On the Theory of Pfaffian Orientations.
I. Perfect Matchings and Permanents.

[*]**Anna Galluccio**
Istituto di Analisi dei Sistemi ed
Informatica - CNR
viale Manzoni 30
00185 Roma
ITALY
galluccio@iasi.rm.cnr.it

[†]**Martin Loebl**
Department of Applied Mathematics
Charles University
Malostranske n. 25
118 00 Praha 1
CZECH REPUBLIC
loebl@kam.ms.mff.cuni.cz

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Abstract

Kasteleyn stated that the generating function of the perfect matchings of a graph of genus g may be written as a linear combination of 4^g Pfaffians. Here we prove this statement. As a consequence we present a combinatorial way to compute the permanent of a square matrix.

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1 Introduction

The theory of Pfaffian orientations of graphs has been introduced by Kasteleyn [7, 6, 5] in early sixties to solve some enumeration problems arising from statistical physics [4, 10]. He proved fundamental results in the planar case and extended his approach to toroidal grids [5, 6, 7]. The case of general toroidal graphs was also considered in an unpublished manuscript by Barahona [1].

In the present paper we extend the method proposed by Kasteleyn and we prove that the generating function of the perfect matchings of a graph of genus g may be obtained as a linear combination of 4^g Pfaffians. As a consequence, we provide a new technique to compute permanents of square matrices, which completes the scheme proposed by Pólya in [9].

A graph is a pair $G = (V, E)$ where V is a set of *vertices* and E is a set of unordered pairs of elements of V, called *edges*. In this paper we shall consider only graphs with finite number of vertices. If $e = xy$ is an edge then the vertices x, y are called *endvertices* of e. We associate with each edge e of G a variable x_e and we let $x = (x_e : e \in E)$. For each $M \subset E$, let $x(M)$ denote the product of the variables of the edges of M.

A graph $G' = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. A *perfect matching* of a graph is a set of disjoint edges, whose union equals the set of the vertices.

Let $\{v_1, e_1, v_2, e_2, ..., v_i, e_i, v_{i+1}, ..., e_n, v_{n+1}\}$ be a sequence such that each v_j is a vertex of a graph G, each e_j is an edge of G and $e_j = v_jv_{j+1}$, and $v_i \neq v_j$ for $i < j$ except if $i = 1$ and $j = n + 1$. If also $v_1 \neq v_{n+1}$ then P is called *a path* of G. If $v_1 = v_{n+1}$ then P is called *a cycle* of G. In both cases the *length* of P equals n. When no confusion arises we shall also denote paths by simply listing their edges, namely $P = (e_1, e_2, \ldots, e_n)$.

A graph $G = (V, E)$ is *connected* if it has a path between any pair of vertices, and it is 2-*connected* if the graph $G_v = (V - \{v\}, \{e \in E; v \notin e\})$ is connected for each vertex v of G. Each maximal 2-connected subgraph of G is called a 2-*connected component* of G.

Let $A\Delta B$ denote the symmetric difference of the sets A and B and let $a \stackrel{2}{=} b$ denote $a = b$ modulo 2.

Let M, N be two perfect matchings of a graph G. Then $M\Delta N$ consists of vertex disjoint cycles of even length. These cycles are called *alternating cycles* of M and N.

An *orientation* of a graph $G = (V, E)$ is a *digraph* $D = (V, A)$ obtained from G by fixing an orientation of each edge of G, i.e., by ordering the elements of each edge of G. The elements of A are called *arcs*.

Let C be a cycle of G and let D be an orientation of G. C is said to be *clockwise even* in D if it has an even number of edges directed in D in agreement with the clockwise traversal. Otherwise C is called *clockwise odd*.

Definition 1.1 *The generating function of the perfect matchings of G is the polynomial $\mathcal{P}(G, x)$ which equals the sum of $x(P)$ over all perfect matchings P of G.*

Definition 1.2 *Let G be a graph and let D be an orientation of G. Let M be a perfect matching of G. For each perfect matching P of G let $sgn(D, M\Delta P) = (-1)^n$ where n is the number of clockwise even alternating cycles of M and P, and let $\mathcal{P}(D, M)$ be the sum of $sgn(D, M\Delta P)x(P)$ over all perfect matchings P of G.*

Definition 1.3 *Let $G = (V, E)$ be a graph with $2n$ vertices and D an orientation of G. Denote by $A(D)$ the skew-symmetric matrix with the rows and the columns indexed by V, where $a_{vw} = x_{vw}$ in case (v, w) is an arc of D, $a_{vw} = -x_{vw}$ in case (w, v) is an arc of D, and $a_{vw} = 0$ otherwise.*

The Pfaffian of the skew-symmetric matrix $A(D)$ is defined as

$$Pf(A(D)) = \sum_P s^*(P) a_{i_1 j_1} \cdots a_{i_n j_n}$$

where $P = \{\{i_1 j_1\}, \cdots, \{i_n j_n\}\}$ is a partition of the set $\{1, \ldots, 2n\}$ into pairs, $i_k < j_k$ for $k = 1, \ldots, n$, and $s^*(P)$ equals the sign of the permutation $i_1 j_1 \ldots i_n j_n$ of $12 \ldots (2n)$.

Each nonzero term of the expansion of the Pfaffian of $A(D)$ equals $x(P)$ or $-x(P)$ where P is a perfect matching of G. If $s(D, P)$ denote the sign of the term $x(P)$, we have that

$$Pf(A(D)) = \sum_P s(D, P)x(P).$$

The following theorem was proved by Kasteleyn [5].

Theorem 1.4 *Let G be a graph and D an orientation of G. Let P, M be two perfect matchings of G. Then*

$$s(D, P) = s(D, M) sgn(D, M\Delta P).$$

Hence,

$$Pf(A(D)) = \sum_P s(D, P)x(P) = s(D, M) \sum_P sgn(D, M\Delta P)x(P) = s(D, M)\mathcal{P}(D, M).$$

The relevance of Pfaffians in our context lies in the fact that, despite their similarity with the permanent, they are polynomial time computable for skew-symmetric matrices (see [2]). In fact, see [7] for a proof.

Theorem 1.5 *Let G be a graph and let D be an orientation of G. Then*

$$Pf^2(A(D)) = det(A(D)).$$

In [5] Kasteleyn introduced the following notion:

Definition 1.6 *A graph G is called Pfaffian if it has a Pfaffian orientation, i.e., an orientation such that all alternating cycles with respect to an arbitrary fixed perfect matching M of G are clockwise odd.*

Hence if a graph G has a Pfaffian orientation D then the signs $s(D, P)$ are equal for all perfect matchings P of G and $\mathcal{P}(G, x)^2 = Pf^2(A(D)) = det(A(D))$.

An *embedding* of a graph on a surface is defined in a natural way: the vertices are embedded as points, and each edge is embedded as a continuous non-self-intersecting curve connecting the embeddings of its endvertices. The interiors of the embeddings of the edges are pairwise disjoint and the interiors of the curves embedding edges do not contain points embedding vertices.

A graph is called *planar* if it may be embedded on the plane. A *plane graph* is a planar graph together with its planar embedding. The embedding of a plane graph partitions the plane into connected regions called *faces*. The (unique) unbounded face is called *outer face* and the bounded faces are called *inner faces*.

