THE ENUMERATION OF PLANAR GRAPHS VIA WICK'S THEOREM

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ABSTRACT. A seminal technique of theoretical physics called *Wick's theorem* interprets the Gaussian matrix integral of the products of the trace of powers of Hermitian matrices as the number of labelled *maps* with a given degree sequence, sorted by their Euler characteristics. This leads to the map enumeration results analogous to those obtained by combinatorial methods. In this paper we show that the enumeration of the graphs *embeddable* on a given 2-dimensional surface (a main research topic of contemporary enumerative combinatorics) can also be formulated as the Gaussian matrix integral of an ice-type partition function. Some of the most puzzling conjectures of discrete mathematics are related to the notion of the *cycle double cover*. We express the number of the graphs with a fixed *directed cycle double cover* as the Gaussian matrix integral of an Ihara-Selberg-type function.

1. INTRODUCTION

Let $M = (M_{ij})$ be an $N \times N$ Hermitian matrix, i.e., $M_{ij} = \overline{M}_{ji}$ for every $1 \le i, j \le N$, where $\overline{M}_{ji}$ denotes the complex conjugate of M_{ji}. Let $\varphi(M) = \sum_I a_I \prod_{ij\in I} M_{ij}$ be a polynomial in its entries, where I ranges over a finite system of multisets of elements of $N \times N$ and a_I's are complex numbers. We start with the following notation.

Notation 1.1. *Denote $\langle\cdot\rangle$ by*

$$\langle\varphi\rangle \;=\; \sum_I a_I \langle \prod_{ij\in I} M_{ij}\rangle \;=\; \sum_I a_I \sum_P \prod_{(p,q)\in P} \langle M_p M_q\rangle,$$

where P ranges over all partitions of I into pairs, and for $p = (p_1, p_2), q = (q_1, q_2)$ we have $\langle M_p M_q\rangle$ is non-zero if and only if $p_1 = q_2$ and $p_2 = q_1$ and in that case $\langle M_p M_q\rangle \;=\; 1/N$.

1.1. The motivation.

Wick's theorem interprets $\langle\varphi\rangle$ as the Gaussian matrix integral (see Section 2). This has been successfully used in enumeration of combinatorial structures and in other fields of mathematics [2, 3, 4, 5, 8, 9, 18, 19, 20, 23, 24, 25, 26, 31].

The basic combinatorial structure studied in this paper is that of a *graph*. A graph G is a pair (V, E) where V is the set of *vertices* and E is the set of pairs of vertices called *edges*. The vertices of an edge are called its *endvertices*. A graph may have *multiple edges*, i.e., different edges with the same endvertices, and also *loops*, i.e., edges whose two endvertices are equal. The graphs without multiple edges and loops are called *simple*. A *map* is a graph together with a fixed cyclic ordering of the incident edges of each vertex: it defines an embedding of the graph on an orientable 2-dimensional surface (see [21]). Note that a map is a graph that is *embedded* on a given 2-dimensional surface.

Principally studied functions φ are products of powers of the trace of M^i for a non-negative integer i. For such functions φ, $\langle\varphi\rangle$ has a useful graphic interpretation as the number of labelled *maps* with given degree sequence, sorted by their Euler characteristics (these maps are the *Feynman diagrams* for the matrix integral). This seminal formula was first described in [5]. While the Gaussian matrix integral of a polynomial in the traces of the powers of M is well understood, the Feynman diagrams expansions of the Gaussian matrix integral of the *exponential* of such polynomials hold in general only on the level of *formal power series*. We will denote this formal equality by $\;' = '$. The expression $f(x)\;' = '\;g(x)$ simply means that all the derivatives of f, g are equal when the variable x is set to zero. Similarly the expression $f(x)\;' \leq '\;g(x)$ means that all the derivatives of f, g satistfy the inequality when the variable x is set to zero. Finally, let $s(N)$ be a positive integer function in N and let $f_N(x), g_N(x)$ be two formal power series in variable x, whose coefficients depend on a positive integer N. The expression $f_N(x)\;' =_{s(N)} '\;g_N(x)$ (resp. $f_N(x)\;' \leq_{s(N)} '\;g_N(x)$) means that all the k-th derivatives of f, g, for $k \leq s(N)$, satistfy the equality (resp. inequality) when the variable x is set to zero. These notations are defined analogously for the multivariate generating functions. We describe the classical application of the matrix integrals in the next subsection.

1.2. **Enumeration of maps.**

Let $\mathcal{H}_N$ denote the set of all $N \times N$ Hermitian matrices.

We consider a function ψ which maps $M \in \mathcal{H}_N$ to

$$\psi(M) := e^{(-N)\sum_{i\in\mathbb{N}} z_i \mathrm{Tr}\left(\frac{M^i}{i}\right)},$$

where $\mathbb{N}$ is the set of all positive integers and z_i's are formal variables.

Taking the logarithm of $\langle\psi\rangle$ we get the following formula for connected maps sorted by genus as discussed by Brézin, Itzykson, Parisi and Zuber in [5].

Theorem 1 ([5]).

$$\log\langle\psi\rangle\;\; ' = '\;\; \sum_{g\geq 0} N^{2-2g} \sum_{(n_1,\cdots,n_k)\in\mathbb{N}^k} \prod_{i=1}^{k} \frac{(-z_i)^{n_i}}{n_i!} M_g(n_1,\cdots,n_k)$$

where $M_g(n_1,\cdots,n_k)$ denotes the number of connected maps with genus g and n_i vertices of degree i for $1 \leq i \leq k$.

The convergence of the generating function in Theorem 1 was studied extensively [6, 7, 13, 14, 16, 22]. For example, the following theorem was proved by Ercolani and McLaughlin in [6]. (For more general results, see a lecture note by Guionnet [15] and references therein.)

Theorem 2 ([6]). *Let a function $\tilde{\psi}$ map each Hermitian matrix $M \in \mathcal{H}_N$ to*

$$\tilde{\psi}(M) := e^{(-N)\sum_{i\in I} z_i \mathrm{Tr}\left(\frac{M^i}{i}\right)},$$

where I is a finite set of positive integers such that its maximum element is even. Then for each $i \in I$ there is $\epsilon_i > 0$ so that for z_i a non-zero real variable with $z_i \in (0, \epsilon_i)$, Theorem 1 holds as equalities between analytic functions *(in variables z_i) when ψ is replaced by $\tilde{\psi}$.*

1.3. **Main contribution.** A natural question is whether the Gaussian matrix integral method may as well be applied to the enumeration of graphs that are *embeddable* on a given 2-dimensional surface. This enumeration has been a long-standing open problem in the enumerative combinatorics, solved only for planar graphs and fairly recently by Giménez and Noy [12].

One of our main results (Theorem 3) is that this enumeration task can as well be formulated as a Gaussian matrix integral of an ice-type partition function (but whose asymptotic analysis is not carried out so far).

Another main result (Theorem 4) is that the number of graphs with a fixed *directed cycle double cover* is expressed as a Gaussian matrix integral of an Ihara-Selberg-type function. Even though the study of the cycle double cover conjectures is extensive, no attempt has been made before to calculate these numbers.

The main contribution of this paper is in applying the matrix integral to *combinatorially defined functions*, in order to loosen the strong connection with maps implied by the integration of the functions of traces. In the rest of this section we will show how to achieve this goal, introduce necessary concepts, and state our main results more precisely.

1.4. **Directed graphs.** A starting idea is to associate directed graphs with matrices. Let $M \in \mathcal{H}_N$ and let $D = D(M) = (N, N \times N)$ be a complete directed graph with weights on directed edges given by M.

Definition 1.2. *A* closed walk *in D is defined as a circular sequence of directed edges of D so that the tail of each edge equals the head of the consecutive edge. A closed walk is called* pointed *if it has a prescribed beginning. A* closed trail *is a closed walk which contains each directed edge* at most once.

