CHROMATIC POLYNOMIAL, Q-BINOMIAL COUNTING AND COLORED JONES FUNCTION

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Abstract. We define a q-chromatic function and q-dichromate on graphs and compare it with existing graph functions. Then we study in more detail the class of general chordal graphs. This is partly motivated by the graph isomorphism problem. Finally we relate the q-chromatic function to the colored Jones function of knots. This leads to a curious expression of the colored Jones function of a knot diagram $\mathcal{K}$ as a chromatic operator applied to a power series whose coefficients are linear combinations of long chord diagrams. Chromatic operators are directly related to weight systems by the work of Chmutov, Duzhin, Lando and Noble, Welsh.

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1. Introduction

The purpose of this paper has been a desire to recast the complicated combinatorial construction of the categorification of flows of [GL] in more common combinatorial terms.

A graph is a pair $G = (V, E)$ where V is a finite set of *vertices* and E is a set of unordered pairs of elements of V, called *edges*. If $e = uv$ is an edge then the vertices u, v are called *end-vertices* of e.

In this paper we study the following *q-chromatic function on graphs*:

Definition 1.1. Let $G = (V, E)$ be a graph and n a positive integer. Let $V = \{1, \dots, n\}$ and for $k \in \mathbb{P} = \{1, 2, \dots\}$ let $V(G, k)$ denote the set of all vertex colourings $s : V \to \{0, \dots, k-1\}$ such that $s(u) \neq s(v)$ whenever $uv \in E$.

$$M_q(G, k) = \sum_{s \in V(G,k)} q^{\sum_{v \in V} s(v)}.$$

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Note that $M_q(G,k)|_{q=1}$ is the classical chromatic polynomial of G.

1.1. **Basic Examples and Observations.** We first recall some notation:

For $k \in \mathbb{P}$, let $(k)_q = q^{k-1} + \cdots + q + 1$ denote a *quantum integer*, with the convention that $(0)_q = 0$, and let $(k)!_q = \prod_{1\le n\le k}(n)_q$, with the convention that $(0)!_q = 1$. For $0 \le n \le k$ the *quantum binomial coefficients* are defined by

$$\binom{k}{n}_q = \frac{(k)!_q}{(n)!_q(k-n)!_q}.$$

The following quantum binomial formula leads to a well-known formula for the summation of the products of distinct powers. We include a proof here in order to keep the paper essentially self-contained.

A simple quantum binomial formula

$$(a-z)(a-qz)\dots(a-q^{k-1}z) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}_q q^{i(i-1)/2}a^{k-i}z^i$$

leads to a well-known formula for the summation of the products of distinct powers (see for example [K]). This gives the q-chromatic function for the complete graph.

Observation 1. *For $k \in \mathbb{P}$, the q-chromatic function of the complete graph on $n \le k$ vertices is given by*

$$M_q(K_n,k) = n!\binom{k}{n}_q q^{n(n-1)/2}$$

and $M_q(K_n,k) = 0$ for $n > k$.

The next theorem provides a natural way to extend the q-chromatic function from positive integers to real numbers, by extending quantum integers $(k)_q$ to *quantum numbers* $(y)_q = \frac{q^y-1}{q-1}$ for real variables y and $q \ne 1$ (and $(y)_1 = y$ by continuity, $\lim_{q\to 1}(y)_q = y$.)

A graph $G = (V,E)$ is *connected* if it has a path between any pair of vertices. If a graph is not connected then its maximum connected subgraphs are called *connected components*. If $G = (V,E)$ is a graph and $A \subset E$ then let $C(A)$ denote the set of the connected components of graph (V,A) and $c(A) = |C(A)|$. If $W \in C(A)$ then let $|W|$ denote the number of vertices of W.

Theorem 1. *For $k \in \mathbb{P}$,*

$$M_q(G,k) = \sum_{A\subset E}(-1)^{|A|}\prod_{W\in C(A)}(k)_{q^{|W|}}.$$

Proof. We use the principle of inclusion and exclusion (PIE): If $\{I_e : e \in E\}$ are finite sets indexed by a finite set E and $\cap_{e\in A}I_e =: I_A$ then

$$|\cup_{e\in E} I_e| = \sum_{1\le a\le |E|}(-1)^{a-1}\sum_{A\subset E,|A|=a}|I_A|.$$

The next considerations connect the PIE with the geometric series formula.

$$M_q(G,k) = \sum_{s:V\to\{0,\dots,k-1\}} q^{\sum_{v\in V}s(v)} - \sum_{s\in\cup_{e\in E}I_e} q^{\sum_{v\in V}s(v)},$$

where $I_e, e = uv \in E$, denotes the set of functions $s : V \to \{0,\dots,k-1\}$ for which $s(u) = s(v)$.

By PIE this equals

$$\sum_{A\subset E}(-1)^{|A|}\sum_{s\in I_A} q^{\sum_v s(v)} =$$

$$\sum_{A\subset E}(-1)^{|A|}\prod_{W\in C(A)}\sum_{0\le i\le k-1} q^{i|W|} = \sum_{A\subset E}(-1)^{|A|}\prod_{W\in C(A)}(k)_{q^{|W|}}.$$

□

Next we extend the definition of the q-chromatic function to *q-dichromate*.

1.2. **q-Dichromate.** The following function called *dichromate* is extensively studied in combinatorics. It is equivalent to the Tutte polynomial.

$$B(G,x,y) = \sum_{A\subset E} x^{|A|} y^{c(A)}.$$

The formula of Theorem 1 leads naturally to a definition of *q-dichromate*.

Definition 1.2. For variables x, y with $y \in \mathbb{R}$,

$$B_q(G,x,y) = \sum_{A\subset E} x^{|A|} \prod_{W\in C(A)} (y)_{q^{|W|}}.$$

Note that $B_{q=1}(G,x,y) = B(G,x,y)$ and $M_q(G,k) = B_q(G,-1,k)$ by Theorem 1.

1.3. **Previous work.** Let $x_1, x_2, \ldots$ be commuting indeterminates and let $G = (V,E)$ be a graph. The q-chromatic function $M_q(G,y)$ restricted to non-negative integer y is the principal specialization of X_G, the *symmetric function generalisation of the chromatic polynomial* (see [S1], [S2]). This is defined as follows:

Definition 1.3.

$$X_G(x_0, x_1, \ldots) = \sum_{s\in\cup_{k\in\mathbb{N}} V(G,k)} \prod_{v\in V} x_{s(v)},$$

the sum over all proper colourings of G by $\{0,1,\ldots\}$.