Let G be a plane graph. A subgraph of G consisting of the vertices and the edges embedded on the boundary of a face will also be called *a face*. If a plane graph is 2-connected then each face is a cycle.

Kasteleyn [5] observed that the planar graphs have a Pfaffian orientation; more specifically, he proved that

Theorem 1.7 *Every plane graph has a Pfaffian orientation such that all inner faces are clockwise odd.*

Proof. Let G be a plane graph, and let M be its perfect matching. Each alternating cycle of M belongs to a 2-connected component of G.

Observe that G has an orientation so that each inner face of each 2-connected component of G is clockwise odd. Each such face 'encircles' no vertex of the corresponding 2-connected component. Let W be a 2-connected component of G. Observe that the orientation we constructed has the property that a cycle C of W is clockwise odd if and only if C encircles an even number of vertices of W. Let C be an alternating cycle of M and let W be a 2-connected component of G which contains C. Then C encircles an even number of vertices of W and hence it is clockwise odd. □

2 Embeddings and Pfaffian orientations

The *genus* g of a graph G is that of the orientable surface $\mathcal{S} \subset I\!R^3$ of minimal genus on which G may be embedded. Any orientable surface of genus g has a *polygonal representation* obtained by cutting the g handles of its space model. In what follows we base our working definition of a surface on this concept.

Definition 2.1 *A surface S_g of genus g consists of a base B_0 and $2g$ bridges B^i_j, $i = 1, ..., g$ and $j = 1, 2$, where* *i)* B_0 *is a convex* $4g$*-gon with vertices* $a_1, ..., a_{4g}$ *numbered clockwise;*

ii) B_1^i, $i = 1, \ldots, g$, *is a* 4*-gon with vertices* $x_1^i, x_2^i, x_3^i, x_4^i$ *numbered clockwise. It is glued with* B_0 *so that the edge* $[x_1^i, x_2^i]$ *of* B_1^i *is identified with the edge* $[a_{4(i-1)+1}, a_{4(i-1)+2}]$ *of* B_0 *and the edge* $[x_3^i, x_4^i]$ *of* B_1^i *is identified with the edge* $[a_{4(i-1)+3}, a_{4(i-1)+4}]$ *of* B_0*;*

iii) B_2^i, $i = 1, \ldots, g$, *is a* 4*-gon with vertices* $y_1^i, y_2^i, y_3^i, y_4^i$ *numbered clockwise. It is glued with* B_0 *so that the edge* $[y_1^i, y_2^i]$ *of* B_2^i *is identified with the edge* $[a_{4(i-1)+2}, a_{4(i-1)+3}]$ *of* B_0 *and the edge* $[y_3^i, y_4^i]$ *of* B_2^i *is identified with the edge* $[a_{4(i-1)+4}, a_{4(i-1)+5(mod 4g)}]$ *of* B_0*.*

Observe that in Definition 2.1 we denote by $[a, b]$ edges of polygons and not edges of graphs. The usual representation in the space of an orientable surface $\mathcal{S}$ of genus g may be then obtained from its polygonal representation S_g by the following operation: for each bridge B, glue together the two segments which B shares with the boundary of B_0, and delete B.

Definition 2.2 *A graph G is called a g-graph if it may be embedded on* S_g *so that all the vertices belong to the base* B_0*, and the embedding of each edge uses at most one bridge. The set of the edges embedded entirely on the base will be denoted by* E_0 *and the set of the edges embedded on the bridge* B_j^i *will be denoted by* E_j^i, $i = 1, \ldots, g$, $j = 1, 2$. *If a g-graph G satisfies the following further conditions:*

1. *the outer face of* $G_0 = (V, E_0)$ *is a cycle, and it is embedded on the boundary of* B_0*,*
2. *if* $e \in E_1^i$ *then e is embedded entirely on* B_1^i *and one endvertex of e belongs to* $[x_1^i, x_2^i]$ *and the other one belongs to* $[x_3^i, x_4^i]$*. Similarly, if* $e \in E_2^i$ *then e is embedded entirely on* B_2^i *and one endvertex of e belongs to* $[y_1^i, y_2^i]$ *and the other one belongs to* $[y_3^i, y_4^i]$*.*
3. *each vertex is incident with at most one edge which does not belong to* E_0*,*
4. G_0 *has a perfect matching,*

then we say that G is a proper g-graph.

Given a proper g-graph G, we denote by C_0 the cycle which forms the outer face of E_0; then, we fix a perfect matching of G_0 and denote it by M_0.

Definition 2.3 *Let G be a proper g-graph and let* $G_j^i = (V, E_0 \cup E_j^i)$*. If we draw* $B_0 \cup B_j^i$ *on the plane as follows:* B_0 *is unchanged, and the edge* $[x_1^i, x_4^i]$ *(*$[y_1^i, y_4^i]$ *respectively) of* B_j^i *is drawn so that it belongs to the external boundary of* $B_0 \cup B_j^i$*, we obtain a planar embedding of* G_j^i*. This embedding will be called planar projection of* E_j^i *outside* B_0*.*

Definition 2.4 *Let $G = (V, E)$ be a proper g-graph. A Pfaffian orientation D_0 of G_0 such that each inner face of each 2-connected component of G_0 is clockwise odd in D_0 is called a basic orientation of G_0.*

Note that a basic orientation always exists for a planar graph by Theorem 1.7.

Definition 2.5 *Let $G = (V, E)$ be a proper g-graph and D_0 a basic orientation of G_0. We define the orientation D^i_j of each G^i_j as follows: We consider G^i_j embedded on the plane by the planar projection of E^i_j outside B_0 (see Definition 2.3), and complete the basic orientation D_0 of G_0 to an orientation of G^i_j so that each inner face of each 2-connected component of G^i_j is clockwise odd.*
The orientation $-D^i_j$ is defined by reversing the orientation D^i_j of G^i_j.

Observe that after fixing a basic orientation D_0, the orientation D^i_j is uniquelly determined for each i, j.

Definition 2.6 *Let G be a proper g-graph, $g \geq 1$. An orientation D of G which equals the basic orientation D_0 on G_0 and which equals D^i_j or $-D^i_j$ on E^i_j is called relevant. We define its type $r(D) \in \{+1, -1\}^{2g}$ as follows: For $i = 0, \dots, g-1$ and $j = 1, 2$, $r(D)_{2i+j}$ equals $+1$ or -1 according to the sign of D^{i+1}_j in D.*

Definition 2.7 *Let G be a proper g-graph and let A be a subset of its edges. The type of A is a vector $t(A) \in \{0, 1\}^{2g}$ defined as follows: For $i = 0, ..., g-1$ and $j = 1, 2$, we let $t(A)_{2i+j}$ equals the number of edges of A which belong to E^{i+1}_j, modulo 2.*

Let $CR(A) \stackrel{2}{=} \sum_{i=0}^{g-1} t(A)_{2i+1} \cdot t(A)_{2i+2}$ denote the number of crossings of the embeddings of the edges of A, after making planar projections of E^i_j for all i, j.