Now $\mathrm{Tr}(M^n)$ can be interpreted as $\sum_p \prod_{e\in p} M_e$, where the sum is over all pointed closed walks in D of length n. Similarly, $\mathrm{Tr}(M^3)^4\mathrm{Tr}(M^2)^3$ can be interpreted as a function which maps each Hermitian matrix $M \in \mathcal{H}_N$ to

$$(\sum_{p_1}\prod_{e\in p_1} M_e)^4(\sum_{p_2}\prod_{e\in p_2} M_e)^3,$$

where the first sum is over all pointed closed walks p_1 in D of length 3, and the second sum is over all pointed closed walks p_2 in D of length 2.

Definition 1.3. *A subset A of directed edges of D is called* eulerian *if for each vertex its indegree in A is equal to its outdegree in A. We denote by $\mathcal{E} = \mathcal{E}(D)$ the collection of all eulerian sets in D.*

Definition 1.4. *A subset A of directed edges of D is called* even *if A can be written as a union of edge-disjoint directed cycles of length bigger than two.*

For a set A of directed edges of D, a subset $C \subset A$ is called a *component* of A if C is a maximal subset of A with respect to inclusion that induces a connected underlying undirected graph.

1.5. **Enumeration of embeddable graphs.** We say that an integer function $f(N)$ is of type $O(N^{-1})$ if $f(N)$ is bounded above by cN^{-1} for a constant c and all large enough N.

Let η be a function which maps each Hermain matrix $M \in \mathcal{H}_N$ to

$$\eta(M, Nzy^{-1}, y) = \sum_{A\in\mathcal{E}} (\prod_{e\in A} M_e) \prod_{C\text{: component of }A} \frac{(Nz)^{|C|/2+1} - Nzy^{|C|/2}}{Nz - y}. \qquad (1)$$

The first result of the paper is the following expression for planar graphs.

Theorem 3. *Let $s(N) \leq \frac{N}{2}$ be any function in N which approaches to ∞ slower than N when $N \to \infty$. The Gaussian integral of the function η defined in (1) satisfies the following.*

$$\exp\left(\sum_{n\geq 1}\sum_{r\geq 0} [\, p(n,r) + e_1(N,n,r)\,] \frac{z^r y^{n-2}}{n!}\right) \ '\leq_{s(N)}' $$
$$e^{-N^2} \langle \eta(M, Nzy^{-1}, y)\rangle$$
$$'\leq_{\frac{N}{2}}' \ \exp\left(\sum_{n\geq 1}\sum_{r\geq 0} [\, p(n,r) + e_2(N,n,r)\,] \frac{z^r y^{n-2}}{n!}\right),$$

where $p(n,r)$ denotes the number of labelled connected simple graphs on n vertices which have planar *embeddings with r faces, and $e_1(N,n,r), e_2(N,n,r)$ are functions of type $O(N^{-1})$.*

Let $p(n)$ denote the number of labelled planar connected simple graphs on n vertices. Clearly, we have that $p(n) = \sum_{r\geq 0} p(n,r)$ and that $p(n)/n!$ is equal to the coefficient of y^{n-2} in $\lim_{N\to\infty} \frac{1}{N^2} \log\langle \eta(M, Nzy^{-1}, y)\rangle|_{z=1}$.

It would be very interesting to see if we can recover

$$p(n) \sim c\, n^{-7/2}\, \gamma^n\, n!$$

with $c \doteq 0.478$ and $\gamma \doteq 27.2$, which is computed by Giménez and Noy [12].

The embeddable graphs of higher genus contribute to the subleading terms $e_1(N,n,r), e_2(N,n,r)$ in Theorem 3; it should be possible to derive their enumeration from our formulas in Section 4.

The function η in (1) is an ice-type partition function. Ice-type partition functions play a seminal role in modern mathematics and theoretical physics (see [30]), and we hope that the asymptotic analysis of the matrix integral of the function η is not out of reach.

1.6. **Enumeration of directed cycle double covers.** Observe that an eulerian set does not need to be even. Even sets are closely related to the *cycle double cover conjectures.*

Definition 1.5. *Let G be an undirected graph. A collection of its cycles is called a* cycle double cover (CDC) *if each edge belongs to exactly two of the cycles. Moreover it is called a* directed cycle double cover (DCDC) *if it is possible to orient the cycles so that they go oppositely on each edge.*

Some of the most puzzling conjectures of discrete mathematics are centred around this notion. A graph is *bridgeless* if it cannot be disconnected by deletion of a single edge. Clearly a graph with a bridge does not have a CDC. On the other hand, there are

- *Cycle double cover conjecture*: Is it true that each bridgeless graph has a CDC?
- *Directed cycle double cover conjecture*: Is it true that each bridgeless graph has a DCDC?

Our next main result concerns directed cycle double covers.

Theorem 4. *Let $\mathcal{A}$ be the set of all pairs (q, K) where q is an even set of edges of D and K is a decomposition of q into directed cycles of length at least three. Then*

$$\langle \sum_{(q,K)\in\mathcal{A}} \prod_{e\in q} M_e\rangle \;=\; \sum_{[(G,C)]} \frac{N(N-1)\dots(N-|V(G)|+1)}{|\mathrm{Aut}(G,C)|N^{e(G)}},$$

where the sum is over all isomorphism classes of pairs (G, C) where G is a simple graph with at most N vertices and no isolated vertex, and C is a specified *DCDC of G.*

We show in Section 5.1 that the integrand ξ in Theorem 4 which maps $M \in \mathcal{H}_N$ to

$$\xi(M) \;:=\; \sum_{(q,K)\in\mathcal{A}} \prod_{e\in q} M_e \tag{2}$$

is an Ihara-Selberg-type function (see [27], [28]). Thus again, we hope that an analysis of the matrix integral of the function ξ may be achievable.

The rest part of the paper is organized as follows. In Section 2 we review the Gaussian matrix integral method and Wick's theorem. In Section 3 we introduce identities essential to the enumeration of embeddable graphs on a 2-dimensional surface. In Sections 4 and 5 we prove our main Theorems 3 and 4.

2. Gaussian matrix integral and Wick's theorem

In this section, based on [9], we recall the definition of Gaussian matrix integral, useful identities, and Wick's theorem in particular.

Let $M \in \mathcal{H}_N$ and let

$$dM = \prod_i dM_{ii} \prod_{i<j} d\,\mathrm{Re}(M_{ij}) d\,\mathrm{Im}(M_{ij})$$

denote the standard Haar measure on $\mathcal{H}_N$, where $\mathrm{Re}(M_{ij})$ and $\mathrm{Im}(M_{ij})$ denote the real part and the imaginary part of M_{ij}.

For an arbitrary function $\varphi : \mathcal{H}_N \to \mathbb{C}$, the Gaussian matrix integral of φ is defined as

$$\langle \varphi \rangle \quad = \quad \frac{1}{Z_0(N)} \int_{\mathcal{H}_N} e^{-N\,\mathrm{Tr}(\frac{M^2}{2})} \varphi(M) dM, \tag{3}$$

where $Z_0(N)$ is the normalization factor making $\langle 1 \rangle \ = 1$, that is,

$$Z_0(N) = \int_{\mathcal{H}_N} e^{-N\,\mathrm{Tr}(\frac{M^2}{2})} dM.$$

We are particularly interested in a function φ which maps $M \in \mathcal{H}_N$ to

$$\varphi(M) = \sum_I a_I \prod_{ij \in I} M_{ij}, \tag{4}$$

where I ranges over a finite system of multisets of elements of $N \times N$. To compute $\langle \varphi \rangle$ we introduce the *source integral* $\langle e^{\mathrm{Tr}(MS)} \rangle$ for a given $N \times N$ Hermitian matrix S, where $\mathrm{Tr}(MS)$ denotes the trace of the matrix MS. It can easily be computed by

$$\begin{aligned} \langle e^{\mathrm{Tr}(MS)} \rangle \quad &= \quad \frac{1}{Z_0(N)} \int_{\mathcal{H}_N} e^{-N\,\mathrm{Tr}(\frac{M^2}{2})} e^{\mathrm{Tr}(MS)} dM \\ &= \quad \frac{1}{Z_0(N)} \int_{\mathcal{H}_N} e^{-N\,\mathrm{Tr}(\frac{1}{2}(M - \frac{S}{N})^2)} e^{\frac{\mathrm{Tr}(S^2)}{2N}} dM \\ &= \quad e^{\frac{\mathrm{Tr}(S^2)}{2N}}, \end{aligned} \tag{5}$$

since the trace is linear and $\mathrm{Tr}(MS) = \mathrm{Tr}(SM)$. On the other hand, for any $1 \leq i, j \leq N$,

$$\begin{aligned} \frac{\partial}{\partial S_{ji}} e^{\mathrm{Tr}(MS)} \Big|_{S=0} \quad &= \quad \left(\frac{\partial}{\partial S_{ji}} \mathrm{Tr}(MS) \right) e^{\mathrm{Tr}(MS)} \Big|_{S=0} \\ &= \quad \left(\frac{\partial}{\partial S_{ji}} \sum_{m,n} M_{mn} S_{nm} \right) e^{\mathrm{Tr}(MS)} \Big|_{S=0} \\ &= \quad M_{ij}. \end{aligned}$$