Therefore $M_q(G,k) = X_G(x_i = q^i (0 \le i \le k-1), x_i = 0 (i \ge k))$.
Further we define *symmetric function generalisation of the bad colouring polynomial* (see [S1]):

Definition 1.4.

$$XB_G(t, x_0, x_1, \ldots) = \sum_{s:V\to\{0,1,\ldots\}} (1+t)^{b(s)} \prod_{v\in V} x_{s(v)},$$

where the sum ranges over ALL colourings of G by $\{0,1,\ldots\}$ and $b(s) := |\{uv \in E : s(u) = s(v)\}|$ denotes the number of monochromatic edges of f.

Noble and Welsh define in [NW] the *U-polynomial* (see Definition 1.5) and show that it is equivalent to XB_G. I. Sarmiento proved in [Sa] that the polychromate defined by Brylawski ([Bry]) is also equivalent to the U-polynomial.

Definition 1.5.

$$U_G(z, x_1, x_2 \ldots) = \sum_{A\subset E(G)} x(\tau_A)(z-1)^{|A|-|V|+c(A)},$$

where $\tau_A = (n_1 \ge n_2 \ge \cdots \ge n_l)$ is the partition of $|V|$ determined by the connected components of A, $x(\tau_A) = x_{n_1} \ldots x_{n_l}$.

The motivation for the work of Noble and Welsh is a series of papers by Chmutov, Duzhin and Lando ([CDL]). It turns out (see [NW]) that the U-polynomial evaluated at $z = 0$ and applied to the intersection graphs of chord diagrams satisfies the $4T-$relation of the weight systems. Hence the same is true for $M_q(G,k)$ for each $k \in \mathbb{P}$ since it is this particular evaluation of the U-polynomial:

Observation 2. *For $k \in \mathbb{P}$,*

$$M_q(G,k) = (-1)^{|V|} U_G(0, x_1, x_2, \ldots)|_{x_i = -(k)_{q^i}}.$$

On the other hand, it seems plausible that the q-dichromate determines the U-polynomial. If true, then the q-dichromate would provide a compact representation of the multivariate generalisations of the Tutte polynomial mentioned above. This is also interesting since Theorem 2 below states that the q-dichromate is equal to the partition function of the Potts model in a magnetic field.

It is not difficult to observe that Conjecture 1 below would imply that the q-dichromate determines the U-polynomial.

Let $\tau = (n_1 \geq n_2 \geq \dots n_k)$ be a partition of n. We let $c(\tau) = (c(\tau, y))_{y=0,1,\dots}$ be an infinite sequence of polynomials in q defined by

$$c(\tau, y) = \prod_{i=1}^{k} (q^{n_i y} + q^{n_i (y-1)} + \dots + 1).$$

Conjecture 1. *Only trivial rational linear combination of $c(\tau)$'s is identically zero.*

1.4. **Main results.** We study the q-dichromate with applications in statistical physics and knot theory in mind. In section 2 we introduce a q-generalisation of the partition function of the Potts model. We prove in Theorem 2 that the q-dichromate equals the partition function of the Potts model in a magnetic field. Such an observation is important for computer simulations of the q-dichromate. The Jones polynomial may be viewed via the Kauffman state sum as an evaluation of the Potts partition function. This leads to a conjecture whether our q-generalisation of the Jones polynomial is a knot invariant.

In section 3 we study the q-chromatic function of chordal graphs. A graph is *chordal* if no cycle of length bigger than three is induced. We are partly motivated by a seminal conjecture of Bollobas, Pebody and Riordan ([BR]): Is it true that the Tutte polynomial distinguishes almost all graphs? Replacing the Tutte polynomial by the q-dichromate should not be too rude a generalization in view of Theorem 2. The isomorphism problem for general graphs may be naturally embedded in the class of chordal graphs: if $G = (V, E)$ is a graph then we can construct a chordal graph $ch(G) = (V', E')$ defined by $V' = V \cup E$ and $E' = E \cup \{ve : v \text{ incident with } e \text{ in } G\}$. Clearly two graphs G, G' are isomorphic if and only if $ch(G), ch(G')$ are isomorphic. R. Stanley conjectures in [S1] that the symmetric function generalisation of the chromatic polynomial distinguishes all trees. We ask in Conjecture 2 whether the q-dichromate distinguishes all chordal graphs. An important consequence of Conjecture 2 would be that chordal graphs are edge and vertex reconstructable. The edge-reconstructability has already been proved by B. D. Thatte [TH].

Finally in section 4 we derive in Theorem 9 a new formula for the colored Jones function. It may be interesting to compare them with the Kontsevich integral expression for the colored Jones function. A description of the theory of Vassiliev knot invariants and weight systems may be found in [BN]. The formula of Theorem 9 may be briefly described as follows:

We start with the $R-$matrix state sum of the Jones polynomial (see [Tur]) and apply it to the n-cabling of a knot. We associate to each state σ a triple $(f_\sigma, S_\sigma, s_\sigma)$. The first element f_σ is a non-negative integer flow; you can imagine that the flow lives on the reduced knot diagram K, although it turns out to be more convenient to define it on the *arc graph* of the reduced knot diagram.

The second element S_σ is a set system on a set of $\sum_e f_\sigma(e)$ elements where the sum is over all 'jump-up' transitions e of K (which are later associated to 'red edges' of the arc graph). Each such set system is called f_σ*-structure*. The number of f_σ-structures for a flow f_σ is given by a product of binomial coefficients.

Finally, the third element s_σ of the triple is a vector of length $\sum_e f_\sigma(e)$ with entries in $\{0, \dots, n-1\}$.

We then represent S_σ as a long chord diagram D_σ (by *long chord diagram* we mean a system of intervals on a line with pairwise disjoint end-points) with $\sum_e f_\sigma(e)$ chords. Let G_{S_σ} be the intersection graph of the chords of D_σ, i.e. its vertices are the intervals and have edges between pairs of intervals with a non-empty intersection. Note that the difference from the intersection graph of a chord diagram is that here we also have edges between pairs of intervals where one contains the other.

Let us denote by $w(f, S)$ the sum of the state sum contributions of all states σ with $f_\sigma = f$ and $S_\sigma = S$. We observe that $w(f, S)$ may be written conveniently using a chromatic operator. For instance in Theorem 9 it is

$$w(f, S) = Z_{n,f}(t) M_t^{\tilde{\text{def}}}(G_S, n),$$

where $M_t^{\tilde{\text{def}}}$ is a 'defected' q-chromatic function and $Z_{n,f}(t)$ is a Laurent polynomial in t whose precise form is given in Theorem 9.

