - \lambda I)^t(v) = 0$, and hence $w \in W_{2t}$. But since $W_t = W_{2t}$, it is also $w \in W_t = \mathrm{Ker}(\Phi - \lambda I)^t$, and hence $v = (\Phi - \lambda I)^t(w) = 0$.

Part 3
We already know that $\dim(V) = \dim(\mathrm{Ker}(\Phi - \lambda I)^t) + \dim(\mathrm{Im}(\Phi - \lambda I)^t)$. Moreover, we know that if V_1 and V_2 are subspaces of V, then the subspace that spans both V_1 and V_2 has the dimension $\dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$. Applying this to $V_1 = \mathrm{Ker}(\Phi - \lambda I)^t$ and $V_2 = \mathrm{Ker}(\Phi - \lambda I)^t$ (i.e., $\dim(V_1 \cap V_2) = 0$), we

obtain that the dimension of the subspace of V that spans both $\text{Ker}(\Phi - \lambda I)^t$ and $\text{Im}(\Phi - \lambda I)^t$ is equal to $\dim(V)$, and hence

$$V = \text{Ker}(\Phi - \lambda I)^t) \oplus \text{Im}(\Phi - \lambda I)^t.$$

Part 4
Both $\text{Ker}(\Phi - \lambda I)^t$ and $\text{Im}(\Phi - \lambda I)^t$ are invariant subspaces of Φ (a subspace U of V is an invariant subspace of Φ, if $v \in U$ implies $\Phi(v) \in U$).
Note that

$$\Phi(\Phi - \lambda I) = \Phi\Phi - \lambda(\Phi I) = \Phi\Phi - \lambda(I\Phi) = (\Phi - \lambda I)\Phi.$$

This implies that
if $v \in \text{Ker}(\Phi - \lambda I)^t$, then $(\Phi - \lambda I)^t(v) = 0$, and

$$0 = \Phi(0) = \Phi(\Phi - \lambda I)^t(v) = (\Phi - \lambda I)^t\Phi(v),$$

and hence $\Phi(v) \in \text{Ker}(\Phi - \lambda I)^t$, and
if $v \in \text{Im}(\Phi - \lambda I)^t$, then $v = (\Phi - \lambda I)^t(w)$ for some $w \in V$, and

$$\Phi(v) = \Phi(\Phi - \lambda I)^t(w) = (\Phi - \lambda I)^t\Phi(w),$$

i.e., $\Phi(v) \in \text{Im}(\Phi - \lambda I)^t$.

Part 5
Now, the lemma can be proved by the induction on the number of different eigenvalues of Φ: if $\lambda_1, \ldots, \lambda_r$ are different eigenvalues of Φ and we put λ of Parts 1-4 to be λ_1, then the eigenvalues of the restriction of Φ to $\text{Im}(\Phi - \lambda I)^t$ are $\lambda_2, \ldots, \lambda_r$, and, by the induction hypothesis,

$$\text{Im}(\Phi - \lambda I)^t = \text{Ker}(\Phi - \lambda_2 I)^{s_2} \oplus \cdots \oplus \text{Ker}(\Phi - \lambda_r I)^{s_r}$$

for some $s_2, \ldots, s_r$. ♣

The second lemma that we will use in order to prove the Jordan form theorems is

Lemma 2 (Mark Wildon[1]) *Let V be an n-dimensional vector space and $T : V \to V$ be a linear mapping of V into itself such that $T^s = \mathbf{0}$ for some natural number s. Then there are vectors $u_1, \ldots, u_k$ and natural numbers $a_1, \ldots, a_k$ such that*

$$T^{a_i}(u_i) = \mathbf{0} \quad \textit{for } i = 1, \ldots, k,$$

and the vectors

$$u_1, T(u_1), \ldots, T^{a_1-1}(u_1), \ldots, u_k, T(u_k), \ldots, T^{a_k-1}(u_k)$$

are non-zero vectors that form a basis of V.

Proof If T itself maps all vectors to $\mathbf{0}$, then it is sufficient to put $u_1, \dots, u_k$ to be a basis of V and $a_1 = \dots = a_k = 1$.