Thus the derivatives of the source integral becomes

$$\begin{aligned} & \frac{\partial}{\partial S_{ji}} \frac{\partial}{\partial S_{lk}} \cdots \langle e^{\mathrm{Tr}(MS)} \rangle \Big|_{S=0} \\ = \quad & \frac{1}{Z_0(N)} \int_{\mathcal{H}_N} e^{-N\,\mathrm{Tr}(\frac{M^2}{2})} \frac{\partial}{\partial S_{ji}} \frac{\partial}{\partial S_{lk}} \cdots e^{\mathrm{Tr}(MS)} \Big|_{S=0} dM \\ = \quad & \frac{1}{Z_0(N)} \int_{\mathcal{H}_N} e^{-N\,\mathrm{Tr}(\frac{M^2}{2})} M_{ij} M_{kl} \cdots dM \\ = \quad & \langle M_{ij} M_{kl} \cdots \rangle, \end{aligned} \tag{6}$$

where the first equality is due to the Leibniz integral rule.

Using (6) and (5), we obtain

$$\begin{aligned}\langle M_{ij}M_{kl}\cdots\rangle &\overset{(6)}{=} \frac{\partial}{\partial S_{ji}}\frac{\partial}{\partial S_{lk}}\cdots\langle e^{\mathrm{Tr}(MS)}\rangle\Big|_{S=0}\\ &\overset{(5)}{=} \frac{\partial}{\partial S_{ji}}\frac{\partial}{\partial S_{lk}}\cdots e^{\frac{\mathrm{Tr}(S^2)}{2N}}\Big|_{S=0}\end{aligned} \tag{7}$$

and in particular

$$\begin{aligned}\langle M_{ij}M_{kl}\rangle &= \frac{\partial}{\partial S_{ji}}\frac{\partial}{\partial S_{lk}}e^{\frac{\mathrm{Tr}(S^2)}{2N}}\Big|_{S=0}\\ &= \frac{\partial}{\partial S_{ji}}\left(\frac{\partial}{\partial S_{lk}}\frac{\mathrm{Tr}(S^2)}{2N}\right)e^{\frac{\mathrm{Tr}(S^2)}{2N}}\Big|_{S=0}\\ &= \frac{\partial}{\partial S_{ji}}\left(\frac{\partial}{\partial S_{lk}}\frac{\sum_{m,n}S_{mn}S_{nm}}{2N}\right)e^{\frac{\mathrm{Tr}(S^2)}{2N}}\Big|_{S=0}\\ &= \frac{\partial}{\partial S_{ji}}\frac{S_{kl}}{N}e^{\frac{\mathrm{Tr}(S^2)}{2N}}\Big|_{S=0}\\ &= \frac{\delta_{il}\delta_{jk}}{N}.\end{aligned} \tag{8}$$

Further, it is clear that the derivatives in (7) and (8) must be taken in pairs (e.g., S_{ji} and S_{lk} with $l=i$ and $k=j$) to get a non-zero contribution. This yields the following result known as *Wick's theorem.*

Theorem 5 (Wick's theorem). *Let $M \in \mathcal{H}_N$ and I be a multiset of elements of $N \times N$. Then*

$$\begin{aligned}\langle\prod_{ij\in I}M_{ij}\rangle &= \sum_{\textit{pairing } P\subset I^2}\ \prod_{(ij,kl)\in P}\langle M_{ij}M_{kl}\rangle\\ &= \sum_{\textit{pairing } P\subset I^2}\ \prod_{(ij,kl)\in P}\frac{\delta_{il}\delta_{jk}}{N}.\end{aligned} \tag{9}$$

Due to the linearity of the integral and (9), for the function φ defined in (4) we have

$$\langle\varphi\rangle = \sum_I a_I\langle\prod_{ij\in I}M_{ij}\rangle = \sum_I a_I\sum_P\prod_{(p,q)\in P}\langle M_pM_q\rangle,$$

where P ranges over all partitions of I into pairs, and for $p=(p_1,p_2), q=(q_1,q_2)$ we have $\langle M_pM_q\rangle$ is non-zero if and only if $p_1=q_2$ and $p_2=q_1$ and in that case $\langle M_pM_q\rangle = 1/N$. This is how we arrive at the notation of $\langle\cdot\rangle$ in Notation 1.1.

3. Trail double cover and pseudoembedding

In this section we derive several identities, which are essential for applying the Gaussian matrix integral method to the enumeration of embeddable graphs on a 2-dimensional surface. We recall that a *map* is a graph together with a fixed cyclic ordering of the incident edges of each vertex. This ordering (of each vertex) defines an embedding of the graph on an orientable 2-dimensional surface (see [21]). A map is also called a *fat graph*; we prefere this term since it corresponds to a helpful graphic representation of Wick's theorem. In a fat graph F the vertices are made into discs (islands) and connected by fattened edges (bridges) prescribed by the cyclic orders. This defines a two-dimensional orientable surface with boundary which we also denote by F. Each component of the boundary of F will be called a *face* of F. Each face is an embedded circle (see e.g., [21]). We will denote by $G(F)$ the underlying graph of F.

We denote by $e(F), v(F), f(F), c(F)$, and $g(F)$ the number of the edges, vertices, faces, connected components, and genus of F. We recall that $2g(F) = 2c(F)+e(F)-v(F)-f(F)$.

In the next sections we will count fat graphs and their relatives. To avoid confusion we assume that a fat graph has *labelled* vertices, i.e., two fat graphs are equal if they are equal as sets. We speak about *unlabelled* fat graphs if the equality is up to isomorphism.

Let $M \in \mathcal{H}_N$ and we recall that $D = D(M) = (N, N \times N)$ is a directed graph with weights on directed edges given by M. A closed walk in D is defined as a circular sequence of directed edges of D so that the tail of each edge equals the head of the consecutive edge. As we have seen in Section 1.4, $\mathrm{Tr}(M^3)^4\mathrm{Tr}(M^2)^3$ can be interpreted as

$$(\sum_{p_1}\prod_{e\in p_1} M_e)^4(\sum_{p_2}\prod_{e\in p_2} M_e)^3,$$

where the first sum is over all pointed closed walks p_1 in D of length 3, and the second sum is over all pointed closed walks p_2 in D of length 2.

A *proper pairing* of a subset A of directed edges of D is a partition of A into pairs $ij, ji \in N \times N$ of oppositely directed edges. Hence if $\varphi(M) = \mathrm{Tr}(M^3)^4\ \mathrm{Tr}(M^2)^3$ we get

$$\langle \varphi \rangle \ = \sum_{q=q_1q_2\ldots q_7}\ \sum_{P}\ \prod_{(e,e')\in P} 1/N,$$

where the second sum is over all *proper pairings* P of the directed edges of the disjoint union $q = q_1q_2 \ldots q_7$ of 7 pointed closed walks, from which four have length 3 and remaining three have length 2. Two directed edges form a proper pairing if one is reversed the other. We also say that such a pair (q, P) *contributes* to $\langle \varphi \rangle$.