This leads to a curious expression of the colored Jones function of a knot diagram $\mathcal{K}$ as a power series whose coefficients are equal to a chromatic operator applied to linear combinations of long chord diagrams constructed from flows on reduced $\mathcal{K}$.

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2. q-Potts

It is well known that the dichromate counts several interesting things in statistical physics. Next we discuss the Potts and Ising partition functions, and their q-extensions.

2.1. Potts partition function.

Definition 2.1. Let $G = (V, E)$ be a graph, $k \in \mathbb{P}$ and $J_e = J_{uv}$ a weight (coupling constant) associated with edge $e = uv \in E$. The Potts model partition function is defined as

$$P^k(G, J_e) = \sum_{s:V\to\{0,\dots,k-1\}} e^{E(P^k)(s)},$$

where

$$E(P^k)(s) = \sum_{uv\in E} J_{uv}\delta(s(u), s(v)).$$

Following [FK, KF], we may write

$$P^k(G, J_e) = \sum_{s:V\to\{0,\dots,k-1\}} \prod_{uv\in E} (1 + z_{uv}\delta(s(u), s(v))) = \sum_{A\subset E} k^{c(A)} \prod_{uv\in A} z_{uv},$$

where $z_{uv} = e^{J_{uv}} - 1$. The RHS is sometimes called *multivariate Tutte polynomial*; see [S] for recent survey. If J_{uv} are all equal to x we get an expression of the Potts partition function in the form of the dichromate:

$$P^k(G, x) = \sum_s \prod_{uv\in E} e^{x\delta(s(u),s(v))} = \sum_{A\subset E} k^{c(A)}(e^x - 1)^{|A|} = B(G, e^x - 1, k).$$

What happens if we replace $B(G, e^x - 1, k)$ by $B_q(G, e^x - 1, k)$? It turns out that this introduces an additional external field to the Potts model.

Theorem 2.

$$\sum_{A\subset E} \prod_{W\in C(A)} (k)_{q^{|W|}} \prod_{uv\in A} z_{uv} = \sum_s q^{\sum_{v\in V} s(v)} e^{E(P^k)(s)},$$

where $z_{uv} = e^{J_{uv}} - 1$ as above.

Proof. Let $W(A, k)$ denote the set of all colourings $s : V \to \{0, \dots, k-1\}$ whose set of monochromatic edges contains A.

We have

$$P^k_q(G, J_e) = \sum_s q^{\sum_{v\in V} s(v)} \prod_{uv\in E} (1 + z_{uv}\delta(s(u), s(v))) =$$

$$\sum_s q^{\sum_{v\in V} s(v)} \sum_{A\subset E} \prod_{uv\in A} z_{uv}\delta(s(u), s(v)) =$$

$$\sum_{A\subset E} \sum_{s\in W(A,k)} q^{\sum_{v\in V} s(v)} \prod_{uv\in A} z_{uv} =$$

$$\sum_{A\subset E} \prod_{W\in C(A)} (k)_{q^{|W|}} \prod_{uv\in A} z_{uv}.$$

□

2.2. **Ising partition function.** The Ising partition function $Z(G)$ of a graph G is defined by:

$$Z(G, J_e) = \sum_{s:V \to \{-1,1\}} e^{E(Z)(s)},$$

where

$$E(Z)(s) = \sum_{uv \in E} J_{uv} s(u)s(v).$$

We immediately have

$$Z(G, J_e) = e^{-\sum_{uv \in E} J_{uv}} P^2(G, 2J_e).$$

Not surprisingly, the q-dichromate again adds an external field to the Ising partition function.

Corollary 2.2.

$$Z(G, x) = e^{-|E|x} B(G, e^{2x} - 1, 2)$$

and

$$\sum_s q^{\sum_{v \in V} s(v)} e^{E(Z)(s)} = q^{-3|V|x} e^{-|E|x} B_q(G, e^{2x} - 1, 2).$$

2.3. **Van der Waerden Theorem.** A remarkable fact about the Ising partition function is a theorem of Van der Waerden which expresses it using the generating function of even subgraphs. It is not hard to formulate its q-generalisation. We use the following notation:

$$sinh(x) = \frac{e^x - e^{-x}}{2}, cosh(x) = \frac{e^x + e^{-x}}{2}, th(x) = \frac{sinh(x)}{cosh(x)}.$$

Theorem 3.

$$\sum_{s:V \to \{-1,1\}} q^{\sum_{v \in V} s(v)} e^{\sum_{uv \in E} J_{uv} s(u) s(v)} =$$

$$\prod_{uv \in E} cosh(J_{uv}) \sum_{A \subset E} \prod_{uv \in A} th(J_{uv}) (q - q^{-1})^{o(A)} (q + q^{-1})^{|V| - o(A)},$$

where $o(A)$ denotes the number of vertices of odd degree in the subgraph (V, A).

The proof uses a lemma:

Lemma 2.3. *Let $G = (V, E)$ be a graph. Then*

$$\sum_{s:V \to \{-1,1\}} q^{\sum_{v \in V} s(v)} \prod_{uv \in E} s(u)s(v) = (q - q^{-1})^{o(E)} (q + q^{-1})^{|V| - o(E)}.$$

Proof. (of Lemma 2.3)

First note that if E is a cycle and s arbitrary then $\prod_{uv \in E} s(u)s(v) = 1$. Hence, we can delete from G any cycle without changing the LHS

$$\sum_s q^{\sum_{v \in V} s(v)} \prod_{uv \in E} s(u)s(v).$$

This reduces the proof to the case that E is acyclic. If E is a path, then it follows from the observation above that for s arbitrary, $\prod_{uv \in E} s(u)s(v) = 1$ if and only if s is constant on the end-vertices of E. Hence, we can delete from E any maximal path and replace it by the edge between its end-vertices, without changing the LHS. Hence it suffices to prove the proposition for the case that each component of G contains at most one edge. This is however simply true.