Let $BR(A)$ denote the subset of edges of A which do not belong to E_0. For each $e \in BR(A)$, let $d(e) = 2i + j$ if $e \in E^{i+1}_j$.

We complete the section with a lemma.

Lemma 2.8 *Let G be a proper g-graph. Let $C_1, ..., C_k$ be vertex-disjoint cycles of G and let $\mathcal{C}$ denote their union. Then*

$$CR(\mathcal{C}) \stackrel{2}{=} \sum_{i=1}^{k} CR(C_i).$$

Proof. Let us embed the cycles $C_1, ..., C_k$ using the planar projections of E^i_j outside B_0 by Definition 2.7. Then $CR(\mathcal{C})$ equals the total number of crossings of $\mathcal{C}$ (modulo 2). Now, each cycle C_l, $l = 1, ..., k$ is represented as a closed curve in the plane and each pair of cycles C_i and C_j, $i \neq j$, intersects an even number of times. Hence the sum (modulo 2) of the number of crossings between pairs of cycles C_i and C_j, $i \neq j$, is 0 and does not affect $CR(\mathcal{C})$. Each of the remaining crossings is a crossing of some C_l, $l = 1, ..., k$, with itself and the lemma follows. $\square$

3 Perfect matchings

Through this section, the graph G will be a proper g-graph embedded on a fixed surface S_g. We also fix a perfect matching M_0 of G_0.

The aim of this section is to prove that, for any perfect matching P, the $sgn(D, M_0\,\Delta P)$ depends only on the vectors $t(M_0\Delta P)$ and $r(D)$.

Given an orientation D of G and an even length cycle C of G, we denote by $l_D(C)$ the number of arcs of C directed in agreement with any of the two possible ways of traversing C, modulo 2. For short, any alternating cycle with respect to M_0 will be simply called an *alternating cycle*. In order to prove our statement, we consider first the case that $M_0\Delta P$ consists of exactly one alternating cycle.

Theorem 3.1 *Let G be a proper g-graph and let D be a relevant orientation of G. If C is an alternating cycle of G, then*

$$l_D(C) \stackrel{2}{=} |BR(C)| - 1 - CR(C) + \frac{1}{2}\sum_{e\in BR(C)} (r(D)_{d(e)} + 1).$$

Proof. We assume without loss of generality that $G = C \cup C_0 \cup M_0$, where C_0 is the outer face of G_0 and M_0 is the fixed perfect matching of G_0. Let D_0 be the basic orientation of G_0.

Claim 1. *If C intersects at most one of E^i_1, E^i_2, for each $i = 1, ..., g$, then*

$$l_D(C) \stackrel{2}{=} |BR(C)| - 1 + \frac{1}{2}\sum_{e\in BR(C)} (r(D)_{d(e)} + 1).$$

A cycle C satisfying the properties of Claim 1 may be embedded without crossings using the planar projection of each E^i_j outside B_0. Hence $l_D(C) = 1$ if and only if $|\{e \in BR(C) : r(D)_{d(e)} = -1\}| \stackrel{2}{=} 0$. **End of Claim 1.**

The proof is by induction on $|BR(C)|$. The case $|BR(C)| = 0$ is proved by Claim 1. By induction we assume that

$$l_W(C') \stackrel{2}{=} |BR(C')| - 1 - CR(C') + \frac{1}{2}\sum_{e\in BR(C')} (r(W)_{d(e)} + 1)$$

for any alternating cycle C' of a proper g-graph H, with relevant orientation W, such that $|BR(C')| < |BR(C)|$.

We distinguish two cases.

Case 1. *There exists a bridge $B = B^i_j$ containing more than one edge of C.*
Let $e = u_1u_2$ and $f = v_1v_2$ be two edges of $C \cap E^i_j$ which see each other on B, i.e., there is no other edge of C drawn between them on B. Without loss of generality, let e be nearer to the edge $[a_{2(i-1)+j}, a_{2(i-1)+j+3}]$ of $B = B^i_j$ than f and let u_1, v_1 and u_2, v_2 belong to the edge $[a_{2(i-1)+j}, a_{2(i-1)+j+1}]$ and $[a_{2(i-1)+j+2}, a_{2(i-1)+j+3}]$, respectively. Since e, f do not belong to E_0, they are not edges of $M_0 \subset E_0$.

Let R_i be the subpath of C_0 from u_i to v_i, $i = 1, 2$, and let R be the cycle of G consisting of (R_1, f, R_2, e). By the choice of e, f, the cycle R is the boundary of a face of the planar projection of $G^i_j = (V, E_0 \cup E^i_j)$ outside B_0. Observe that $l_W(R) = 1$ for each relevant orientation W of G, since R contains two edges embedded outside B_0.

Let us introduce a new edge h (not belonging to G), between the endvertices of e, f such that one of two cycles $\bar{H}_1, \bar{H}_2$ formed by h and C and containing h is alternating. Without loss of generality, let h have u_1 as an endvertex. Hence we have that $h = u_1v_1$ or $h = u_1v_2$.

We may assume without loss of generality that $\bar{H}_2$ is alternating. Hence $\bar{H}_1$ contains both e, f. Note that $\bar{H}_1$ consists of an even number of edges. We denote by h_1, h_2 the two arcs with the same endvertices as h, directed opositely. Let $D' = D \cup \{h_1, h_2\}$. Let H_i be the subdigraph of D' which is the orientation of $\bar{H}_i$ using h_i, $i = 1, 2$. Observe that $l_D(C) = l_{D'}(H_1) + l_{D'}(H_2)$.

Subcase 1.1: $h_1 = u_1v_1$.

We adjust the boundary of B_0 by replacing $\{R_1\}$ with h_1, h_2. Observe that $CR(C) \overset{2}{=} CR(H_1) + CR(H_2)$: attention should be drawn to the question of how crossings of C with itself are manifested as crossings of H_1 or H_2, when all E^i_j are projected outside of B_0 (see Definition 2.3). If two edges of C cross and they are not separated in C by the endvertices of h_1, then that crossing counts as a crossing with in H_1 or H_2. We must therefore consider the parity of the number of crossings of C where the crossed edges are separated in C by the endvertices of h_1. These crossings are counted as crossings of H_1 with H_2. If the number of such crossings of C is odd, then there must be an additional crossing of H_1 with H_2, since the total number of crossings of H_1 with H_2 must be even. Since h_1 and h_2 do not cross, this additional crossing must occur at an endvertex of h_1. It is easy to see that in the present case there is no such crossing, and so, there are an even number of crossings of C where the crossed edges are separated in C by the ends of h. The required congruence therefore follows in this case.

We construct now two digraphs D_1, D_2 as follows:

- D_1 is obtained from $D - \{e, f\}$ by adding new vertices u'_1, v'_1 of degree 2, incident with new arcs e', f', h'_1. The arcs e', f', h'_1 are obtained from e, f, h_1 by replacing u_1 by u'_1 and v_1 by v'_1. We adjust the boundary of B_0 by replacing $\{R_2\}$ with $\{e', f', h'_1\}$. Finally we add h'_1 to M_0. Let H'_1 be the cycle of D_1 obtained from H_1 by replacing e, f, h_1 by e', f', h'_1. Then $l_{D_1}(H'_1) = l_{D'}(H_1)$ and $CR(H'_1) \overset{2}{=} CR(H_1)$;
- D_2 is obtained from $D - \{e, f\}$ by adding arc h_2. We remind that h_2 is embedded on the adjusted B_0 parallel to R_1. Let $H'_2 = H_2$. Then $l_{D_2}(H'_2) = l_{D'}(H_2)$ and $CR(H'_2) \overset{2}{=} CR(H_2)$.