Next we recall that a closed trail is a closed walk which does not repeat edges and we denote by $\mathcal{W}(r)$ the set of all subsets of edges of D that can be decomposed into r edge-disjoint closed trails, and we let $\mathcal{W} = \cup_{r\geq 0}\mathcal{W}(r)$.

We define a function ω_r which maps each Hermitian matrix $M \in \mathcal{H}_N$ to

$$\omega_r(M) \ := \sum_{q\in\mathcal{W}(r)} y^{|q|/2}\prod_{e\in q} M_e, \tag{10}$$

for each integer $r \geq 1$ and $\omega_0(M) = 1$. We call $y^{|q|/2}\prod_{e\in q} M_e$ the weight of q.

Next two definitions and proposition are crucial.

Definition 3.1. *Let G be a graph. A* trail double cover *(TDC) is a collection of closed trails so that each edge of G is traversed exactly twice, and in the opposite directions.*

We remark that each graph has a TDC. Next we introduce a useful notation. A simple graph will be called *nimple* if it has no vertex of degree 0.

Definition 3.2. *Let G be a finite nimple graph with at most N vertices. Then let $c(G)$ be the set of all pairs (q,P) so that there is a colouring d of the vertices of G by colours $\{1,\dots,N\}$, where each vertex gets a different colour,*

$$q = \{(d(x),d(y)),(d(y),d(x)); \{x,y\} \in E(G)\},$$

and P consists of all the pairs $\{[(d(x),d(y)),(d(y),d(x))]; \{x,y\} \in E(G)\}$.

We remark that each such q belongs to $\mathcal{W}$ since G has a TDC and that

$$|c(G)| \;=\; \frac{N(N-1)\cdots(N-|V(G)|+1)}{|\mathrm{Aut}(G)|}.$$

Proposition 3.3. *Let $r \geq 0$. A pair (q,P) with P a proper pairing of q contributes to $\langle \omega_r \rangle$ if and only if there is a nimple graph G with a TDC consisting of r closed trails such that $(q,P) \in c(G)$.*

Proof. If $(q,P) \in c(G)$, then any TDC consisting of r closed trails provides a partition of q into its trails and hence (q,P) contributes to $\langle \omega_r \rangle$. On the other hand, if (q,P) contributes to $\langle \omega_r \rangle$, then letting G be the graph with the vertices from $\{1,\dots,N\}$ and the edges given by P, we get that G is nimple since q consists of edge-disjoint closed trails. Furthermore, G has a TDC consisting of r closed trails and hence $(q,P) \in c(G)$. □

Proposition 3.4. *If $c(G) \cap c(G') \neq \emptyset$, then G is isomorphic to G'. Moreover, if G is isomorphic to G', then $c(G) = c(G')$.*

Proof. If $(q,P) \in c(G) \cap c(G')$, then the construction of q induces a function between the sets of vertices of G and G', and P gives the edges of both G, G'. Hence they are isomorphic. The second part is true since the definition of $c(G)$ does not depend on 'names' of the vertices. □

As a consequence we obtain the following.

Theorem 6. *Let ω_r be the function defined in (10). Then*

$$\langle \omega_r \rangle \;=\; \sum_{[G]} y^{e(G)} \frac{N(N-1)\cdots(N-|V(G)|+1)}{|\mathrm{Aut}(G)|N^{e(G)}},$$

where the sum is over all isomorphism classes $[G]$ of nimple graphs with at most N vertices that have a TDC consisting of r closed trails.

Next we need to extend the notion of the embedding. A *pseudosurface* is obtained from a surface by identifying finitely many (not necessarily disjoint) pairs of vertices. The identified vertices are called *singularities* of the pseudosurface. A *pseudoembedding* of a graph G is defined to be an embedding on a pseudosurface S such that each singularity of S has a vertex of G embedded in it.

Proposition 3.5. *Let G be a nimple graph. There is a TDC of G consisting of r closed trails iff there is a pseudoembedding of G with r faces.*

Proof. The first implication: we construct, from each trail w of TDC, a planar disc bounded by a polygon with $|w|$ edges. Let the boundary be directed according to the direction of w. Then we glue the discs together by identifying the oppositely oriented edges with the same endvertices. The result is a pseudosurface.

The other implication: given a pseudoembedding, orient the faces so that each edge appears twice, and with the opposite orientation. The orientations of the faces clearly form a TDC.

□

As a consequence of Theorem 6 and Proposition 3.5 we have

Corollary 3.6. *Let ω_r be the function defined in (10). Then*

$$\langle \omega_r \rangle \;=\; \sum_{[G]} y^{e(G)} \frac{N(N-1)\dots(N-|V(G)|+1)}{|\mathrm{Aut}(G)|N^{e(G)}},$$

where the sum is over all isomorphism classes $[G]$ of nimple graphs with at most N vertices that have pseudoembeddings with r faces.

If W is a fat graph, then its *dual* is the abstract graph $W^* = (V^*, E^*)$ with vertex set V^* and edge set E^*, where V^* is the set of (the duals of) the faces of W and E^* is the set of (the duals of) the edges e of W such that in W^*, $e = e(fg)$ has endvertices $f, g \in V^*$, and in W, e lies on the boundary of faces f, g of W. Note that there is a natural bijection between the set of the edges of W^* and that of W.

Next we consider the geometric duals of the pseudoembeddings.

Definition 3.7. *Given a fat graph W and a partition Q of the set of its faces, we denote by $G_Q(W^*)$ the (abstract) graph obtained from W^* by the contraction of the classes of Q into single vertices.*

A pair (W, Q), where W is a fat graph and Q a partition of the set of its faces, is called r-relevant *if W has r vertices, Q has at most N classes, and $G_Q(W^*)$ is a nimple graph.*

We say that (W, Q) is equivalent to (W', Q'), denoted by $(W, Q) \sim (W', Q')$, if $G_Q(W^*)$ is isomorphic to $G_{Q'}(W'^*)$.

We call a partition Q of the face set of W *trivial* and denoted it by $\emptyset$ if each face of W forms a single partition class of Q.

The following is a consequence of Corollary 3.6.

Corollary 3.8. *Let ω_r be the function defined in (10). Then*

$$\langle \omega_r \rangle \;=\; \sum_{[(W,Q)]} y^{e(W)} \frac{N(N-1)\dots(N-\alpha(Q)+1)}{|\mathrm{Aut}(G_Q(W^*))|N^{e(W)}}, \tag{11}$$

where the sum is over all the equivalence classes $[(W, Q)]$ of the r-relevant pairs where Q has $\alpha(Q) \le N$ parts.

4. Counting planar graphs (Proof of Theorem 3)

In this section we prove Theorem 3. To this end we first derive a relation (Theorem 7) between the enumeration of planar graphs and the Gaussian integration of the generating function for edge-disjoint closed trails. We then (in Theorem 8 and Corollary 4.1) derive an identity between this Gaussian integration and the Gaussian integration of the ice-type partition function for eulerian sets defined in (1). Theorem 3 follows from Corollary 4.1.

We consider the generating function for edge-disjoint closed trails definded by

$$\zeta(M, x, y) := \sum_{r \geq 0} \omega_r(M) x^r \tag{12}$$

where $\omega_r(M) = \sum_{q \in \mathcal{W}(r)} y^{|q|/2} \prod_{e \in q} M_e$ (as defined in (10)) and $\mathcal{W}(r)$ is the set of all subsets of edges of $D(M)$ that can be decomposed into r edge-disjoint closed trails.