□

Proof. (of Theorem 3)

Using the identity

$$e^{x(s(u)s(v))} = cosh(x) + s(u)s(v)sinh(x),$$

we have:

$$\sum_s q^{\sum_{v \in V} s(v)} e^{\sum_{uv \in E} J_{uv} s(u) s(v)} =$$

$$\sum_s q^{\sum_{v\in V} s(v)} \prod_{uv\in E} [cosh(J_{uv}) + s(u)s(v)sinh(J_{uv})] =$$
$$\prod_{uv\in E} cosh(J_{uv}) \sum_s q^{\sum_{v\in V} s(v)} \prod_{uv\in E} [1 + s(u)s(v)th(J_{uv})] =$$
$$\prod_{uv\in E} cosh(J_{uv}) \sum_s q^{\sum_{v\in V} s(v)} \sum_{A\subset E} \prod_{uv\in A} s(u)s(v)th(J_{uv}) =$$
$$\prod_{uv\in E} cosh(J_{uv}) \sum_{A\subset E} \prod_{uv\in A} th(J_{uv})U(A),$$

where

$$U(A) = \sum_s q^{\sum_{v\in V} s(v)} \prod_{uv\in A} s(u)s(v).$$

Theorem 3 now follows from Lemma 2.3. □

2.4. **Kauffman state sum of the Jones polynomial.** We first show by a well-known construction how the Jones polynomial may be derived from the Potts partition function. We follow [VN]. Let $G = (V, E)$ be a plane directed graph which is an oriented knot diagram; G is 4-regular and each vertex (crossing) v has a *directed sign* $\mathrm{sign}(v)$ associated with it, and two arcs entering it and two arcs leaving it.

Given G, we construct a plane *signed graph* $\tilde{G} = (\tilde{V}, \tilde{E})$ as follows. First color the faces of G white and black so that neighbouring faces receive a different color. This is the *checkerboard* or *Tait* colouring of $\tilde{G}$. Let $b(v)$ be the *undirected sign* of crossing v, induced by this colouring (see e.g. [W] for the definition of $\mathrm{sign}(v), b(v)$).

Assume the outer face is white; then let $\tilde{V}$ be the set of the black faces, and two vertices are joined by an edge if the corresponding faces share a crossing. Note that $\tilde{G}$ is also a planar graph. An edge $e \in \tilde{E}$ is given a sign $b(e)$ defined to be equal to the undirected sign $b(v)$ of the crossing v shared by its endpoints (black faces in G).

Let us now describe what a *state* will be: we can split a vertex v of G so that the white faces incident with v are joined into one face and the black faces are disconnected, or vice versa. The first case corresponds in $\tilde{G}$ to deleting the edge whose endpoints share the crossing v, and the second case to keeping this edge. Let $|V| = n$. There are 2^n ways to split all the vertices of G: the ways are called *states*. After performing all the splittings of a state s, we are left with a set of disjoint non-self-intersecting cycles in the plane; let $S(s)$ denote their number. The state s will be identified with a function $s : V \to \{-1, 1\}$ where $s(v) = 1$ if the black faces are joined at v and $s(v) = -1$ if the white faces are joined at v. The following statement consists of Theorems 2.6 and 2.8 by Kauffman (see [K1]).

Theorem 4. *Let $G = (V, E)$ be an oriented knot diagram. The following function $f_K(G)$ is a knot invariant:*

$$f_K(G, A) = (-A)^{-3w(G)} \sum_s (-A^2 - A^{-2})^{S(s)-1} A^{\sum_{v\in V} b(v)s(v)},$$

where $w(G) = \sum_{v\in V} \mathrm{sign}(v)$ is the writhe of G. Moreover the Jones polynomial equals

$$J(G, A^{-4}) = f_K(G, A).$$

Each state s determines a subset of edges $\tilde{E}(s)$ of edges of $\tilde{G}$ which are not deleted by the splittings of s, and it is easy to see that this gives a natural bijection between the set of states and the subsets of edges of $\tilde{G}$. Moreover for each state s

$$\sum_{v\in V} b(v)s(v) = -\sum_{e\in\tilde{E}} b(e) + 2 \sum_{e\in\tilde{E}(s)} b(e).$$

The following proposition appears in [K2].

Proposition 2.4.

$$S(s) = 2c(\tilde{E}(s)) + |\tilde{E}(s)| - |\tilde{V}|,$$

where $c(\tilde{E}(s))$ denotes the number of connected components of $(\tilde{V}, \tilde{E}(s))$.

Proof. Note that $S(s) = f(\tilde{E}(s)) + c(\tilde{E}(s)) - 1$, where $f(\tilde{E}(s))$ denotes the number of faces of $(\tilde{V}, \tilde{E}(s))$. Hence the formula follows from the Euler formula for the planar graphs. □

Corollary 2.5.
$$f_K(G,A) = (-A)^{-3w(G)}(-A^2-A^{-2})^{-|\tilde{V}|-1}A^{-\sum_{e\in\tilde{E}} b(e)}\times \sum_s (-A^2-A^{-2})^{2c(\tilde{E}(s))}\prod_{e\in\tilde{E}(s)}(-A^2-A^{-2})A^{2b(e)}.$$

This obviously provides an expression for the Jones polynomial of a knot diagram which has the same undirected sign at each crossing (i.e. an alternating diagram) as a dichromate.

Question 1. Is
$$\sum_s \prod_{e\in\tilde{E}(s)}(-A^2-A^{-2})A^{2b(e)}\prod_{W\in C(\tilde{E}(s))}((-A^2-A^{-2})^2)_{q^{|W|}}$$
times an appropriate prefactor (equal to that for $f_K(G,A)$ when $q=1$) also a knot invariant?

3. q-Chromatic function of chordal graphs

Next we study chordal graphs, i.e. graphs such that each cycle of length at least four has a chord. The main result is Theorem 5 which gives a formula for $M_q^{\mathrm{def}}(G,k)$ where $k\in\mathbb{P}$, G is a chordal graph of a given 'shape', and $M_q^{\mathrm{def}}(G,k)$ is one of the 'defect' versions of the q-chromatic function (defined below) which appears in the study of the Jones polynomial in the next section.

Our motivation here partly comes from a conjecture of Bollobas, Pebody and Riordan ([BR]) which asks whether almost all graphs are determined by the Tutte polynomial, and by a conjecture of Stanley ([S1]) which asks whether X_G, the symmetric function generalisation of the chromatic polynomial, determines all trees. If the following conjecture is true, then q-dichromate provides a graph function close to Tutte polynomial (by Theorem 2, the q-dichromate is the partition function of Potts model in a magnetic field), which, with simple graph preprocessing, tells non-isomorphic graphs apart.