We remind that $l_D(R) = 1$. Hence, exactly one of h_i is oriented so that both cycles it makes with R are clockwise odd. Let it be h_2. Then D_2 is a relevant orientation and D_1 becomes relevant after reversing the orientation of h'_1: this digraph, obtained from D_1 by reversing the orientation of h'_1, we denote by D^*_1, and its subdigraph corresponding to H'_1 we denote by H^*_1. Then, $l_{D^*_1}(H^*_1) \overset{2}{=} l_{D_1}(H'_1) + 1$.

Note that both D_2 and D_1^* are relevant orientations of proper g-graphs, H_2' is an alternating cycle of D_2, H_1^* is an alternating cycle of D_1^* and $CR(H_1^*) < CR(C)$ and $CR(H_2') < CR(C)$. Hence, by the induction assumption, we have that:

$$l_D(C) \stackrel{2}{=} l_{D'}(H_1) + l_{D'}(H_2) \stackrel{2}{=} l_{D_1}(H_1') + l_{D_2}(H_2') \stackrel{2}{=} l_{D_1^*}(H_1^*) + 1 + l_{D_2}(H_2') \stackrel{2}{=}$$

$$|BR(H_1^*)| - 1 - CR(H_1^*) + \frac{1}{2} \sum_{p \in BR(H_1^*)} (r(D_1^*)_{d(p)} + 1) +$$

$$|BR(H_2')| - 1 - CR(H_2') + \frac{1}{2} \sum_{p \in BR(H_2')} (r(D_2)_{d(p)} + 1) + 1.$$

Now, the theorem follows by observing that $|BR(C)| \stackrel{2}{=} |BR(C - \{e, f\})| \stackrel{2}{=} |BR(H_1^*)| + |BR(H_2')| - 2$, $CR(C) \stackrel{2}{=} CR(H_1^*) + CR(H_2')$ and $r(D_1^*)_{d(p)}$, $r(D_2)_{d(p)}$ and $r(D)_{d(p)}$ coincide for any $p \in BR(C) - \{e, f\}$. Hence,

$$l_D(C) \stackrel{2}{=} |BR(C)| - 1 - CR(C) + \frac{1}{2} \sum_{p \in BR(C)} (r(D)_{d(p)} + 1).$$

(End of Subcase 1.1)

Subcase 1.2: $h_1 = u_1 v_2$.

Let h_1 and h_2 be embedded on the bridge B. Observe that $CR(C) \stackrel{2}{=} CR(H_1) + CR(H_2) + 1$: attention again should be drawn to the question of how crossings of C with itself are manifested as crossings of H_1 or H_2, when all E_j^i are projected outside of B_0 (see Definition 2.3). To see this clearly, we introduce some notation. Let A be a subset of arcs of H_1 and B a subset of arcs of H_2. We denote by $CR(A \times B)$ the number of crossings between arcs of A and B, mod 2. We also denote by $cr(i, j)$ the number of crossings of arcs of $H_i \cap C$ with h_j. Hence, we have:

$$CR(H_1) \stackrel{2}{=} CR(H_1 \cap C) + cr(1, 1),$$
$$CR(H_2) \stackrel{2}{=} CR(H_2 \cap C) + cr(2, 2),$$
$$CR(C) \stackrel{2}{=} CR(H_1 \cap C) + CR(H_2 \cap C) + CR((H_1 \cap C) \times (H_2 \cap C)),$$
$$CR(H_1 \times H_2) \stackrel{2}{=} 0,$$

and

$$\sum_{i,j=1}^{2} cr(i, j) \stackrel{2}{=} 0$$

since each arc which crosses h_1 crosses also h_2.

Hence it remains to show that

$$CR(H_1 \times H_2) \stackrel{2}{=} CR((H_1 \cap C) \times (H_2 \cap C)) + cr(1, 2) + cr(2, 1) + 1 :$$

this follows since in this case one additional crossing between H_1 and H_2 must occur at an endvertex of h. The required congruence follows.

We construct two digraphs D_1, D_2 as follows:

- D_1 is obtained from $D - \{e, f\}$ by adding a new arc h_1' between v_1 and the endvertex u_2 of e. If $l_{D'}(fh_1e) = 1$ then we let $h_1' = (v_1, u_2)$. If $l_{D'}(fh_1e) = 0$ then we let $h_1' = (u_2, v_1)$.

  We consider h_1' embedded on the bridge B. Let H_1' be obtained from H_1 by replacing $\{f, h_1, e\}$ by h_1'. We have $l_{D'}(H_1) = l_{D_1}(H_1')$ and $CR(H_1) = CR(H_1')$.

- D_2 is obtained from $D - \{e, f\}$ by adding the arc h_2. We consider h_2 embedded on the bridge B. We let $H_2 = H_2'$. Thus again we have $l_{D'}(H_2) = l_{D_2}(H_2')$ and $CR(H_2) = CR(H_2')$.

We remind that $l_D(R) = 1$ and thus exactly one of h_i is oriented so that both cycles it makes with R are clockwise odd. Let it be h_2. Let R_3 be the subpath of C_0 from v_1 to v_2 such that (e, R_1, R_3, R_2) is a cycle. We have $l_{D_1}(h_1', R_3, R_2) \stackrel{2}{=} l_{D'}(e, h_1, f, R_3, R_2) \stackrel{2}{=} l_{D'}(f, R_3) + l_{D'}(e, h_1, R_2)$.

We show now that both D_1 and D_2 are relevant orientations with $r(D_1) = r(D_2) = r(D)$. We only need to show that h_1' and h_2 are correctly oriented in D_1 and D_2. This follows easily for D_2, since both cycles h_2 makes with R are clockwise odd.

For D_1 we distinguish two cases. First, let $r(D)_{2(i-1)+j} = 1$. In this case we have $l_{D'}(f, R_3) = 1$ and $l_{D'}(e, h_2, R_2) = 1$. Hence $l_{D'}(e, h_1, R_2) = 0$. It follows that $l_{D_1}(h_1', R_3, R_2) = 1$ and D_1 is relevant with $r(D_1) = r(D)$. Secondly, let $r(D)_{2(i-1)+j} = -1$. In this case we have $l_{D'}(f, R_3) = 0$ and $l_{D'}(e, h_2, R_2) = 1$. Hence $l_{D'}(e, h_1, R_2) = 0$. It follows that $l_{D_1}(h_1', R_3, R_2) = 0$ and D_1 is relevant with $r(D_1) = r(D)$.