Theorem 7. *Let $s(N) \leq \frac{N}{2}$ be any function in N which approaches to ∞ slower than N when $N \to \infty$. The Gaussian matrix integral of the function ζ defined in (12) satisfies the following.*

$$\exp\left(\sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r}\left(E_1(N,r,e) + \sum_{[G_{r,e}]} \frac{1}{|\mathrm{Aut}(G_{r,e})|}\right)\right) \ '\leq_{s(N)}'$$
$$e^{-N^2}\langle \zeta(M, Nzy^{-1}, y)\rangle$$
$$'\leq_{\frac{N}{2}}' \ \exp\left(\sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r}\left(E_2(N,r,e) + \sum_{[G_{r,e}]} \frac{1}{|\mathrm{Aut}(G_{r,e})|}\right)\right),$$

where the sum is over all isomorphism classes $[G_{r,e}]$ of connected simple graphs which have planar embeddings with r faces, e edges and $\leq N$ vertices, and $E_1(N,r,e), E_2(N,r,e)$ are of type $O(N^{-1})$.

The following theorem formulates the enumeration of planar graphs as the Gaussian integration of an ice-type partition function via eulerian sets.

Theorem 8. *Let ζ be the function defined in (12) and consider a function η which maps $M \in \mathcal{H}_N$ to*

$$\eta(M, x, y) := \sum_{A \subset N \times N \ \text{eulerian}} U(A, x) y^{|A|/2} \prod_{e \in A} M_e, \tag{13}$$

where $U(A,x) = \prod_{C\text{: component of } A}(\sum_{i=1}^{|C|/2} x^i)$. Then

$$\langle \zeta \rangle \ = \ \langle \eta \rangle.$$

Proof. This is true from the Euler theorem without $\langle\cdot\rangle$ when we restrict the eulerian subsets of D to the subsets consisting of pairs of oppositely directed edges with the same endvertices. In the integration the remaining terms of both ζ, η disappear. □

Note that the function $\eta(M,x,y)$ defined in (13) is equal to

$$\sum_{A\subset N\times N \text{ eulerian}} (\prod_{e\in A} M_e) \prod_{C:\text{ component of } A} \frac{x(x^{|C|/2}-1)}{x-1} y^{|C|/2}.$$

Thus we have

$$\eta(M,Nzy^{-1},y) = \sum_{A\subset N\times N \text{ eulerian}} (\prod_{e\in A} M_e) \prod_{C:\text{ component of } A} \frac{(Nz)^{|C|/2+1}-(Nz)y^{|C|/2}}{Nz-y},$$

as defined in (1). As a consequence of this and Theorem 8 we have the following.

Corollary 4.1. *Theorem 7 is true when we replace $\zeta(M,Nzy^{-1},y)$ by $\eta(M,Nzy^{-1},y)$.*

In the rest of this section we will prove Theorem 7 and then Theorem 3.

Proof. (of Theorem 7) From Corollary 3.8 we get

$$\begin{aligned}
&\langle\zeta(M,Nzy^{-1},y)\rangle\\
=\ &\sum_{r\geq 0}\langle\omega_r\rangle(Nzy^{-1})^r\\
\overset{(11)}{=}\ &\sum_{r\geq 0}\sum_{[(W,Q)]} N^{r-e(W)}\frac{N(N-1)\dots(N-\alpha(Q)+1)}{|\mathrm{Aut}(G_Q(W^*))|}z^r y^{e(W)-r}\\
=\ &\sum_{r\geq 0}\sum_{[(W,Q)]} N^{r-e(W)}N^{\alpha(Q)}\prod_{1\leq i\leq\alpha(Q)-1}\left(1-\frac{i}{N}\right)\frac{z^r y^{e(W)-r}}{|\mathrm{Aut}(G_Q(W^*))|},
\end{aligned} \tag{14}$$

where the sum is over all equivalence classes $[(W,Q)]$ of the r-relevant pairs where Q has $\alpha(Q)\leq N$ parts.

To see how (14) gets non-zero contributions from equivalence classes of the r-relevant pairs, observe the following. Each equivalence class $[(W,Q)]$ can be represented by a fat graph W with some number $f(W)$ of faces satisfying $\alpha(Q)\leq f(W)\leq 2e(W)$.

For further computation we take any representation $(W,Q)_R$ of each $[(W,Q)]$ and get

$$\begin{aligned}
&\langle\zeta(M,Nzy^{-1},y)\rangle\\
&=\sum_{r\geq 0}\sum_{(W,Q)_R} N^{r-e(W)+f(W)}N^{\alpha(Q)-f(W)}\prod_{1\leq i\leq\alpha(Q)-1}\left(1-\frac{i}{N}\right)\frac{z^r y^{e(W)-r}}{|\mathrm{Aut}(G_Q(W^*))|}.
\end{aligned} \tag{15}$$

We will bound (15) from above and from below.

Let $S_1(N,y,z)$ and $S_2(N,y,z)$ be defined as follows:

$$\begin{aligned}
S_1(N,y,z) &= \sum_{r\geq 0}\sum_{(W,Q)_R} N^{r-e(W)+f(W)}N^{\alpha(Q)-f(W)}\left(1-\frac{2s(N)}{N}\right)^{\alpha(Q)}\frac{z^r y^{e(W)-r}}{|\mathrm{Aut}(G_Q(W^*))|}\\
S_2(N,y,z) &= \sum_{r\geq 0}\sum_{(W,Q)_R} N^{r-e(W)+f(W)}N^{\alpha(Q)-f(W)}\frac{z^r y^{e(W)-r}}{|\mathrm{Aut}(G_Q(W^*))|}
\end{aligned} \tag{16}$$

where $s(N)$ is any function in N which approaches to ∞ slower than N when $N \to \infty$.

It is straightforward (we recall that that $\alpha(Q) \leq 2e(W)$) that

$$S_1(N,y,z) \,' \leq_{s(N)} {}' \, \langle \zeta(M, Nzy^{-1}, y) \rangle \,' \leq {}' \, S_2(N,y,z).$$

First we study $S_2(N,y,z)$. Recall that for each representation $(W,Q)_R$, W is a fat graph with r labelled vertices and $f(W)$ faces. Define the set $S_{(W,Q)_R}$ of siblings $(W',Q)_R$ of each $(W,Q)_R$, where W' is obtained from W by relabelling of its vertex labels. Since $\sum_{W' \in S_{(W,Q)_R}} 1 = r!/|\mathrm{Aut}(W)|$ (and notice that $v(W) = v(W') = r, e(W) = e(W'), f(W) = f(W')$, $|\mathrm{Aut}(W)| = |\mathrm{Aut}(W')|$ and $|\mathrm{Aut}(G_Q(W^*))| = |\mathrm{Aut}(G_Q(W'^*))|$), we have that

$$\begin{aligned} S_2(N,y,z) &= \sum_{r \geq 0} \sum_{(W,Q)_R} \left(\frac{\sum_{W' \in S_{(W,Q)_R}} 1}{r!/|\mathrm{Aut}(W)|} \right) N^{r-e(W)+f(W)} N^{\alpha(Q)-f(W)} \frac{z^r y^{e(W)-r}}{|\mathrm{Aut}(G_Q(W^*))|} \\ &= \sum_{r \geq 0} \sum_{(W,Q)_R} \sum_{W' \in S_{(W,Q)_R}} N^{r-e(W')+f(W')} N^{\alpha(Q)-f(W')} \frac{|\mathrm{Aut}(W')|}{|\mathrm{Aut}(G_Q(W'^*))|} \frac{z^r y^{e(W')-r}}{r!}. \end{aligned} \tag{17}$$

Let $\{G_1, \cdots, G_c\}$ be the set of the connected components of $G_Q(W^*)$. This induces a partition of the components of W into $W_1, \ldots, W_c$, and a partition of the classes of Q into $Q_1, \ldots, Q_c$. Each Q_i is a partition of the faces of W_i.