Conjecture 2. *Is it true that q-dichromate distinguishes all chordal graphs?*

A consequence of Conjecture 2 would be that chordal graphs are edge and vertex reconstructable. The edge-reconstructability has been proved by B. D. Thatte [TH]. An important feature of chordal graphs is their *tree structure*, and each tree is obviously chordal. Conjecture 2 for trees is weaker than the conjecture of Stanley mentioned above (here we assume validity of Conjecture 1), and it is still open. It was tested extensively by Mertens [MM], who used the equivalence of the q-dichromate to the partition function of the Potts model with a variable external magnetic field (recall Theorem 2). Mertens was able to verify Conjecture 2 for all trees up to 16 vertices.

Let $G=(V,E)$ be a chordal graph. We fix a linear ordering $v_1<\dots<v_n$ of the vertices of G so that for each i, vertex v_i is a simplicial vertex, i.e. its neighbourhood is complete in the subgraph induced by the vertices $v_1,\dots,v_i$. Such an order is called a perfect elimination order and it is well-known that the existence of such an order characterises chordal graphs.

A tree is an acyclic connected graph (W,F). We consider trees *rooted*, i.e. a vertex r is distinguished in each tree, and denote the rooted tree by $T=(W,F,r)$. For each vertex $x\neq r$ of T there is unique path $P(x)$ in T connecting x to r. The neighbour of x on $P(x)$ is called *predecessor of* x and denoted by $p(x)$. We say that a subtree of a rooted tree *starts* at its unique nearest vertex to the root.

Another well-known characterisation says that a graph is chordal if and only if it is an intersection graph of subtrees of a tree. Let $G=(V,E)$ be a chordal graph and $v_1<\dots<v_n$ the specified ordering of its vertices. Let $T=(W,F)$ be the tree whose subtrees 'represent' G, i.e. there are subtrees $T_v, v\in V$ of T so that T_u,T_v have a vertex in common if and only if uv is an edge of G. We can choose a vertex $r\in W$ arbitrarily as a root of T and define sets A_w,B_w for each $w\in W$ by $A_w=\{v\in V; T_v \text{ starts at } w\}$ and $B_w=\{v\in V; T_v \text{ contains but does not start at } w\}$. The sizes of these sets will be denotes by $a_w=|A_w|$, $b_w=|B_w|$.

Then these have the following properties:

1. The sets A_w are disjoint and $V=\cup_{w\in W}A_w$.
2. $B_w\subset\cup_{x\in P(w)}A_x$. In particular $B_r=\emptyset$.
3. if x,y,z are distinct vertices of T such that $x,z\in P(y), x\in P(z)$ and $v\in A_x\cap B_y$ then $v\in B_z$.
4. if $u\in A_x, v\in A_y$, where $x\neq y$ and $u<v$ then $y\notin P(x)$.

5. $uv \in E$ if and only if $\{u, v\} \subset A_w \cup B_w$ for some $w \in W$.

This leads to the following definition of a *tree structure.*

Definition 3.1. Let $T = (W, F)$ be a tree, $V = \{v_1 < \cdots < v_n\}$ an ordered set and $\{A(w), B(w) : w \in W\}$ be sets that satisfy the above properties 1.,2.,3.,4.,5. Then $(B_w : w \in W)$ is called a $(T, V, (A_w, b_w : w \in W))-$ tree structure (tree structure for short). The set of all structures is denoted by $\Sigma(T, V, (A_w, b_w : w \in W))$.

What distinguishes tree structures are the sets B_w. B_w is an arbitrary subset of $A_{p(w)} \cup B_{p(w)}$ of b_w elements. Hence we get the following observation.

Proposition 3.2. *The number of* $(T, V, (A_w, b_w : w \in W))$*-tree structures is* $\prod_{r \neq w \in W} \binom{a_{p(w)}+b_{p(w)}}{b_w}$.

Remark 3.3. On the other hand, each tree structure on $T = (W, F, r), V = \{v_1 < \cdots < v_n\}$ determines a set of subtrees $\{T_v : v \in V\}$ so that $T_u \cap T_v \neq \emptyset$ if and only if $u, v \in A_w \cup B_w$ for some $w \in W$ (i.e. ensuring condition (5) above holds), reversing the above construction of a tree structure from a chordal graph.

Definition 3.4. Let $S = (B_w : w \in W)$ be a $(T, V, (A_w, b_w : w \in W))$-tree structure, $k \in \mathbb{P}$ and $s : V \to \{0, \ldots, k-1\}$.

- Denote by $G(S)$ the unique chordal graph with tree structure S (see the remark above).
- For $v \in V$ belonging to A_w, let $m(S, v) = \#\{u \in A_w \cup B_w ; u < v\}$. Note that $m(S, v) = b_w + \#\{u \in A_w : u < v\}$. Hence $m(S, v)$ does not depend on the sets B_w but only on their sizes b_w, and we let $m(S, v) = m(v)$.
- For $v \in A_w$, let $\operatorname{def}(S, s, v) = \#\{u \in A_w \cup B_w : u < v, s(u) < s(v)\}$.

Finally, in the notation of section 1 let $V(G(S), k) = \{s : V \to \{0, \ldots, k-1\} :$ if $\{u, v\} \subset A_w \cup B_w$ for some $w \in W$ then $s(u) \neq s(v)\}$ denote the set of proper colourings of $G(S)$ by $\{0, \ldots, k-1\}$, and

$$M_q^{\operatorname{def}}(G(S), k) = \sum_{s \in V(G(S), k)} q^{\sum_{v \in V} s(v) - \operatorname{def}(S, s, v)}.$$

Theorem 5. *Let S be a tree structure, $G(S)$ the unique chordal graph with tree structure S. Then*

$$M_q^{\operatorname{def}}(G(S), k) = \prod_{v \in V}^{k} (k - m(v))_q.$$

Theorem 5 follows from the following Claim.