Hence, D_i is a relevant orientation of a proper g-graph, H_i' is an alternating cycle of D_i and $|BR(H_i')| < |BR(C)|$, for $i = 1, 2$, and, by the induction hypothesis, we have that:

$$l_D(C) \stackrel{2}{=} l_{D'}(H_1) + l_{D'}(H_2) \stackrel{2}{=} l_{D_1}(H_1') + l_{D_2}(H_2') \stackrel{2}{=}$$

$$|BR(H_1')| - 1 - CR(H_1') + \frac{1}{2} \sum_{p \in BR(H_1')} (r(D_1)_{d(p)} + 1) + \frac{1}{2}(r(D_1)_{d(h_1)} + 1) +$$

$$|BR(H_2')| - 1 - CR(H_2') + \frac{1}{2} \sum_{p \in BR(H_2')} (r(D_2)_{d(p)} + 1) + \frac{1}{2}(r(D_2)_{d(h_2)} + 1).$$

The theorem follows by observing that $|BR(C)| \stackrel{2}{=} |BR(C - \{e, f\})| \stackrel{2}{=} |BR(H_1')| + |BR(H_2')| - 2$, $CR(C) + 1 \stackrel{2}{=} CR(H_1') + CR(H_2')$ and $r(D_1) = r(D_2) = r(D)$.

(End of Subcase 1.2)

End of Case 1

Case 2. *There exists i such that C contains exactly one edge from both E_1^i and E_2^i.* Let $e \in E_1^i$ and $f \in E_2^i$ and let C_1 and C_2 be two paths such that $C = (C_1, e, C_2, f)$. The endvertices of e, f belong to C_0. Let us assume that along the boundary of B_0 from $a_{4(i-1)+1}$ to a_{4i+1}, the endvertices of e, f appear in the order v_1, u_1, v_2, u_2 where $e = u_1u_2$ and $f = v_1v_2$.

Let R_1, R_2 be the two disjoint subpaths of the segment of C_0 between $a_{4(i-1)+1}$ and a_{4i+1}, which cover the endvertices of e, f. Note that R_1, R_2 contain no other vertex of G incident with an edge out of E_0, by the choice of i. Let R denote the cycle (R_1, e, R_2, f) and let R_3 denote the segment of C_0 between u_1 and v_2.

Let us introduce a new edge h (not belonging to G), between endvertices of e, f such that one of two cycles $\bar{I}_1, \bar{I}_2$ formed by h and C and containing h is alternating.

Without loss of generality let h have u_1 as an endvertex. Hence we have that $h = u_1v_1$ or $h = u_1v_2$. We may also assume without loss of generality that $\bar{I}_2$ is alternating. Hence $\bar{I}_1$ contains both e, f. Note that $\bar{I}_1$ consists of an even number of edges.

We denote by h_1, h_2 the two arcs with the same endvertices as h, directed opositely. Let $D' = D \cup \{h_1, h_2\}$. Let I_i be the subdigraph of D' which is the orientation of $\bar{I}_i$ using h_i, $i = 1, 2$. Observe that $l_D(C) = l_{D'}(I_1) + l_{D'}(I_2)$.

Again we distinguish two subcases.

Subcase 2.1: $h_1 = u_1v_1$.

In this case h forms a cycle with R_1.

As in Subcase 1.1, we extend B_0 along R_1 and consider h_1, h_2 as embedded on the extended B_0.

Observe that $CR(C) \stackrel{2}{=} CR(I_1) + CR(I_2)$: attention should be drawn to the question of how crossings of C with itself are manifested as crossings of I_1 or I_2, when all E^i_j are projected outside of B_0 (see Definition 2.3). In this case, the arguments are identical to those used in the proof of Subcase 1.1, and we omit them.

We construct two digraphs D_1, D_2 as follows:

- D_1 is obtained from $D - \{e, f\}$ by adding new vertices u'_1, v'_1 of degree 2, incident with new arcs e', f', h'_1. The arcs $e'f', h'_1$ are obtained from e, f, h_1 by replacing u_1 by u'_1 and v_1 by v'_1. We extend B_0 along R_2 and we embed the path (e', f', h'_1) on the extended B_0. Finally we add h'_1 to M_0. Let I'_1 be the cycle of D_1 obtained from I_1 by replacing e, f, g_1 by e', f', g'_1. We have $l_{D'}(I_1) = l_{D_1}(I'_1)$ and $CR(I_1) - 1 \stackrel{2}{=} CR(I'_1)$.

- D_2 is obtained from $D - \{e, f\}$ by adding the arc h_2. We consider h_2 embedded on the extended B_0 along R_1. We let $I'_2 = I_2$. Hence, $l_{D'}(I_2) = l_{D_2}(I'_2)$ and $CR(I_2) \stackrel{2}{=} CR(I'_2)$.

Hence, for $i = 1, 2$, D_i is an orientation of a proper g-graph and I'_i is an alternating cycle of D_i. Moreover $|BR(I'_i)| < |BR(C)|$.

Let us assume without loss of generality that h_2 is directed so that the cycle $l_{D'}(R_1, h_2) = 1$. Hence D_2 is a relevant orientation with $r(D_2) = r(D)$.

We show now that D_1 is a relevant orientation with $r(D) = r(D_1)$ if and only if $r(D)_{d(e)} = r(D)_{d(f)}$. We first prove that if $r(D)_{d(e)} = r(D)_{d(f)} = 1$ then D_1 is a relevant orientation.

In this case it suffices to show that $l_{D_1}(R_2, f', h'_1, e') \stackrel{2}{=} 1$. We have $l_{D_1}(h'_1, f', R_3) \stackrel{2}{=} l_{D'}(h_1, f, R_3) \stackrel{2}{=} l_{D'}(h_2, f, R_3) + 1 \stackrel{2}{=} l_{D_2}(h_2, f, R_3) + 1 \stackrel{2}{=} 0$, since $r(D_2)_{d(f)} = r(D)d(f) = 1$, and thus, $l_{D_2}(h_2, f, R_3) = 1$.

Moreover $l_{D_1}(R_2, R_3, e') = l_{D'}(R_2, R_3, e) = 1$, since $r(D)_{d(e)} = 1$ and D is a relevant orientation. Replacing $f'h'_1$ for R_3 gives what we claimed.

Similarly, we can prove that if $r(D)_{d(e)} = r(D)_{d(f)} = -1$ then again $l_{D_1}(R_2, f', h'_1, e') = 1$, and so, D_1 is a relevant orientation.

On the other hand, if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then D_1 is obtained from a relevant orientation by reversing one arc, and so, it is not relevant.

Summarizing, if $r(D)_{d(e)} = r(D)_{d(f)}$ then D_1 is a relevant orientation with $r(D) = r(D_1)$, and if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then D_1 becomes relevant after reversing the orientation of h'_1: this digraph, obtained from D_1 by reversing the orientation of h'_1, we denote by D^*_1, and its subdigraph corresponding to H'_1 we denote by H^*_1. Then $l_{D^*_1}(I'_1) \overset{2}{=} l_{D_1}(I'_1) + 1$.