Notice in (17) that each of the factors, $N^{r-e(W')+f(W')} N^{\alpha(Q)-f(W')} = N^{v(W')-e(W')+\alpha(Q)}$, $|\mathrm{Aut}(W')|$, $|\mathrm{Aut}(G_Q(W'^*))|$, $z^r y^{e(W')-r} = z^{v(W')} y^{e(W')-v(W')}$ is multiplicative under disjoint union of (W_i, Q_i). We have $r!/|\mathrm{Aut}(W)|$ ways to relabelling the vertices of W, which is equal to

$$\binom{r}{r_1, r_2, \cdots, r_c} \times \frac{r_1! \times r_2! \times \cdots \times r_c!}{|\mathrm{Aut}(W_1)| \times |\mathrm{Aut}(W_2)| \times \cdots \times |\mathrm{Aut}(W_c)|},$$

that is, the number of ways of choosing vertex sets for each W_i and labelling W_i. Thus we can use the relation between the generating function for not-necessarily connected representations and the exponent of the generating function for connected ones to obtain

$$S_2(N,y,z) =_{\frac{N}{2}} \exp(T_2(N,y,z)) \tag{18}$$

with

$$T_2(N,y,z) = \sum_{r \geq 0} \sum_{(W,Q)_R^c} \sum_{W' \in S_{(W,Q)_R^c}} N^{r-e(W')+f(W')} N^{\alpha - f(W')} \frac{|\mathrm{Aut}(W')|}{|\mathrm{Aut}(G_Q(W'^*))|} \frac{z^r y^{e(W')-r}}{r!}$$

where the second sum is over all representations $(W,Q)_R^c$ where $G_Q(W^*)$ is *connected*. The bound $\frac{N}{2}$ in (18) is imposed, since in S_2 the total degree in y and z (which equals $e(W') = e(W)) \leq \frac{N}{2}$ implies that $f(W) \leq N$. Observe that for any nimple graph, $f(W) \leq 2e(W)$.

Since $\sum_{W'\in S_{(W,Q)^c_R}} 1 = r!/|\mathrm{Aut}(W')|$, we have

$$\begin{aligned}
&T_2(N,y,z)\\
&=\sum_{r\geq 0}\sum_{(W,Q)^c_R} N^{r-e(W)+f(W)}N^{\alpha(Q)-f(W)}\frac{z^r y^{e(W)-r}}{|\mathrm{Aut}(G_Q(W^*))|}\sum_{W'\in S_{(W,Q)^c_R}}\frac{|\mathrm{Aut}(W')|}{r!}\\
&=\sum_{r\geq 0}\sum_{(W,Q)^c_R} N^{r-e(W)+f(W)}N^{\alpha(Q)-f(W)}\frac{z^r y^{e(W)-r}}{|\mathrm{Aut}(G_Q(W^*))|}\\
&=\sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r}\sum_{(W,Q)^c_R} N^{2-2g(W)}N^{\alpha(Q)-f(W)}\frac{1}{|\mathrm{Aut}(G_Q(W^*))|},
\end{aligned}\tag{19}$$

where W has r vertices, e edges, and $\leq N$ faces.

We recall that for each $r\geq 0$,

$$2-2g(W)+\alpha(Q)-f(W)=v(W)-e(W)+\alpha(Q)=r-e(G_Q(W^*))+v(G_Q(W^*))$$

is independent of the choice of the representation $(W,Q)_R$.

If $(W,Q)^c_R$ has $g(W)>0$, or $g(W)=0$ and Q is non-trivial, then

$$N^{-2g(W)}N^{\alpha(Q)-f(W)}=O(N^{-1}).$$

The contribution of such representations to $T_2(N,y,z)$ is

$$N^2\sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r}\quad E_2(N,r,e),\tag{20}$$

where

$$E_2(N,r,e)\qquad =\sum\nolimits^{\mathrm{err}}_{[(W,Q)^c]} O(N^{-1})\frac{1}{|\mathrm{Aut}(G_Q(W^*))|},\tag{21}$$

where the sum is over all the equivalence classes $[(W,Q)^c]$ of the r-relevant pairs where $G_Q(W^*)$ is connected, W has e edges, and $g(W)>0$ or Q non-trivial.

On the other hand, for the remaining representations $(W,Q)^c_R$ (that is, $g(W)=0$ and Q is trivial),

$$N^{-2g(W)}N^{\alpha(Q)-f(W)}=1.$$

Hence the contribution of such representations to $T_2(N,y,z)$ is

$$N^2\sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r}\sum_{[(W,\emptyset)^c]\,planar}\frac{1}{|\mathrm{Aut}(W^*)|}.\tag{22}$$

We can also sum over the duals in (22). The dual of a *connected* fat graph is *connected* and the dual of a *relevant* fat graph is *nimple*. Moreover, there is exactly one nimple graph on at most one vertex (with no edge) and at most one simple connected graph with one vertex and no edge. Hence (22) is equal to

$$N^2\sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r}\sum_{[G_{r,e}]}\frac{1}{|\mathrm{Aut}(G_{r,e})|},\tag{23}$$

where the sum is over all isomorphism classes $[G_{r,e}]$ of *connected simple* graphs which have planar embedding (we recall that Q is trivial) with r faces, e edges and $\leq N$ vertices.

Putting (18), (20), and (23) together, we obtain that

$$S_2(N,y,z) =_{\frac{N}{2}} \exp\left(N^2 \sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r}\left(E_2(N,r,e) + \sum_{[G_{r,e}]} \frac{1}{|\mathrm{Aut}(G_{r,e})|}\right)\right). \tag{24}$$

The analysis for $S_1(N,y,z)$ is analogous as to $S_2(N,y,z)$; we only note that the factor $\left(1-\frac{2s(N)}{N}\right)^{\alpha(Q)}$ is multiplicative. We obtain that

$$S_1(N,y,z) =_{\frac{N}{2}} \exp\left(N^2 \sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r}\left(E_1(N,r,e) + \sum_{[G_{r,e}]} \frac{1}{|\mathrm{Aut}(G_{r,e})|}\right)\right),$$

where

$$\begin{aligned} E_1(N,r,e) \quad &= \sum\nolimits^{\mathrm{err}}_{[(W,Q)^c]} O(N^{-1})\left(1-\frac{2s(N)}{N}\right)^{\alpha(Q)} \frac{1}{|\mathrm{Aut}(G_Q(W^*))|} \\ &+ \sum_{[G_{r,e}]}\left(-1+\left(1-\frac{2s(N)}{N}\right)^{\alpha(Q)}\right)\frac{1}{|\mathrm{Aut}(G_{r,e})|}, \end{aligned} \tag{25}$$

where the sum in the first term is as in $E_2(N,r,e)$, and the sum in the second term is as in (23). □

Proof. (of Theorem 3) From Corollary 4.1 we have that

$$\begin{aligned} &\exp\left(\sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r}\left(E_1(N,r,e) + \sum_{[G_{r,e}]} \frac{1}{|\mathrm{Aut}(G_{r,e})|}\right)\right)\ '\leq_{s(N)}\ ' \\ &e^{-N^2}\langle \eta(M, Nzy^{-1}, y)\rangle \\ &'\leq_{\frac{N}{2}}\ '\ \exp\left(\sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r}\left(E_2(N,r,e) + \sum_{[G_{r,e}]} \frac{1}{|\mathrm{Aut}(G_{r,e})|}\right)\right). \end{aligned} \tag{26}$$

We first study the leading term $\sum_{[G_{r,e}]} 1/|\mathrm{Aut}(G_{r,e})|$ in (26). We note that to each isomorphism class $[G_{r,e}]$ of connected simple graphs which have planar embeddings with r faces, e edges, and $v(G_{r,e}) \leq N$ vertices, there are $v(G_{r,e})!\ /\ |\mathrm{Aut}(G_{r,e})|$ many labelled corresponding ones. From this, together with the Euler relation, $v(G_{r,e}) - e + r = 2$, we have that

$$\begin{aligned}
&\sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r} \sum_{[G_{r,e}]} \frac{1}{|\mathrm{Aut}(G_{r,e})|}\\
&= \sum_{r\geq 0}\sum_{e\geq 0} \sum_{G_{r,e} \text{ labelled}} \frac{z^r y^{v(G_{r,e})-2}}{|\mathrm{Aut}(G_{r,e})|} \times \frac{|\mathrm{Aut}(G_{r,e})|}{v(G_{r,e})!}\\
&= \sum_{r\geq 0}\sum_{e\geq 0} \sum_{G_{r,e} \text{ labelled}} \frac{z^r y^{v(G_{r,e})-2}}{v(G_{r,e})!}. \qquad (27)
\end{aligned}$$

Being sorted by a possible number n of vertices, (27) is equal to

$$\sum_{n=1}^{N}\sum_{r\geq 0} p(n,r)\frac{z^r y^{n-2}}{n!},$$

where $p(n,r)$ denotes the number of labelled connected simple graphs on n vertices that have planar embeddings with r faces.