Now, the proof is by induction on the dimension of V. Suppose first that the dimension of V is 1: in this case T^s could be a constant mapping to $\mathbf{0}$ only if T is, and we use the previous statement.

Let us suppose that the lemma holds for all cases when the dimension is smaller than n, and we will prove the lemma for n. Consider the vector space $\mathrm{Im}(T)$. If $\dim(\mathrm{Im}(T)) = 0$, then T is a zero mapping and the lemma follows. The assumption $\dim(\mathrm{Im}(T)) = n$ would imply that T is a one-to-one mapping, which would contradict to the assumption that $T^s = \mathbf{0}$ for some s. Thus, we can assume that $0 < \dim(\mathrm{Im}(T)) < n$ and, by the induction hypothesis, there are vectors $v_1, \dots, v_\ell$ and natural numbers $b_1, \dots, b_\ell$ such that

$$T^{b_i}(v_i) = \mathbf{0} \quad \text{for } i = 1, \dots, \ell, \text{ and}$$

$$v_1, T(v_1), \dots, T^{b_1-1}(v_1), \dots, v_\ell, T(v_\ell), \dots, T^{b_\ell-1}(v_\ell) \qquad (1)$$

form a basis of $\mathrm{Im}(T)$.

For each $i = 1, \dots, \ell$, $v_i \in \mathrm{Im}(T)$, and hence we can choose $w_i \in V$ such that $T(w_i) = v_i$. Vectors $T^{b_1-1}(v_1), \dots, T^{b_\ell-1}(v_\ell)$ are linearly independent vectors in $\mathrm{Ker}(T)$. Steinitz theorem says that we can extend these vectors to a basis

$$T^{b_1-1}(v_1), \dots, T^{b_\ell-1}(v_\ell), z_1, \dots, z_m \qquad (2)$$

of $\mathrm{Ker}(T)$.

Note that in our notation, $T^j(w_i) = T^{j-1}(v_i)$ for all relevant i and j.

Now it is sufficient to prove that the vectors

$$w_1, T(w_1), \dots, T^{b_1}(w_1), \dots, w_\ell, T(w_\ell), \dots, T^{b_\ell}(w_\ell), z_1, \dots, z_m \qquad (3)$$

form a basis of V.

We will first prove their linear independence. Assume that

$$\alpha_{1,0}w_1 + \alpha_{1,1}T(v_1) + \dots + \alpha_{1,b_1}T^{b_1}(w_1) + \dots + \alpha_{\ell,0}w_\ell + \dots + \alpha_{\ell,b_\ell}T^{b_\ell}(w_\ell) +$$
$$+ \beta_1 z_1 + \dots + \beta_m z_m = 0.$$

Apply the linear mapping T to the equation to get

$$\alpha_{1,0}T(w_1) + \alpha_{1,1}T^2(w_1) + \dots + \alpha_{1,b_1-1}T^{b_1}(w_1) + \dots + \alpha_{\ell,0}T(w_\ell) + \dots + \alpha_{\ell,b_\ell-1}T^{b_\ell}(w_\ell) = 0$$

i.e.,

$$\alpha_{1,0}v_1 + \alpha_{1,1}T(v_1) + \dots + \alpha_{1,b_1-1}T^{b_1-1}(v_1) + \dots + \alpha_{\ell,0}v_\ell + \dots + \alpha_{\ell,b_\ell-1}T^{b_\ell-1}(v_\ell) = 0$$

and since the left side of the last equation is a linear combination of elements of a basis (1) of $\mathrm{Im}(T)$, the corresponding α's must be 0.

Putting $\alpha_{1,0} = \alpha_{1,1} = \cdots = \alpha_{1,b_1-1} = \cdots = \alpha_{\ell,0} = \cdots = \alpha_{\ell,b_\ell-1} = 0$ into the original equation, we get

$$\alpha_{1,b_1} T^{b_1}(w_1) + \cdots + \alpha_{\ell,b_\ell} T^{b_\ell}(w_\ell) + \beta_1 z_1 + \cdots + \beta_m z_m = 0,$$

but the left side of this equation is a linear combination of elements of a basis (2) of $\mathrm{Ker}(T)$, and hence even α's in the last equation are equal t 0, which proves the linear independence of the original system of vectors listed in (3).