Using the induction assumption of 3.1 for D^*, I^*_1, D_1, I'_1 and D_2, I'_2 we get:

$$l_D(C) \overset{2}{=} l_{D'}(I_1) + l_{D'}(I_2) \overset{2}{=} l_{D_1}(I'_1) + l_{D_2}(I'_2) \overset{2}{=}$$

$$|BR(C)| - 4 - CR(C) + 1 + \frac{1}{2} \sum_{p \in BR(C) - \{e,f\}} (r(D)_{d(p)} + 1) + \frac{1}{2}(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1) \overset{2}{=}$$

$$|BR(C)| - 1 - CR(C) + \frac{1}{2} \sum_{p \in BR(C)} (r(D)_{d(p)} + 1).$$

(End of Subcase 2.1)

Subcase 2.2: $h = u_1 v_2$.

In this case h forms a cycle with R_3. We extend B_0 along R_3 and consider h_1, h_2 embedded on the extended B_0.

Observe that $CR(C) \overset{2}{=} CR(I_1) + CR(I_2)$:attention should be drawn to the question of how crossings of C with itself are manifested as crossings of I_1 or I_2, when all E^i_j are projected outside of B_0 (see Definition 2.3). In this case, the arguments are identical to those used in the proof of Subcase 1.1 and Subcase 2.1, and we omit them.

We construct two digraphs D_1, D_2 as follows:

- D_1 is obtained from $D - \{e, f\}$ by adding new vertices u'_1, v'_2 of degree 2, incident with new arcs e', f', h'_1. The arcs e', f', h'_1 are obtained from e, f, h_1 by replacing u_1 by u'_1 and v_2 by v'_2. We extend B_0 along $R_1 R_3 R_2$ and we embed e', f', h'_1 on the extended B_0. Finally we add h'_1 to M_0. Let I'_1 be the cycle of D_1 obtained from I_1 by replacing e, f, h_1 by e', f', h'_1. We have that $l_{D'}(I_1) \overset{2}{=} l_{D_1}(I'_1)$ and $CR(I'_1) \overset{2}{=} CR(I_1) - 1$.

- D_2 is obtained from $D - \{e, f, \}$ by adding arc h_2. We again extend B_0 along R_3 and consider h_2 embedded on the extended B_0. We let $I'_2 = I_2$. Hence, $l_{D'}(I_2) \overset{2}{=} l_{D_2}(I'_2)$ and $CR(I'_2) \overset{2}{=} CR(I_2)$.

Hence for $i = 1, 2$, D_i is an orientation of a proper g-graph and I'_i is an alternating cycle of D_i. Moreover $|BR(I'_i)| < |BR(C)|$.

Let us assume without loss of generality that h_2 is directed so that $l(R_3, h_2) = 1$. Hence D_2 is a relevant orientation with $r(D_2) = r(D)$.

As in Subcase 2.1, we shall show that D_1 is a relevant orientation if and only if $r(D)_{d(e)} = r(D)_{d(f)}$: It again suffices to consider the case that $r(D)_{d(e)} = r(D)_{d(f)} = 1$. In this case it suffices to show that $l_{D_1}(R_2, R_3, R_1, f', h'_1, e') = 1$. In fact, we have $l_{D_2}(R_1, f, h_2) = 1$ since $r(D_{d(f)}) = 1$ and D_2 is a relevant orientation. Hence $l_{D_1}(R_1, f', h'_1) = 0$. Moreover $l_{D_1}(R_2, R_3, e') \overset{2}{=} l_D(R_2, R_3, e) = 1$ since $r(D)_{d(e)} = 1$. Hence $l_{D_1}(R_2, R_3, R_1, f', h'_1, e') = 1$.

Summarizing, if $r(D)_{d(e)} = r(D)_{d(f)}$ then D_1 is a relevant orientation with $r(D) = r(D_1)$, and if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then D_1 becomes relevant after reversing the orientation of h'_1.

The proof then proceeds analogously as in Subcase 2.1. *(End of Subcase 2.2)*

End of Case 2

It is not difficult to see that the two cases complete the proof. □

Next we show that a statement analougous to that of Theorem 3.1 holds for the set of the alternating cycles of $M_0 \Delta P$ as well.

Theorem 3.2 *Let G be a proper g-graph and let D be a relevant orientation of G. Let P be a perfect matching of G. Then*

$$sgn(D, M_0\Delta P) = (-1)^q,$$

where

$$q \overset{2}{=} |BR(M_0\Delta P)| - CR(M_0\Delta P) + \frac{1}{2} \sum_{e \in BR(M_0\Delta P)} (r(D)_{d(e)} + 1).$$

Proof. Let $C_1, ..., C_k$ be the alternating cycles of $M_0\Delta P$. We have that $sgn(D, M_0\Delta P) = (-1)^q$, where $q \overset{2}{=} l(C_1) + ... + l(C_k) - k$.

Using Theorem 3.1 for $C_1, ..., C_k$, it remains to show that:

$$CR(M_0\Delta P) \overset{2}{=} \sum_{j=1}^{k} CR(C_j),$$

but this holds by Lemma 2.8 and the theorem follows. □

Corollary 3.3 *Let G be a proper 1-graph and D a relevant orientation of G. Let $\mathcal{C}$ be a set of disjoint alternating cycles of M_0. Then:*

1. *If $r(D) = (1, 1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) \in \{(0, 0), (0, 1), (1, 0)\}$.*
2. *If $r(D) = (1, -1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) \in \{(0, 0), (1, 1), (1, 0)\}$.*
3. *If $r(D) = (-1, 1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) \in \{(0, 0), (0, 1), (1, 1)\}$.*
4. *If $r(D) = (-1, -1)$ then $sgn(D, \mathcal{C}) = 1$ if and only if $t(\mathcal{C}) = (0, 0)$.*

Definition 3.4 *Let G be a proper g-graph and D a relevant orientation of G. Let $r(D) = (r_1, ..., r_{2g})$. We let $c(r(D))$ equal to the product of c_i, $i = 0, ..., g-1$, where $c_i = c(r_{2i+1}, r_{2i+2})$ and $c(1,1) = c(1,-1) = c(-1,1) = 1/2$ and $c(-1,-1) = -1/2$.*

Observe that $c(r(D)) = (-1)^n 2^{-g}$, where $n = |\{i; r_{2i+1} = r_{2i+2} = -1\}|$.

Corollary 3.5 *Let G be a proper 1-graph. Let D_1, D_2, D_3, D_4 be the relevant orientations of G. Then*

$$\mathcal{P}(G, x) = \sum_{i=1}^{4} c(r(D_i))\mathcal{P}(D_i, M_0).$$

A result analogous to Corollary 3.5 holds for all proper g-graphs, $g > 1$. In order to deduce it we start with another corollary of Theorem 3.2.