Claim. Let $G = G(S) = (V, E)$ be a chordal graph with tree structure S. Let v be the largest element of the linearly ordered set V and $V' = V - v$. Fix a partial proper colouring $s' : V' \to \{0, \ldots, k-1\}$ of G (i.e. a proper colouring of the induced graph on V'). Then

$$\sum_{s \in V(G, k) : s_{|V'} = s'} q^{s(v) - \operatorname{def}(S, s, v)} = (k - m(v))_q.$$

Proof. Suppose that $v \in A_w$. Then $m(v) = \#\{u \in A_w \cup B_w : u \neq v\} = \{u \in V : uv \in E\}$ since, for all $u \in V'$, $u < v$ and $uv \in E$ if and only if $u \in A_w \cup B_w$. Also, $\operatorname{def}(S, s, v) = \#\{uv \in E : s(u) < s(v)\}$. Let $\{s(u) : uv \in E\} = \{k_1 < k_2 < \cdots < k_{m(v)}\}$ and set $k_0 = -1$, $k_{m(v)+1} = k$. Then

$$\sum_{s \in V(G, k) : s_{|V'} = s'} q^{s(v) - \#\{uv \in E : s(u) < s(v)\}} = \sum_{0 \leq i \leq m(v)} \sum_{k_i < j < k_{i+1}} q^{j-i},$$

since s can extend s' by assigning any colour not in $\{k_1, \ldots, k_{m(v)}\}$ to the vertex v. The latter sum is equal to

$$\sum_{0 \leq i \leq m(v)} q^{k_i+1-i} \frac{q^{k_{i+1}-k_i-1} - 1}{q-1} = (q-1)^{-1} \sum_{0 \leq i \leq m(v)} (q^{k_{i+1}-i} - q^{k_i-(i-1)}),$$

and this telescopes to $\frac{q^{k-m(v)}-1}{q-1} = (k - m(v))_q$.

$\square$

Hence

$$\sum_{s\in V(G(S),k)} \prod_{v\in V} q^{s(v)-\mathrm{def}(S,s,v)}$$

is invariant for $S \in \Sigma(T, V, (A_w, b_w : w \in W))$. Note that the same is not true for the non-defected version: the path with three edges and the star with three edges, with their tree being the path on three edges, provide a counterexample.

4. A motivation from the quantum knot theory: colored Jones function

The motivation to the previous discussion comes from a study of the colored Jones function J_n which is a joint work with Stavros Garoufalidis in [GL]. The colored Jones function is the quantum group invariant of knots that corresponds to the $(n+1)$-dimensional irreducible representation of $\mathfrak{sl}_2$. In [GL] a non-commutative formula for the colored Jones function is presented.

Fix a generic planar projection $\mathcal{K}$ of an oriented knot with r crossings. Let c_i for $i = 1, \dots, r$ denote an ordering of the crossings of $\mathcal{K}$. Then $\mathcal{K}$ consists of r arcs a_i, which we label so that each arc a_i ends at the crossing i. We will single out a specific arc of $\mathcal{K}$ which we decorate by $\star$. Without loss of generality, we may assume that the crossings of a knot appear in increasing order, when we walk in the direction of the knot, and that the last arc is decorated by $\star$. We denote by K the long knot obtained by removing $\star$ from $\mathcal{K}$.

Given $\mathcal{K}$, we define a weighted directed graph $G_{\mathcal{K}}$. In a directed graph, each edge $e = uv$ is directed, making an arc (uv) starting at u and ending at v.

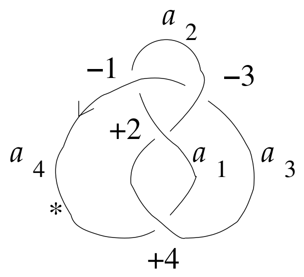
Definition 4.1. The *arc-graph* $G_{\mathcal{K}}$ has r vertices $1, \dots, r$, r *blue directed edges* $(v, v+1)$ (v taken modulo r) and r *red directed edges* (u, v), where at the crossing u the arc that crosses over is labeled by a_v.

The vertices of $G_{\mathcal{K}}$ are equipped with a sign, where $\mathrm{sign}(v)$ is the sign of the corresponding crossing v of $\mathcal{K}$, and the edges of $G_{\mathcal{K}}$ are equipped with a weight β, where the weight of the blue edge $(v, v+1)$ is $t^{-\mathrm{sign}(v)}$, and the weight of the red edge (u, v) is $1 - t^{-\mathrm{sign}(u)}$. Here t is a variable.

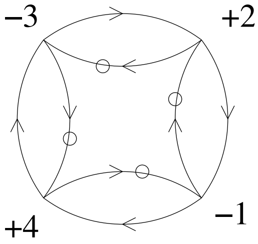
Finally, G_K denotes the digraph obtained by deleting vertex r from $G_{\mathcal{K}}$.

It is clear from the definition that from every vertex of $G_{\mathcal{K}}$, the blue outdegree is 1, the red outdegree is 1, and the blue indegree is 1. It is also clear that $G_{\mathcal{K}}$ has a Hamiltonian cycle that consists of all the blue edges. We denote by e_i^b (e_i^r) the blue (red) edge *leaving vertex* i.

Example 4.2. For the figure 8 knot we have:



Its arc-graph $G_{\mathcal{K}}$ with the ordering and signs of its vertices is given by

$G_{\mathcal{K}} =$ 

where the blue edges are the ones with circles on them.

Definition 4.3. A *flow* on a digraph $G = (V, E)$ is a function $f : E \to \mathbb{N}$ that satisfies Kirchhoff's conservation law

$$\sum_{e \text{ begins at } v} f(e) = \sum_{e \text{ ends at } v} f(e)$$

at all vertices v of G. Let $\mathcal{F}(G)$ denote the set of all flows of G.

If β is a weight function on the set of edges of G and f is a flow on G, then the *weight* $\beta(f)$ of f is given by $\beta(f) = \prod_e \beta(e)^{f(e)}$, where $\beta(e)$ is the weight of the edge e.

In order to express the Jones polynomial $J(\mathcal{K})(t)$ as a function of the reduced arc graph G_K, we need to add the following two structures, which may be read off from the knot diagram.

- We associate in a standard way a *rotation* $\mathrm{rot}(e)$ to each edge e of G_K; the exact definition is not relevant here; it may be found in [GL].
- We linearly order the set of edges of G_K terminating at the vertex $v \in \{1, \dots, r-1\}$ as follows: if we travel along the arc of K corresponding to the vertex v, we 'see' one by one the arcs corresponding to the starting vertices of the red edges entering v: this gives the linear order of the red edges entering v. Finally there is at most one blue edge entering v, and we make it smaller than all the red edges entering v. The linear order will be denoted by $<_v$ and let $P(e)$ denote the set of predecessors of an edge e in the corresponding linear order.

With these decorations we define

$$\mathrm{rot}(f) = \sum_{e \in E} f(e)\mathrm{rot}(e), \qquad \mathrm{exc}(f) = \sum_v \mathrm{sign}(v) f(e_v^b)(\sum_{e \in P(e_v^r)} f(e)), \qquad \delta(f) = \mathrm{exc}(f) - \mathrm{rot}(f).$$

A subgraph C of G is *admissible* if each component of C is a directed cycle. The set of all admissible subgraphs is denoted by $\mathcal{S}(G)$ and it may be identified with a finite subset of $\mathcal{F}(G)$ since the characteristic function of C is a flow.