In order to prove that the system (3) forms a basis of V we just need to prove that the number of vectors in (3) is equal to the dimension of V. The system (1) is a basis of $\mathrm{Im}(T)$, which means that $\dim(\mathrm{Im}(T)) = b_1 + \cdots + b_\ell$. Moreover, the system (2) is a basis of $\ker(T)$, i.e., $\dim(\mathrm{Ker}(T)) = \ell + m$. Using the theorem on the dimension of the image and the kernel of a linear mapping, we get that

$$\dim(V) = \dim(\mathrm{Im}(T)) + \dim(\mathrm{Ker}(T)) = b_1 + \cdots + b_\ell + \ell + m =$$

$$= (1 + b_1) + \cdots + (1 + b_\ell) + m,$$

which is exactly the number of vectors of the system (3). ♣

An example for the Wildon's lemma: Let V be a vector space of the dimension 3 and $T(x_1, x_2, x_3) = (x_2 + x_3, 0, 0)$. Then $\mathrm{Im}(T)$ is one-dimensional vector space generated by the vector $(1, 0, 0)$. We can easily choose $\ell = 1$, $v_1 = (1, 0, 0)$, and $a_1 = 1$.

Now, there are two important vectors that T maps to v_1, namely $(0, 1, 0)$ and $(0, 0, 1)$. Moreover, any vector $(x_1, x_2, 1 - x_2)$ maps into v_1 as well. We choose one of them as w_1, e.g., $(0, 0, 1)$. Now, what about the vector $(0, 1, 0)$ and other vectors that map into v_1? If $T(w) = v_1$ for some vector w other than w_1 (e.g., if $w = (0, 1, 0)$), then $T(w - w_1) = v_1 - v_1 = \mathbf{0}$, and hence $w - w_1$ is a member of $\mathrm{Ker}(T)$ that was not included in $\mathrm{Im}(T)$, and we can choose that vector as z_1, an additional member of a basis of $\mathrm{Ker}(T)$. Thus, we obtain the basis $w_1 = (0, 0, 1)$, $v_1 = (1, 0, 0)$, and $z_1 = (0, 1, -1)$, and we know that $T(w_1) = v_1$, $T(v_1) = \mathbf{0}$, and we also have $T(z_1) = \mathbf{0}$. ♣

Proof of the Theorem:
Using the first lemma, there are integer $s_1, \ldots, s_r$ such that

$$V = \mathrm{Ker}(\Phi - \lambda_1 I)^{s_1} \oplus \cdots \oplus \mathrm{Ker}(\Phi - \lambda_r I)^{s_r},$$

where $\lambda_1, \ldots, \lambda_s$ are different eigenvalues of Φ.

Assume a basis of V obtained so that we concatenate bases of $\mathrm{Ker}(\Phi - \lambda_1 I)^{s_1}, \ldots, \mathrm{Ker}(\Phi - \lambda_r I)^{s_r}$. With respect to such a basis, the matrix representation of Φ is a matrix of the form

$$\begin{pmatrix} \mathcal{J}_1 & & & & \\ & \mathcal{J}_2 & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & \mathcal{J}_k \end{pmatrix},$$

where $\mathcal{J}_1, \ldots, J_k$ are general square matrices; $\mathcal{J}_i$ is the matrix of the restriction of Φ to $\mathrm{Ker}(\Phi - \lambda_i I)^{s_i}$ with respect to the chosen basis.

However, if the basis of $\mathrm{Ker}(\Phi - \lambda_i I)^{s_i}$ was constructed using Wildon's lemma, then each $\mathcal{J}_i$ turns to be

$$\begin{pmatrix} J_{i,1} & & & & \\ & J_{i,2} & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & J_{i,\ell_i} \end{pmatrix},$$

where each $J_{i,j}$ is a Jordan block with λ_i on the diagonal; each Jordan block corresponds to one chain of vectors $v_j, T(v_j), \ldots, T^{a_j-1}$, where $T = (\Phi - \lambda_i I)^{s_i}$. ♣

References

[1] Mark Wildon, Royal Holloway, University of London,
A short proof of the existence of Jordan Normal Form,
https://www.math.vt.edu/people/renardym/class_home/Jordan.pdf