Corollary 3.6 *Let G be a proper g-graph and D a relevant orientation of G. Let P be a perfect matching of G. Then $sgn(D, M_0 \Delta P)$ is a function of $r(D)$ and $t(M_0 \Delta P)$ only. Let us denote this function by $\sigma(r(D), t(M_0 \Delta P))$.*

Lemma 3.7 *Let $r = (r_1, \ldots, r_{2g})$ and $t = (t_1, \ldots, t_{2g})$ be $2g$-dimensional vectors. Let $r(j) = (r_{2j+1}, r_{2j+2})$ and $t(j) = (t_{2j+1}, t_{2j+2})$, $j = 0, \ldots, g-1$. Then*

$$\sigma(r, t) = \prod_{j=0}^{g-1} \sigma(r(j), t(j)).$$

Proof. By Corollary 3.6, we have that $sgn(D, \mathcal{C}) = sgn(D', \mathcal{C}')$ if and only if $r(D) = r(D')$ and $t(\mathcal{C}) = t(\mathcal{C}')$. This implies that we can restrict ourselves to consider the following case: $G = C_0 \cup M_0 \cup \mathcal{C}$ is a proper g-graph, D is a relevant orientation of G such that $r(D) = r$ and $\mathcal{C}$ consists of a set of vertex-disjoint cycles $C_1, \ldots, C_k$ satisfying the following properties:

1. each C_i is alternating with respect to the perfect matching M_0,
2. for each i, j $|E^i_j| \leq 1$,
3. for each i there is at most one j such that $|C_j \cap (E^i_1 \cup E^i_2)| \geq 1$,
4. for each C_j there is exactly one i such that C_j intersects $E^i_1 \cup E^i_2$,
5. $t(\mathcal{C}) = t$.

Hence,

$$\sigma(r, t) = sgn(D, \mathcal{C}) = \prod_{i=1}^{k} sgn(D, C_i) = \prod_{i=1}^{k} sgn(D_i, C_i)$$

where D_i is the restriction of D to $C_0 \cup C_i$. Observe that, by Corollary 3.3, $\sigma(z_1, z_2) = 1$ if $z_2 = (0, 0)$. Hence, using Corollary 3.6, we have that $\prod_{i=1}^{k} sgn(D_i, C_i) = \prod_{j=0}^{g-1} \sigma(r^j, t^j)$ as claimed.

□

Theorem 3.8 *Let G be a proper g-graph. Then*

$$\mathcal{P}(G,x)=\mathcal{L}_g(G,x)=\sum_{i=1}^{4^g} c(r(D_i))\mathcal{P}(D_i,M_0)$$

where D_i, $i=1,\dots,4^g$, are the relevant orientations of G.

Proof. Let P be a perfect matching of G. In each term $\mathcal{P}(D_i,M_0)$, the coefficient of $x(P)$ is $sgn(D_i,M_0\Delta P)$. By Corollary 3.6, $sgn(D_i,M_0\Delta P)=\sigma(r(D_i),t(M_0\Delta P))$.

Let

$$\mathcal{K}_g(t(M_0\Delta P))=\sum_{i=1}^{4^g} c(r(D_i))\sigma(r(D_i),t(M_0\Delta P))$$

denote the coefficient of $x(P)$ in $\mathcal{L}_g(G,x)$.

To prove the theorem it suffices to prove the following claim:

Claim. $\mathcal{K}_g(t(M_0\Delta P))=1$ *for each* $t(M_0\Delta P)$.

The proof of the claim is by induction on g. The basis of the induction when $g=1$ is proved in Corollary 3.5.

To prove the inductive step we introduce the following notation: if z is a $2g$-dimensional vector then we let $z=(z(0),\dots,z(g-1))$ where $z(i)=(z_{2i+1},z_{2i+2})$.

We call two relevant orientations D and D' of G *equivalent* if $(r(D)(1),\dots,r(D)(g-1))=(r(D')(1),\dots,r(D')(g-1))$. Clearly, the equivalence classes consist of 4 elements; let $\mathcal{R}_1,\dots,\mathcal{R}_{4^{g-1}}$ be the equivalence classes of the relevant orientations of G and let $\mathcal{R}_j=\{D_1^j,D_2^j,D_3^j,D_4^j\}$, $j=1,\dots,4^{g-1}$.

Finally let $r(D_i^j)(k)=r_i^j(k)$, $k=0,\dots,g-1$ and let $t=t(M_0\Delta P)$. We have that

$$\mathcal{K}_g(t)=\sum_{j=1}^{4^{g-1}}\sum_{i=1}^{4} c(r(D_i^j))\sigma(r(D_i^j),t).$$

Now, by Lemma 3.7, this equals

$$\sum_{j=1}^{4^{g-1}}\sum_{i=1}^{4} c(r_i^j(0))c(r_i^j(1),\dots,r_i^j(g-1))\prod_{k=0}^{g-1}\sigma(r_i^j(k),t(k)).$$

By the definition of the equivalence classes, $r_1^j(k)=r_2^j(k)=r_3^j(k)=r_4^j(k)$ for $k\geq 1$ and $j=1,\dots,4^{g-1}$. Hence, we let $r_i^j(k)=r^j(k)$ and write the above summation as:

$$\sum_{j=1}^{4^{g-1}} c(r^j(1),\dots,r^j(g-1))\prod_{k=1}^{g-1}\sigma(r^j(k),t(k))\sum_{i=1}^{4} c(r_i^j(0))\sigma(r_i^j(0),t(0))$$

.

The internal sum equals to 1 for each $j=1,\dots,4^{g-1}$ by the basis step of the induction, and hence, using Lemma 3.7 in the external sum, we can write the above summation as

$$\sum_{j=1}^{4^{g-1}} c(r^j(1),\ldots,r^j(g-1))\sigma((r^j(1),\ldots,r^j(g-1)),(t(1),\ldots,t(g-1))) =$$

$$\mathcal{K}_{g-1}(t(1),\ldots,t(g-1)) = 1,$$

by the inductive hypothesis for $g-1$. **End of Claim.**

□

As a consequence of Theorem 1.4 and Theorem 3.8, we have:

Corollary 3.9 *Let G be a proper g-graph. Then $s(D_i, M_0) = s(D_j, M_0)$ for each $i, j \in \{1, \ldots, 4^g\}$ and*

$$\mathcal{P}(G,x) = \mathcal{L}_g(G,x) = s(D_1, M_0)\sum_{i=1}^{4^g} c(r(D_i))Pf(A(D_i))$$

where D_i, $i = 1, \ldots, 4^g$, are the relevant orientations of G.

Theorem 3.10 *Let G be a graph embeddable on an orientable surface of genus g. Then $\mathcal{P}(G,x)$ may be expressed as a linear combination of 4^g Pfaffians of matrices $A(D)$, where each D is an orientation of G.*

Proof. As observed in the previous section, any orientable surface $\mathcal{S}$ of genus g may be obtained from its polygonal representation S_g as follows: for each bridge B, glue together the two segments in which B intersects the boundary of B_0, and delete B.

If a graph G is embedded on an orientable surface $\mathcal{S}$ of genus g, then without loss of generality no vertex belongs to the boundary of B_0. In this way we get an embedding of G on S_g such that all vertices of G belong to B_0 but the embeddings of some edges may use several bridges.

We construct a graph G' by replacing each edge $e = uv$ which uses k bridges, $k \geq 1$, by a path $P_e = (u, e_1, v_1, \ldots, v_{2k}, e_{2k+1}, v)$. The new vertices $v_1, ..., v_{2k}$ are embedded on the embedding of e so that each new edge uses at most one bridge. Moreover, we let $x'_{e_1} = x_e$ and $x'_{e_i} = 1$ for each $i > 1$. We do a similar construction when G_0 has no perfect matching. In fact, take any perfect matching M of G and replace any edge $e = uv \in M$ embedded on a bridge by a path $u, e_1, y, e_2, z, e_3, v$ and let $x'_{e_1} = x_e$ and $x'_{e_2} = x'_{e_3} = 1$. Then leave the only edge e_2 to be embedded on the bridge B.