Concerning the subleading term $E_2(N,r,e)$ in (26), due to the Euler relation, $e-r = f(W)+2g(W)-2$, we have

$$\begin{aligned}
&\sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r} E_2(N,r,e)\\
&\overset{(21)}{=} \sum_{r\geq 0}\sum_{e\geq 0} z^r y^{e-r} {\sum}^{\mathrm{err}}_{[(W,Q)^c]} \frac{O(N^{-1})}{|\mathrm{Aut}(G_Q(W^*))|}\\
&= \sum_{r\geq 0} z^r y^{f(W)+2g(W)-2} {\sum}^{\mathrm{err}}_{[(W,Q)^c]} \frac{O(N^{-1})}{|\mathrm{Aut}(G_Q(W^*))|}. \qquad (28)
\end{aligned}$$

If (28) is sorted by a possible number n where $n = f(W)+2g(W)$, it is equal to

$$\sum_{n\geq 1}\sum_{r\geq 0} e_2(N,n,r)\frac{z^r y^{n-2}}{n!},$$

with

$$e_2(N,n,r) := {\sum}^{\mathrm{err}}_{[(W,Q)^c]} O(N^{-1})\frac{n!}{|\mathrm{Aut}(G_Q(W^*))|},$$

where the sum is over all equivalent classes $[(W,Q)^c]$ of the r-relevant pairs where $G_Q(W^*)$ is connected, W satisfies $n = f(W)+2g(W)$, and $g(W) > 0$ or Q non-trivial. Note that the sum is finite, since $e(W) = n+r-2$.

The analysis of $E_1(N,r,e)$ and $e_1(N,n,r)$ is analogous. □

5. Counting directed cycle double covers (Proof of Theorem 4)

In this section we prove Theorem 4. To this end we first formulate the enumeration of graphs with a *specified* directed cycle double cover (DCDC) as the Gaussian matrix integral of the generating function ξ for even sets with its specified cycle decomposition (defined in (2)). We further show that ξ is indeed an Ihara-Selberg type function, via a lemma on coin arrangements (Lemma 5.5).

First we recall from the introduction that a subset A of directed edges of D is *even* if A can be written as a union of edge-disjoint directed cycles of length bigger than two. We further denote by $\mathcal{A}$ the set of all pairs (q, K) where q is an even set of edges of D and K is a decomposition of q into directed cycles of length at least three. The following observation is straightforward.

Observation 5.1. *Let $(q, K) \in \mathcal{A}$ and let P be a proper pairing of q. Then the cycles of K form a DCDC of a nimple graph with vertices $\{1, \dots, N\}$ and the edges given by the pairing P.*

Definition 5.1. *Let G be a finite nimple graph with at most N vertices and let C be a DCDC of G. Then let $c(G, C)$ be the set of all triples (q, P, C) so that there is a colouring d of the vertices of G by colours $\{1, \dots, N\}$, where each vertex gets a different colour,*

$$q = \{(d(x), d(y)), (d(y), d(x)); \{x, y\} \in E(G)\},$$

and P consists of all the pairs $([(d(x), d(y)), (d(y), d(x))]; \{x, y\} \in E(G))$.

We remark that each such pair (q, C) in Definition 5.1 belongs to $\mathcal{A}$ and that

$$|c(G, C)| \ = \ \frac{N(N-1)\cdots(N-|V(G)|+1)}{|\mathrm{Aut}(G, C)|}.$$

Proposition 5.2. *For the function ξ defined in (2) which maps $M \in \mathcal{H}_N$ to*

$$\xi(M) := \sum_{(q,C)\in\mathcal{A}} \prod_{e\in q} M_e,$$

a term $((q, C), P)$ with P a proper pairing of q contributes to $\langle \xi \rangle$ if and only if there is a nimple graph G with at most N vertices where C is its DCDC and such that $(q, P, C) \in c(G, C)$.

Proof. If $(q, P, C) \in c(G, C)$, then C provides a partition of q into its directed cycles and hence $((q, C), P)$ contributes to $\langle \xi \rangle$. On the other hand if $((q, C), P)$ contributes to $\langle \xi \rangle$, then letting G be the graph with the vertices from $\{1, \dots, N\}$ and the edges given by P we get that G is nimple. Moreover q comes with a decomposition C into the directed cycles and thus $(q, P, C) \in c(G, C)$. □

Proposition 5.3. *If $c(G, C) \cap c(G', C') \neq \emptyset$, then (G, C) is isomorphic to (G', C'). Moreover, if (G, C) is isomorphic to (G', C'), then $c(G, C) = c(G', C')$.*

Proof. If $(q, P, C) \in c(G, C) \cap c(G', C')$, then the construction of q induces a function between the sets of vertices of G and G', and P gives the edges of both G, G'. Hence there is an isomorphism from G to G' which preserves the fixed DCDCs. The second part is true since the definition of $c(G, C)$ does not depend on 'names' of the vertices. $\square$

As a consequence we have the following.

Theorem 9. *Let ξ be the function defined in (2). Then*

$$\langle \xi \rangle = \sum_{[(G,C)]} \frac{N(N-1)\dots(N-|V(G)|+1)}{|\mathrm{Aut}(G,C)|N^{e(G)}},$$

where the sum is over all isomorphism classes of pairs (G, C) where G is a nimple graph with at most N vertices and C a specified DCDC *of G.*

5.1. **Calculations.** The integral $\langle \xi \rangle$ counts all the directed cycle double covers of graphs on at most N vertices and hence its calculation is an attractive task which need not be hopeless. We show next a curious formula for $\xi(M)$ which identifies it with an Ihara-Selberg-type function (see Theorem 10).

Construction of digraph D'. We first construct a directed graph D' with the weights on the *transitions between the edges.*

First we split each vertex of D, i.e., we replicate each vertex v, and then we connect two v's by a new edge $e(v)$ and we let all the edges of D entering v enter the initial vertex of $e(v)$, and all the edges of D leaving v leave the terminal vertex of $e(v)$. If an edge g_1 enters v in D then we define the weight of the transition $w(g_1, e(v)) = M_{g_1}$. We let all the remaining transition be equal to one (see Figure 1, the first two parts).

Finally, for each pair g_1, g_2 of oppositely directed edges of D, say $g_1 = (uv), g_2 = (vu)$ we introduce a new vertex v_g (with $g = \{u, v\}$) and we let both g_1, g_2 pass through it; equivalently, we subdivide both g_1, g_2 by one vertex and identify this pair of vertices into a unique vertex called v_g (and thus we have new edges $(uv_g), (v_g v)$ from $g_1 = (uv)$, and new edges $(vv_g), (v_g u)$ from $g_2 = (vu)$) (see Figure 1, the last two parts).

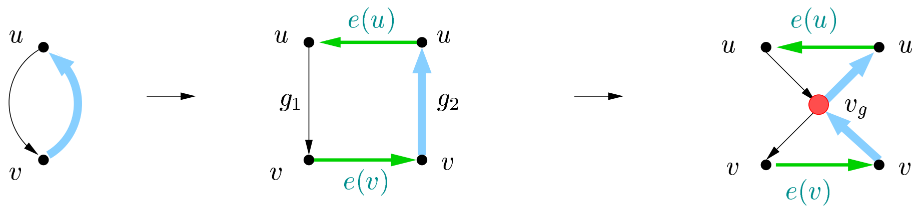

FIGURE 1. Construction of $e(v)$ and v_g

We let the weights of the transitions at the vertex v_g *between* g_1 and g_2 (i.e., between (uv_g) and $(v_g u)$ and between (vv_g) and $(v_g v)$) be equal to zero, the transitions *along* g_1 and g_2 (i.e., between (uv_g) and $(v_g v)$ and between (vv_g) and $(v_g u)$) be equal to one, and the transitions between $(v_g v)$ and $e(v)$ be equal to M_{g_1} and between $(v_g u)$ and $e(u)$ be equal to M_{g_2}. See an example in Figure 2.