Let $\mathcal{K}$ be a planar projection. The *writhe* of $\mathcal{K}$, $w(\mathcal{K})$, is the sum of the signs of the crossings of $\mathcal{K}$, and $\mathrm{rot}(K)$ is the *rotation number* of K, defined as follows: *smoothen* all crossings of $\mathcal{K}$, and consider the oriented circles that appear; one of them is special, marked by $\star$. The number of circles different from the special one whose orientation agrees with the special one, minus the number of circles whose orientation is opposite to the special one is defined to be $\mathrm{rot}(K)$. We further let $\delta(K, n) = 1/2(n^2 w(\mathcal{K}) + n\mathrm{rot}(K))$, and $\delta(K) = \delta(K, 1)$.

The following theorem appears in [LW] (see also [GL]); recall the definition of the weight function β in Definition 4.1.

Theorem 6.

$$J(\mathcal{K})(t) = t^{\delta(K)} \sum_{c \in \mathcal{S}(G_K)} t^{\delta(c)} \beta(c).$$

The colored Jones function $J_n(\mathcal{K})(t)$ is equal to the Jones polynomial of a proper 'cabling' of the knot diagram. In the language of graph theory, this may be described as follows.

Definition 4.4. Fix a red-blue digraph G. Let $G^{(n)}$ denote the digraph with vertices a_j^k for $k = 1, \dots, r$ and $j = 1, \dots, n$. $G^{(n)}$ contains blue directed edges (a_j^l, a_j^{l+1}) with weight $t^{-\epsilon n}$ (where $\epsilon \in \{-1, +1\}$ is the sign of the crossing l) for each $l = 1, \dots, r$ ($l+1$ considered modulo r) and $j = 1, \dots, n$. Moreover, if (a_k, a_l) is a red directed edge of G, then $G^{(n)}$ contains red edges (a_i^k, a_j^l) for all $i, j = 1, \dots n$ with weight $t^{(j-1)}(1-t)$ (resp. $t^{-(n-j)}(1-t^{-1})$), if the sign of the i crossing is -1 (resp. $+1$).

We will denote the set of admissible subgraphs of $G^{(n)}$ by $\mathcal{S}_n(G)$. The following theorem appears in [GL] as Theorem 10.

Theorem 7. *For every knot diagram $\mathcal{K}$ and every $n \in \mathbb{N}$, we have*

$$J_n(\mathcal{K})(t) = t^{\delta(K,n)} \sum_{c \in \mathcal{S}_n(G_K)} t^{\delta(c)} \beta(c).$$

Recall from 1.1 the definition of quantum integers and quantum binomial coefficients.

For an arc-graph $G_{\mathcal{K}} = (V, E)$ of an oriented knot $\mathcal{K}$ and $f : E \to \mathbb{N}$ define

$$\mathrm{mult}_q(f) = \prod_v \binom{f(v)}{f(e_v^b)}_{q^{\mathrm{sign}(v)}}.$$

One of the key propositions of [GL] (Theorem 7) is the following expression of the colored Jones function (as a deformed zeta function of the reduced arc graph).

Theorem 8. *For any oriented knot diagram $\mathcal{K}$ we have:*

$$J_n(\mathcal{K})(t) = t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} \mathrm{mult}_t(f) t^{\delta(f)} \prod_{v\in\{1,\dots,r-1\}} t^{-\mathrm{sign}(v)nf(e^b_v)} \prod_{e=(uv),\ e\ red} \prod_{j=0}^{f(e)-1} (1-t^{-\mathrm{sign}(u)(n-j-\sum_{e'<_v e} f(e))}).$$

The proof of Theorem 8 is based on a rather complicated combinatorial construction. In this paper we present a curious interpretation of this construction as a chromatic operator applied to a power series whose coefficients are linear combinations of chord diagrams.

Definition 4.5. Given a reduced knot diagram K and a flow f on G_K, we define:

- A collection $I_1,\dots,I_p$ of subintervals of $\{1,\dots,r-1\}$ is *relevant* (for K,f) if for each $1\le v\le r-1$, the number of intervals starting at v equals $\sum\{f(e) : e=(u,v)$ a red edge ending at $v\}$, and the number of intervals ending at v equals $f(e^r_v)$. We fix an order on the intervals of the relevant collection starting in the same vertex v, according to $<_v$.
- Each relevant collection of intervals defines a set of *long chord diagrams* as follows:

  For each $1\le v\le r-1$ we introduce ordered vertices $v_1<\dots<v_{o(v)}<v^1<\dots<v^{i(v)}$ where $o(v)$ is the number of intervals starting at v and $i(v)$ is the number of intervals ending at v.

  We assume that all the new vertices appear in the introduced order along a line. For each interval we introduce a *chord* on this line. The chords corresponding to intervals starting at v will start at $v_1,\dots,v_{o(v)}$, in agreement with the fixed ordering of the intervals. The chords corresponding to intervals terminating at v will terminate at $v^1,\dots,v^{i(v)}$, in *an arbitrary order*. If D is a resulting long chord diagram, then we denote by $\deg(D)$ the number of long chord diagrams obtained from the same relevant collection of intervals as D. We assume that the chords (intervals) in a long chord diagram are ordered by their starting vertices.
- We denote by $\Delta(K,f)$ the set of all long chord diagrams obtained in this way from a relevant collection of intervals for K,f.

Definition 4.6. Let D be a long chord diagram. We define intersection graph of its chords $G(D)=(D,E(D))$ so that the chords of D form the set of vertices of $G(D)$, and there is an edge between two vertices if the corresponding chords intersect or one contains the other.

Definition 4.7. Let D be a long chord diagram, , $c\in D$ and $s\in V(G(D),n)$.

- We denote by $P(D,c)$ the set of chords $c'\in D$ which encircle the starting vertex of c,
- $\mathrm{def}_1(D,s,c)$ equals the number of chords c' of $P(D,c)$ satisfying $s(c')<s(c)$,
- We denote by $Q(D,c)$ the set of chords $c'\in D$ which encircle the terminal vertex z of c and at least one starting vertex of a chord after z,
- $\mathrm{def}_2(D,s,c)$ equals the number of chords c' of $Q(D,c)$ satisfying $s(c')<s(c)$,
- $$M^{\tilde{\mathrm{def}}}_t(G(D),n) = \sum_{s\in V(G(D),n)} \prod_{c\in D} t^{s(c)-\mathrm{def}_1(D,s,c)-\mathrm{def}_2(D,s,c)}.$$

Remark 4.8. We remark that def_1 above is equivalent to the definition for a tree structure, see Definition 3.4. It is very close to the definition of sign for quantum determinants, first introduced by L. Fadeev, N. Reshetikhin and L. Takhtadjian in [FRT]. Indeed, Theorem 8 is used in [GL] to give a non-commutative formula for the colored Jones function.