Finally, we add edges so that the outer face of the planar part is a cycle and we let $x'_e = 0$ for each such edge e.

It is easy to see that G' is a proper g-graph and that $\mathcal{P}(G', x') = \mathcal{P}(G, x)$.

Now, by Theorem 3.8, $\mathcal{P}(G', x')$ may be written as a linear combination of 4^g terms $Pf(A(D'))$, where each D' is a relevant orientation of G'.

It remains to show that for each relevant orientation D' of G' there is an orientation D of G such that $Pf(A(D')) = Pf(A(D))$ or $Pf(A(D')) = -Pf(A(D))$.

We construct D from D' in two steps:

1. delete the edges e of $G' - G$ with $x'_e = 0$,
2. for each edge e of G which was changed into a path P_e of odd length in the construction of G', orient e in the direction in which an odd number of edges of P_e is directed in D': this is uniquelly determined since P_e has an odd length.

If P is a perfect matching of G then there is a unique perfect matching P' of G' such that $x(P) = x'(P')$.

Observe that $sgn(D, P\Delta Q) = sgn(D', P'\Delta Q')$ for each pair of perfect matchings P, Q of G. The claim now follows from Theorem 1.4.

This finishes the proof of the theorem. □

4 Pfaffian Graphs, Exact Matching, and Permanents

The results of the previous section have interesting algorithmic implications.

Theorem 4.1 *Let g and k be fixed positive integers. Let $\mathcal{G}$ be the class of graphs of genus g whose edges are partitioned into at most k classes and the variables x_e have the same value in each class. Then $\mathcal{P}(G, x)$ may be determined in polynomial time for $G \in \mathcal{G}$.*

Proof. It follows from Theorems 3.8 and 3.10 that $\mathcal{P}(G, x)$ may be expressed as a linear combination of a finite number of Pfaffians.

We show now that if the set of the edges of graph G is partitioned into a bounded number of classes and the variables x_e are equal in each class, then $\mathcal{P}(D, M)$ and $Pf(A(D))$ may be determined efficiently. Let $M = \{\{i_1 j_1\}, \ldots, \{i_n j_n\}\}$, $i_k < j_k$, be a perfect matching of G. Let x' be defined as follows: $x'_e = x_e$ if $e \notin M$ and $x'_f = x_f z$ if $f \in M$, where z is a new variable. Let A' be the matrix obtained from $A(D)$ by replacing each x_e by x'_e. Then $det(A')$ may be viewed as a polynomial $det(A')(x, z)$ in the variables x and z and its coefficients can be determined efficiently.

By Theorem 1.5, $Pf(A')(x, z) = \pm\sqrt{det(A')(x, z)}$. Hence we can determine efficiently a polynomial $Q(x, z)$ such that $Pf(A')(x, z) = \pm Q(x, z))$. Note that $\mathcal{P}(D, M) = \pm Q(x, 1)$.

There is exactly one monomial in $Q(x, z)$ containing z^n and its coefficient is $+1$ or -1. Let $Q'(x, z)$ be the unique polynomial such that $Q'(x, z) = Q(x, z)$ or $Q'(x, z) = -Q(x, z)$ and the coefficient of $Q'(x, z)$ of the term containing z^n equals $+1$. We have $\mathcal{P}(D, M) = +Q'(x, 1)$. Moreover, $Pf(A(D)) = s(D, M)\mathcal{P}(D, M)$ and $s(D, M) = s^*(M)t^*(M)$ where $t^*(M)$ equals the product of the signs of the elements $a_{i_k j_k}$ of the matrix $A(D)$ such that $i_k j_k \in M$. Hence $\mathcal{P}(D, M)$ and $Pf(A(D))$ may be determined efficiently. □

As a consequence, if we are in the hypothesis stated by the theorem, the following problems may be solved efficiently.

1. Recognition of Pfaffian graphs: *given a graph $G \in \mathcal{G}$, decide whether G admits a Pfaffian orientation.*

It was proved by Vazirani and Yannakakis (see the proof of Theorem 3.1 in [13]) that, given a graph G, it is possible to construct efficiently an orientation D of G such that G is Pfaffian if and only if D is its Pfaffian orientation. Hence $Pf(A(D))$ equals to the number of the perfect matchings of G if and only if G is Pfaffian, and it means that we can decide efficiently whether a graph is Pfaffian once we can compute efficiently its number of perfect matchings.

2. Exact Matching Problem: *given a graph $G \in \mathcal{G}$ with some edges coloured red, and a number h, decide whether G has a perfect matching with exactly h red edges.*

It suffices to assign an x variable to red edges and a y variable to the remaining edges and compute $\mathcal{P}(G; x, y)$. If the coefficient of the monomial $x^h y^{t-h}$, where t is the cardinality of a perfect matching of G, is nonzero then the answer to the problem is yes, otherwise no such a matching exists.

3. Computing permanents of square matrices.

In 1913, Pólya [9] suggested computing the permanent of a matrix A by changing the signs of some entries of A so that the determinant of the resulting matrix equals the permanent of A. Let us call a $(0,1)$-matrix A *convertible* if such a change is possible.

Szegö [11] pointed out in the same year that not all matrices are convertible.

This may be explained nowadays using a complexity argument. There is an efficient algorithm to compute the determinant, while Valiant proved that the problem of computing the permanent of a $(0,1)$-matrix is $\#P$-complete [12].

The computational problem of recognition of convertible matrices has been proved recently to admit a polynomial algorithm by McCuaig, Robertson, Seymour and Thomas [8]. An earlier paper of Galluccio and Loebl [3] contains a related algorithmic result, as well as a history of the problem.

The problem of recognizing convertible matrices is equivalent to the problem of recognizing bipartite Pfaffian graphs, and to the *Even Cycle Problem*: given a directed graph, decide whether it contains a directed cycle of even length.

Let A be a square matrix. Denote by $G(A)$ the bipartite graph whose two bipartition classes are indexed by the rows and the columns of A, and for each edge ij, $a_{ij} = x_{ij}$. Then $per(A) = \mathcal{P}(G(A), x)$.

Hence, Theorem 3.10 provides a new combinatorial way to compute permanents of square matrices: $per(A)$ may be written as a linear combination of 4^g terms of form $Pf(A(D))$, where D is an orientation of $G(A)$ and g is the genus of $G(A)$.

Since $G(A)$ is a bipartite graph, the non-zero entries of $A(D)$ belong to two blocks A_1, A_2, where A_1 is obtained from A by changing the sign of some entries and $A_2 = -A_1$. Moreover $|Pf(A(D)| = |det(A_1)| = |det(A_2)|$ by Theorem 1.5.

This means that the method of Pólya may be completed as follows:

Corollary 4.2 *Let A be a square matrix. Then $per(A)$ may be expressed as a linear combination of terms of the form $det(A^i)$, $i = 1, ..., 4^g$, where each A^i is obtained from A by changing the sign of some entries.*

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