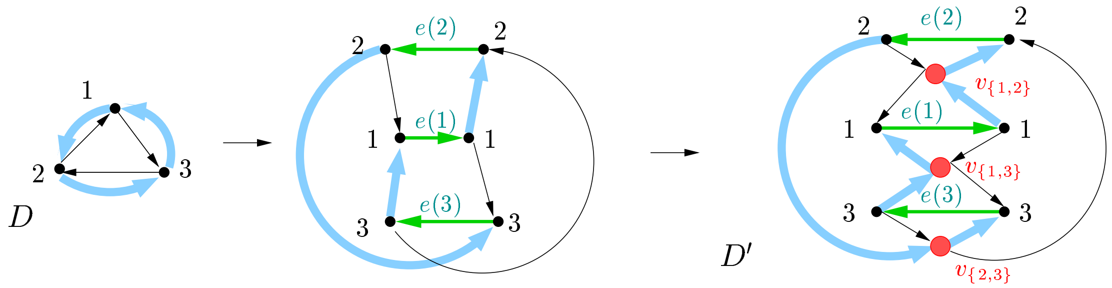

FIGURE 2. An example of the construction of D' from D

In what follows, the directed closed walk is considered *not pointed*. We let the weight of the directed closed walk be the product of the weights of its transitions.

Observation 5.2. *There is a weight preserving bijection between the set of the directed cycles of D of length at least three and a non-zero weight, and the set of the aperiodic directed closed walks of D' of a non-zero weight which go through each directed edge and through each vertex v_g at most once.*

Proof. This follows directly from the construction of D'. □

Definition 5.4. *We define the rotation number $\rho(w)$ for each closed walk w of D' with a non-zero weight by induction as follows: first order the directed edges of D', say as $a_1, \ldots, a_m$, so that the edges $e(v), v \in V(D)$ form the terminal segment. Then*

1. *If w is a directed cycle, then we let $\rho(w) = -1$.*
2. *Let w go at least twice through a directed edge. Let a be the first such edge in the fixed ordering. Hence w is a concatenation of two shorter closed walks w_1, w_2, both containing a. If $a \neq e(v)$ for some v then we let $\rho(w) = \rho(w_1)\rho(w_2)$. If $a = e(v)$, then we let $\rho(w) = 0$.*
3. *If none of 1.,2. applies, w must go through the vertex v_g (introduced in the definition of D') at least twice. Then we again let $\rho(w) = 0$.*

Theorem 10. *Let ξ be the function defined in (2). Then*

$$\xi(M) = \prod_p (1 - \rho(p)w(p)),$$

where the product is over all aperiodic directed closed walks p in D' and $w(p)$ denotes the weight of p.

To prove Theorem 10 we will need a curious lemma on coin arrangements stated below. It has been introduced by Sherman [29] in the study of 2-dimensional Ising problem.

Lemma 5.5 (A lemma on coin arrangements.)**.** *Suppose we have a fixed collection of N objects of which m_1 are of one kind, m_2 are of second kind, $\cdots$, and m_n are of n-th kind. Let b_k be the number of exhaustive unordered arrangements of these symbols into k disjoint, nonempty, circularly ordered sets such that no two circular orders are the same and none are periodic. For example let us have 10 coins of which 3 are pennies, 4 are nickles and 3 are quarters. Then $\{(p,n),(n,p),(p,n,n,q,q,q)\}$ is not a correct arrangement since (p,n) and (n,p) represent the same circular order. If $N>1$ then $\sum_{i=1}^{N}(-1)^{i+1}b_i=0$.*

Proof of Lemma 5.5. The lemma follows immediately if we expand the LHS of the following *Witt Identity* and collect terms where the sums of the exponents of the z_i's are the same.

Witt Identity (see [17]): Let $z_1,...,z_k$ be commuting variables. Then

$$\prod_{m_1,...,m_k\geq 0}(1-z_1^{m_1}...z_k^{m_k})^{M(m_1,...,m_k)}=1-z_1-z_2-...-z_k,$$

where $M(m_1,....,m_k)$ is the number of different nonperiodic sequences of z_i's taken with respect to circular order. □

Proof of Theorem 10. We first show that the coefficients corresponding to the products of variables where at least one $M_e, e\neq e(v)$, appears with the exponent greater than one, are all equal to zero.

Let us denote $W(p)=-\rho(p)w(p)$. Let A_1 be the set of all aperiodic closed walks p such that a_1 appears in p. Each $p\in A_1$ has a unique factorization into words $(W_1,...,W_k)$ each of which starts with a_1 and has no other appearance of a_1.

Let S be a monomial summand in the expansion of $\prod_{p\in A_1}(1+W(p))$. Hence S is a product of finitely many $W(p), p\in A_1$.

Each $p\in A_1$ has a unique factorization into words defined above. Each word may appear several times in the factorization of p and also in the factorization of different aperiodic directed closed walks. Let $B(D')$ be the set-system of all the words (with repetition) appearing in the factorizations of the aperiodic directed closed walks of D'.

It directly follows from Lemma 5.5, the lemma on coin arrangements, that the sum of all monomial summands S in the expansion of $\prod_{p\in A_1}(1+W(p))$, which have the same set of 'coins' of $B(D')$ with more than one element, is zero. Hence the monomial summands S which survive in the expansion of $\prod_{p\in A_1}(1+W(p)$ all have the set of coins of $B(D')$ consisting of exactly one word. Such S cannot have a_1 with exponent bigger than one. Now we can repeat the same consideration for the other edges different from $e(v), v\in V$.

Hence the only terms of the expansion of the infinite product that survive have all $M_e, e\neq e(v)$, with the exponent at most one.

We know from Observation 5.2 that the collections of the edge-disjoint directed cycles of length at least three in D correspond to the collections of the aperiodic directed closed walks of D' where each edge of D' and each vertex v_g appear at most once; by above, these exactly *have chance* to survive in the infinite product.

Each term of $\xi(M)$ may be expressed several times as a product of aperiodic directed closed walks of D', *but* only one such expression survives in the infinite product since if a closed walk goes through an edge $e(v)$ or through a vertex v_g more than once, its rotation is defined to be zero. Hence $\xi(M)$ is counted correctly in the infinite product. □

Remark 5.6. Let us write $\rho(p) = (-1)^{\mathrm{rot}(p)}$. Without the zero values of $\rho(p)$, the function $\mathrm{rot}(p)$ is additive when we 'smoothen' p into directed cycles. The integer lattice generated by the directed cycles has a basis which may be constructed e.g., from the ear-decomposition [11]; the function $\mathrm{rot}(p)$ may be split into contributions of the edge-transitions for the basis, and since it is a basis, it may be split into contributions of the edge-transitions also for all the directed cycles. Hence **if** the additivity property holds, $\mathrm{rot}(p)$ may be split into the contributions $\mathrm{rot}(t)$ of the edge-transitions t for the aperiodic closed walks. Hence we would have

$$\prod_p (1 - \rho(p)w(p)) = \prod_p (1 - \prod_{t \text{ transition of } p} (-1)^{\mathrm{rot}(t)} w(t)).$$

This formula transforms the infinite product into the *Ihara-Selberg function.* It was studied by Bass in [1] who proved that it is equal to a determinant. A combinatorial proof was given by Foata and Zeilberger in [10].

Having the zero values of $\rho(p)$ it is not clear how to split the rotation to individual edge-transitions. A determinant-type formula may however exist.

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