We state the main theorem of this section below.

Theorem 9.

$$J_n(\mathcal{K})(t) = t^{\delta(K,n)} \sum_{f\in\mathcal{F}(G_K)} Z_{n,f}(t) \sum_{D\in\Delta(K,f)} \deg(D)^{-1} M^{\tilde{\mathrm{def}}}_t(G(D),n),$$

where $Z_{n,f}(t)$ is a Laurent polynomial in t with integer coefficients and parameters f and n:

$$Z_{n,f}(t) = t^{\delta(f)} t^{n(f^-_b-f^+_b)} (1-t)^{f^-_r} (1-t^{-1})^{f^+_r} \prod_{e\in F^+_r} t^{-(n-1-|P(f,e)|)} \prod_{v;\mathrm{sign}(v)=+} t^{f(e^r_v)f(e^b_v)}.$$

Moreover for each n only flows bounded by n may contribute a non-zero term to the RHS.
The terms $f^-_b, f^+_b, f^-_r, f^+_r, F^+_r$ are defined in 4.1 and the term $P(f,e)$ is defined in Definition 4.11.

4.1. **Categorification of flows: proof of Theorem 9.** Recall Theorem 7. Each $c \in \mathcal{S}_n(G_K)$ projects to a flow on G_K. An analysis of the contribution of each flow is obtained via the categorification of the flows and their multiplicities in [GL]. Next we briefly describe this.

Let f be a flow on G_K. Let F (resp. F_r) denote the multiset that contains each edge (resp. red edge) e of G_K with multiplicity $f(e)$.

Let F_r^+ denote the set of all red edges of F which leave a vertex with $+$ sign. Let $f_r^+ = |F_r^+|$. Analogously we define F_r^-, f_r^-.

If e is an edge of G_K then we let $F(e) \subset F$ be the set of all copies of e in F, and fix an *arbitrary* total order on each $F(e)$.

Definition 4.9. Fix a flow f on G_K. A *flow configuration* of f is a sequence $C = (C_1, \ldots, C_{r-2})$ so that C_1 is a subset of $\{e \in F_r; e \text{ terminates at vertex } 1\}$ of $f(e_1^b)$ elements and for each $2 \le i < r-1$, C_i is a subset of $C_{i-1} \cup \{e \in F_r; e \text{ terminates at vertex } i\}$ of $f(e_i^b)$ elements.

Let us denote by $\mathcal{C}(f)$ the set of all flow configurations of f.

Definition 4.10. Let f be a flow on G_K, $n > 0$ a natural number, $C \in \mathcal{C}(f)$, and $s : F_r \to \{0, \ldots, n-1\}$. A pair $P = (C, s)$, is *admissible* if, for every two edges $e, e' \in F_r$ such that $s(e) = s(e')$ and e ends at vertex i and e' ends at vertex j and $j \ge i$, there exists an l, $i \le l < j$ such that $e \notin C_l$. We denote the set of admissible flow configurations by $\mathcal{AC}(f)$.

Definition 4.11. Let $e' \in F(e)$. We define the set $P(f, e')$ as follows. Given $e_1' \in F(e_1)$ we say that $e_1' \in P(f, e')$ if $e_1 \in P(e)$ in G_K or $e = e_1$ and $e_1' < e'$ in our fixed total order of $F(e)$.

Definition 4.12. Let $s : F_r \to \{0, \ldots, n-1\}$ and $e \in F_r$ a red edge of G_K terminating at vertex $i \in \{1, \ldots, r-1\}$. We define

- $\mathrm{def}_1(C, s, e) = |\{e' \in P(f, e) : s(e') < s(e)\}|$,
- $\mathrm{def}_2(C, s, e) = |\{e' \in C_{d(e)} : s(e') < s(e)\}|$, where $d(e)$ is the largest index such that $d(e) \ge i$ and $e \notin C_{d(e)}$.

The following formulas appear in paragraph 8.3 of [GL].

Theorem 10.

- $$J_n(K)(t) = t^{\delta(K,n)} \sum_{f \in \mathcal{F}(G_K)} t^{\delta(f)} t^{n(f_b^- - f_b^+)} (1-t)^{f_r^-} (1-t^{-1})^{f_r^+}$$
$$\prod_{e \in F_r^+} t^{-(n-1-|P(f,e)|)} \prod_{v:\mathrm{sign}(v)=+} t^{f(e_v^r) f(e_v^b)} \sum_{(C,s) \in \mathcal{AC}(f,n)} \prod_{e \in F_r} t^{s(e) - \mathrm{def}_1(C,s,e) - \mathrm{def}_2(C,s,e)}.$$
- $$J_n(K)(t) = t^{\delta(K,n)} \sum_{f \in \mathcal{F}(G_K)} t^{\delta(f)} t^{n(f_b^- - f_b^+)} (1-t)^{f_r^-} (1-t^{-1})^{f_r^+} t^{-(n-1) f_r^+}$$
$$\prod_{e \in F_r^+} t^{|P(f,e)|} \prod_{v:\mathrm{sign}(v)=+} t^{f(e_v^r) f(e_v^b)} \prod_{v=1}^{r} \binom{f(v)}{f(e_v^b)}_{t^{-1}} \prod_{e \in F_r} (n - |P(f,e)|)_t.$$

From flow structures to relevant collections of intervals. The formula of Theorem 10 may be interpreted in terms of chordal graphs. The basic observation is that each flow structure is a $(T, V, (A_w, b_w : w \in W))-$ tree structure where T is a path with vertices $1, \ldots, r-1$ rooted at 1, $B_{w+1} = C_w$ and $A_w = \{e \in F_r; e \text{ terminates in vertex } w\}$. Now we recall that each chordal graph is the intersection graph of subtrees of a tree, and this representation may be obtained from its tree structure (see section 3). However, if a tree structure of a graph is a path, then the graph is the intersection graph of subpaths (intervals) of the path. This directly leads to the relevant collection of intervals. Hence the first part of Theorem 10 implies Theorem 9. The second part of Theorem 10 together with Theorem 5 implies an analogous expression for J_n in terms of the function $M_t^{\mathrm{def}}(G, k)$.

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